MAA OMWATI DEGREE COLLEGE HASSANPUR

NOTES

CLASS:- BCA 1ST SEM

SUBJECT: MATHEMATICAL FOUNDATION OF COMPUTER SCIENCE (MC)

* Unit - 1 *

(Sets, Relation and functions)

*** Topic Covered :**

Sets, Subsets, Equal Sets, Universal Sets, Finite and Infinite Sets, Operation on sets, Union and Intersection of Sets, Complements on Sets, Cartesian Product, Cardinality of set, Practical application of Set Theory.

*** Relation and Functions :**

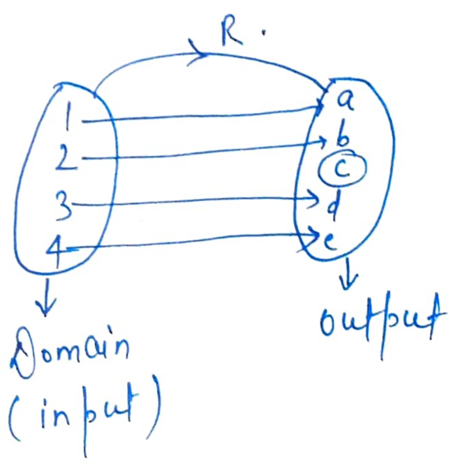
Properties of Relations, Equivalence Relation, Partial order Relation, Function, Domain and Range, Onto, Into and one-one function, Composite and Inverse Functions.

✩ Relation ✩

A relation R from set A to set B is a subset of Cartesian product of $A \times B$.

Every Relation has a property.

→ Domain : Set of all permissible input.
→ Range : Set of all permissible output.
→ Co-domain: All possible output.



Range = $\{a, b, d, e\}$
Co-domain = $\{a, b, c, d, e\}$

Note : The total number of relations that can be defined from set A to a set B is the number of possible subset of $A \times B$ if

$$n(A) = P\ (n(B)) = q$$

Total ~~s~~ number of relations $= 2^{pq}$.

Ex- Let $A = \{1,2\}$ and $B = \{3,4\}$. Find the number of relations from A to B.

Solⁿ:

$n(A) = 2$

$n(B) = 2$

No. of relations $= 2^{2 \times 2}$

$= \underline{\underline{2^4}}.$

Ex- Find the domain of the following real-valued function:

(i) $f(x) = \frac{1}{x+2}$ (ii) $f(x) = \frac{x-1}{x-3}$.

Solⁿ:

(i) $f(x) = \frac{1}{x+2}$

$x + 2 \neq 0$

$x \neq -2$

Domain $\Rightarrow R - \{-2\}$.

(ii) $f(x) = \frac{x-1}{x-3}$

$x - 3 \neq 0$

$x \neq 3$

Domain $\Rightarrow R - \{3\}$.

Find domain of given function:

$$f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$$

Sol^n:

$$f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$$

$4 - x \geq 0$

$4 \geq x$

$x \leq 4$

$x^2 - 1 > 0$

$(x-1)(x+1) > 0$

$x > 1$

$x < -1$

Domain $= (-\infty, -1) \cup (1, 4]$.

☆

Function	Domain	Range
1. Rational function $f(x) = \frac{P(x)}{q(x)}$, $q(x) \neq 0$	$R - \{0\}$	$R - \{0\}$
2. Modulus function $f(x) = \lvert x \rvert = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$	R	$[0, \infty)$
3. Greatest integer function $f(x) = [x]$	R	Z
4. Signum function $f(x) = \text{signum}(x) = \begin{cases} \frac{\lvert x \rvert}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$	R	$\{-1, 0, 1\}$.

$$\begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

★ Algebra of functions :

$$(f+g)(x) = f(x) + g(x) \quad , x \in X$$

$$(f-g)(x) = f(x) - g(x) \quad , x \in X$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \quad , x \in X$$

for functions $f : X \to R$ and $g : X \to R$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad , x \in X \ , \ g(x) \neq 0 .$$

Universal Set :-

A universal set is a set that contains all the elements of all related sets, without repeating any elements. It is denoted by "U".

Ex → $A = \{1, 2, 3\}$
$B = \{1, a, b, c\}$
$\Rightarrow U = \{1, 2, 3, a, b, c\}$.

* Finite Set :

A set having finite number of elements is called finite set.

Note : Cordinal number of a finite set → Number of elements in a set.

Ex → $A = \{1, 2, 3, 4, 5\}$
$n(A) = \underline{5}$.

* Infinite Set :-

A set having infinite number of elements in a set is called Infinite set.

Ex – A set of Natural numbers.

* Sets :-

Set is a well-defined collection of digits.
Set is usually denoted by Capital letters.

Ex- A, B, C, D, X, Z.

Elements of set is denoted by small letters.

Ex- The set of Natural Numbers

$N = \{1, 2, 3, \dots\}$ → Roaster or Tabular form.

* Subset :-

A set is said to be a subset of a set B if all the elements of set A is also an element of set B.

It is denoted by $A \subset B$.
↓
A is a subset of B.

Ex- $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$ $\quad$ $A \subset B$.

* Equal set :

Two sets are said to be equal if they have exactly same element.

Ex- $A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3, 4\}$.

Union of sets :-

Let A and B two sets then union of A and B is the set which Consist of all the elements of A and all the elements of B, the Common elements being taken Only once. It is denoted by "U".

Ex →

$A = \{1, 2, 3\}$

$B = \{3, 4, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$.

★ Intersection of sets:

The intersection of sets A and B is the set of all elements which are common to both A and B. It is denoted by "∩".

Ex →

$A = \{1, 2, 3\}$

$B = \{3, 4, 5, 6\}$

$A \cap B = \{3\}$.

★ Complement of a set :

Let 'U' be the universal set and 'A' be the subset of 'U' then Complement of A is the set of all elements of U which are not the elements of A.

It is denoted by A' or $\bar{A}$.

$$\boxed{A' = U - A}$$

* De-Morgan's Law :-

1. $(A \cup B)' = A' \cap B'$.
2. $(A \cap B)' = A' \cup B'$.

* Cartesian Product of sets :-

Let two non-empty sets 'P' and 'Q' then the Cartesian product $P \times Q$ is the set of all ordered pairs whose first elements is component of P's

$$P \times Q = \{(p, q) : p \in P, q \in Q\}$$

If P and Q is the null set then

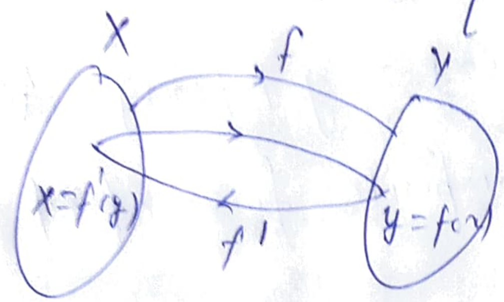
$$P \times Q = \phi.$$

Note:

$$A \times B \neq B \times A.$$

Inverse function:

If $f: X \longrightarrow Y$ is a bijection then there always exist pre image $f^{-1}(y)$ of each element y of Y and this will be a unique element of X.



Let $f: R \to R$ s.t $f(x) = 2x - 3$

We know that it is one-one & onto

So, $y = 2x - 3$

$$x = \frac{y+3}{2}$$

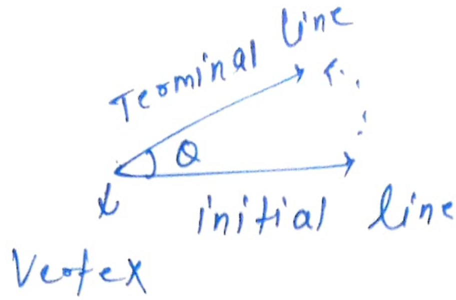
$$f^{-1}(y) = \frac{y+3}{2}$$

Unit - 2nd

* Topic Covered :

1. Introduction
2. Measurement of Angles
3. Limit at a point
4. Properties of limit.
5. Computation of limits of Various functions
6. Continuity of a function at a point.
7. Continuity over an interval
8. Previous year Questions.

* Trigonometry *



1 hour $= 60$ Minute

1 Rotation $= 360^\circ$

$1^\circ = 60'$

$1' = 60''$

* Radian Measure :

$$\theta = \frac{l}{r}.$$

Circumference $= 2\pi r$

$$\theta = \frac{2\pi r}{r}$$

$$\boxed{\theta = 2\pi}$$

Radian Measure $= \frac{\pi}{180^\circ} \times$ Degree Measure

Q. Convert 60° to radian form and Convert into degree. (π).

Soln: Radian Measure $= \frac{\pi}{180} \times$ Degree Measure

$= \frac{\pi}{180} \times 60 = \frac{\pi}{3}$ Radian.

* Limit of a function *

A function $f(x)$ is said to have a limit 'l' as $x \longrightarrow a$ written as

$$\lim_{x \to a} f(x) = l \quad \text{if } \epsilon > 0,$$

however small $\exists$ a +ve real number 'δ' s.t

$$|f(x) - l| < \epsilon \quad \forall \quad 0 < |x - a| < \delta$$

Ex- $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

Solⁿ:-

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1) = 1 + 1 = 2$$

* Left Hand limit :-

$$\lim_{x \to a^-} f(x) = l.$$

* Right Hand limit :

$$\lim_{x \to a^+} f(x) = l.$$

Q1. Solve! $\lim_{x \to 1} \dfrac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}}$

Solⁿ:-

$$\lim_{x \to 1} \frac{\sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} + \lim_{x \to 1} \frac{\sqrt{x - 1}}{\sqrt{x^2 - 1}}$$

$$\lim_{x \to 1} 1 + \lim_{x \to 1} \frac{\sqrt{x + 1}}{\sqrt{(x-1)(x+1)}}$$

$$\Rightarrow 1 + \frac{1}{\sqrt{2}} \Rightarrow \frac{\sqrt{2} + 1}{\sqrt{2}} \text{ Ans.}$$

★ Note:

1. $\lim_{x \to a} l\, f(x) = l \lim_{x \to a} f(x)$

where 'l' is any Constant.

2. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$.

3. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

4. $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$.

Q2. Evaluate:

$$\lim_{x \to 0} \frac{x^2+5x+6}{5}$$

Sol^n:

$$\lim_{x \to 0} \frac{1}{5}\left(x^2+5x+6\right)$$

$$\Rightarrow \frac{1}{5}(0+0+6) \Rightarrow \frac{6}{5} \text{ Ans.}$$

Computation of limits of various type of functions :-

~~Q1.~~

Note:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\tan x}{x} = 1$$

Q1. Evaluate:

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 15x}$$

Soln:- The given function is:

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 15x} = \lim_{x \to 0} \frac{\frac{\sin 5x}{5x} \times 5x}{\frac{\sin 15x}{15x} \times 15x}$$

$$\Rightarrow \frac{\lim_{x \to 0} \frac{\sin 5x}{5x} \times 5x}{\lim_{x \to 0} \frac{\sin 15x}{15x} \times 15x}$$

$\left\{ \because \lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} g(x)} \right.$

$$\Rightarrow \lim_{x \to 0} \frac{5\cancel{x}}{15\cancel{x}} = \frac{1}{3} \text{ Ans.}$$

Q2 Evaluate:

$$\lim_{x \to 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$$

Soln:-

$$\lim_{x \to 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$$

$$\Rightarrow \lim_{x \to 0} \frac{\frac{\tan 3x}{3x} \times 3x - 2x}{3x - \frac{\sin^2 x}{x^2} \times x^2}$$

$$\Rightarrow \lim_{x \to 0} \frac{3x - 2x}{3x - x^2} \qquad \left\{ \because \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\tan x}{x} = 1 \right.$$

$$\Rightarrow \lim_{x \to 0} \frac{x}{3x - x^2}$$

$$\Rightarrow \lim_{x \to 0} \frac{1}{3 - x} = \frac{1}{3-0} = \frac{1}{3} \text{ Ans.}$$

Q3. $\lim_{x \to 0} \frac{x^3 \cot x}{1 - \cos x}$ Evaluate.

Sol^n:- $\lim_{x \to 0} \frac{x^3 \cos x}{\sin x \left(2\sin^2 x/2\right)}$ $\left\{ \because 1 - \cos x = 2\sin^2 x/2 \right.$

$$\Rightarrow \lim_{x \to 0} \frac{x^3 \cos x}{x \cdot \frac{\sin x}{x} \cdot 2 \frac{\sin^2 x/2}{x^2/4} \times x^2/4}$$

$$\Rightarrow \lim_{x \to 0} \frac{\cancel{x^3} \cos x}{\cancel{x^3} \times \cancel{2}} \times \cancel{4}\, 2 = \lim_{x \to 0} 2\cos x$$

$$\Rightarrow 2 \times \cos 0 \Rightarrow 2 \text{ Ans} \qquad \left\{ \because \cos 0^\circ = 1. \right.$$

Determine the intervals over which the following function is Continuous:

$$f(x) = \frac{x-3}{x^2+4x}.$$

Sol^n:

The given function is:

$$f(x) = \frac{x-3}{x^2+4x}$$

$$f(x) = \frac{x-3}{x(x+4)}$$

$$x=0, \quad x=-4$$

$$(-\infty,-4) \cup (-4,0) \cup (0,\infty)$$

Q2. If $f(x) = \begin{cases} -x^2 & \text{if } -1 \le x < 0 \\ 4x-3 & \text{if } 0 \le x \le 1 \\ 5x^2-4x & \text{if } 1 < x \le 2 \end{cases}$ then

test the Continuity in $[-1,2]$.

Sol^n:

The given function is:

$$f(x) = \begin{cases} -x^2 & \text{if } -1 \le x < 0 \\ 4x-3 & \text{if } 0 \le x \le 1 \\ 5x^2-4x & \text{if } 1 < x \le 2 \end{cases}$$

RHL at (-1) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (-x^2)$

$$\Rightarrow \lim_{h\to 0} (-1+h)^2 = -1$$

$$f(-1) = -(-1)^2 = -1.$$

$$\lim_{x \to -1^+} f(x) = f(-1).$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (5x^2 - 4x)$$

$$\Rightarrow \lim_{h \to 0} [5(2-h)^2 - 4(2-h)]$$

$$\Rightarrow \lim_{h \to 0} 5(4+h^2-4h) - (8+4h)$$

$$\Rightarrow 20 - 8 = 12.$$

$$f(2) = 5x^2 - 4x = 5.4 - 4.2 = 12$$

$$\lim_{x \to 2^-} f(x) = f(2)$$

Continuity at $x = 0$

LHL $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (-x^2)$ 0

RHL $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} 4x - 3 = \lim_{h \to 0} 4(0+h) - 3$

$= -3$

RHL $\neq$ LHL
at $x=0$ at $x=0$

So, the above function is discontinuous at ~~x=~~ Interval $[-1, 2]$.

If $f(x) = \begin{cases} \dfrac{\sin 2x}{\sin 3x} & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$ find whether $f(x)$ is continuous at $x = 0$.

Sol^n:

The given function is:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{\sin 3x}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$$

LHL

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$$

$$\Rightarrow \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} \times 2x}{\frac{\sin 3x}{3x} \times 3x} = \lim_{x \to 0} \frac{2x}{3x} \qquad \left\{ \because \lim_{x \to 0} \frac{\sin x}{x} = 1 \right.$$

$$\Rightarrow \frac{2}{3}$$

$\Rightarrow$ RHL

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} \times 2x}{\frac{\sin 3x}{3x} \times 3x} = \frac{2}{3}$$

$f(0) = 2$.

Since, LHL $\neq$ RHL

So, given function is not continuous at $x = 0$.

★ Continuity of a function at a Point :-

A function $f(x)$ is said to be Continuous at a point 'a' of its domain if

$$\lim_{x \to a} f(x) = f(a) \to x = a$$

OR

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

$$RHS = LHS = f(a).$$

Q1. Discuss the Continuity of the function F given by:
$F(x) = |x|$ at $x = 0$.

Sol^n:- The given function is:

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

<u>LHL</u>

$$\lim_{x \to a^-} f(x) = \lim_{x \to 0} (-x) = 0$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to 0} (x) = 0$$

$$f(0) = 0$$

$\because$ LHL = RHL = $f(0)$

Hence, $f(x) = |x|$ is Continuous at $x = 0$.

Determine the value of 'k' for which the following function (x) is continuous at $x=3$

$$f(x) = \begin{cases} \dfrac{(x+3)^2-36}{x-3} & , x \neq 3 \\ k & x = 3 \end{cases}$$

Sol^n:

The given function is:

$$f(x) = \begin{cases} \dfrac{(x+3)^2-36}{x-3} & , x \neq 3 \\ k & , x = 3 \end{cases}$$

LHL

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{(x+3)^2-36}{x-3}$$

Put $x = 3-h$ so $h \to 0$

$$\Rightarrow \lim_{h \to 0} \frac{(3-h+3)^2-36}{\cancel{3}-h-\cancel{3}} = \lim_{h \to 0} \frac{\cancel{36}+h^2-12h-\cancel{36}}{-h}$$

$$\Rightarrow \lim_{h \to 0} 12-h = 12-0 = \underline{12}$$

RHL

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{(x+3)^2-36}{x-3}$$

Put $x = 3+h$, $h \to 0$

$$\Rightarrow \lim_{h \to 0} \frac{(3+h+3)^2-36}{\cancel{3}+h-\cancel{3}} = \lim_{h \to 0} \frac{\cancel{36}+h^2+12h-\cancel{36}}{h}$$

$$\Rightarrow \lim_{h \to 0} h+12 = 0+12 = \underline{12}$$

Since, f(x) is continuous

So, RHL = LHL = f(3)

$$\boxed{12 = k}$$ Ans.

Continuity Over an Interval :-

A function $f(x)$ is said to be Continuous over an open interval (a,b) if it is continuous at each point of (a,b).

If $f(x)$ is defined on a closed interval $[a,b]$ then we can say that:

(i) f is Continuous at a if $\lim_{x \to a^+} f(x) = f(a)$

(ii) f is Continuous at b if $\lim_{x \to b^-} f(x) = f(b)$

(iii) f is Continuous in $[a,b]$ if it is Continuous at a, at b and at at each point of (a,b).

Unit - 3rd

★ Differentiation ★

★ Topic Covered:

1. Derivative of a function
2. Derivative of Sum, Difference, product and Quotient of function.
3. Derivative of Polynomial
4. Derivative of Trigonometric & Exponential fun.
5. Derivative of logarithm and Inverse func.
6. Chain Rule and differentiation by Substituate.
7. Previous year Questions.

$\frac{d}{dx}(x^n) = nx^{n-1}$ Ex→ $\frac{d}{dx}(x^2) = 2x$

$\frac{d}{dx}(\text{constant}) = 0$ $\frac{d}{dx}(x^3) = 3x^2$

$\frac{d}{dx}$ ~~(Sinx~~ $\frac{d}{dx}(kx) = k$ $\frac{d}{dx}(a^x) = a^x . \log a$.

* Differentiation of Sum & Product of functions:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

where 'u' & 'v' are functions of 'x'.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Ex→ 1. If $y = 2x^3 + 5x^2 + 7x + 6$ then find $\frac{dy}{dx}$.

2. If $y = 5x^3 - 8x^2 + 6x - 9$ then find $\frac{dy}{dx}$.

3. If $y = (3x+1)(5x-7)$ then find $\frac{dy}{dx}$.

4. If $y = \frac{x^2+x+1}{x+5}$, then find $\frac{dy}{dx}$.

* Differentiation of Root function:

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

* Differentiation of Trigonometric function *

1. $\frac{d}{dx}(\sin x) = \cos x.$

2. $\frac{d}{dx}(\cos x) = -\sin x$

3. $\frac{d}{dx}(\tan x) = \sec^2 x$

4. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$

5. $\frac{d}{dx}(\sec x) = \sec x \tan x$

6. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Ex→ 1. If $y = \sin^2 x$ then find $\frac{dy}{dx}$.

2. If $y = \sin x^3$, then find $\frac{dy}{dx}$.

3. If $y = x \sin x$, then find $\frac{dy}{dx}$.

4. If $y = x^2 \cos x^2$, then find $\frac{dy}{dx}$.

* Differentiation of Inverse Trigonometric functions:

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}.$

Differentiation of log functions:

1. $\log(m^n) = n \log m$.
2. $\log(mn) = \log m + \log n$
3. $\log\left(\frac{m}{n}\right) = \log m - \log n$
4. $\log_e e = \log_a a = 1$
5. $\log_a b = \frac{\log b}{\log a}$.

$\frac{d}{dx}(\log x) = \frac{1}{x}$.

Ex-1. Find $\frac{dy}{dx}$, if $y = \log x^2$.

2. Find $\frac{dy}{dx}$ if $y = \log x^{x^2}$.

3. Find $\frac{dy}{dx}$ if $y = x^{\sin x}$

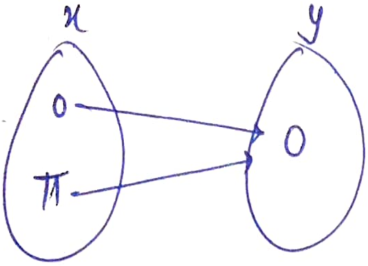
4. Find $\frac{dy}{dx}$, if $y = x^{\cos x}$.

5. Find $\frac{dy}{dx}$ if $y = (\sin x)^{\cos x}$.

→ The function which is one-one and onto Can be written in Inverse form.

Ex→

$y = \sin x$



$f(x) = \sin x$ is not one-one function.

Domain = $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Range = $[-1, 1]$

Inverse can be determined within a limit.

$f : x \to y$

2 *

1. Differentiate $\sqrt{1+x^2}$ w.r.t 'x'.
2. Find $\frac{dy}{dx}$ if $y = \cos^{-1} x^3$.
3. Find derivative of $\cot^{-1}\left(\frac{1+x}{1-x}\right)$
4. Find $\frac{dy}{dx}$ if $y = x^x + x^{1/x}$.
5. Differentiate $x^{\cos x}$ w.r.t $(\cos x)^x$.
6. Find $\frac{dy}{dx}$, if $y = \frac{\sqrt{1-x}}{\sqrt{1+x}}$.

Differentiation of Exponential function :-

$$\frac{d}{dx}(e^x) = e^x.$$

Ex→ Find $\frac{dy}{dx}$ if $y = e^{5x}$.

Soln:-

$$y = e^{5x}$$

diff. w.r.t 'x'

$$\frac{dy}{dx} = e^{5x}\frac{d}{dx}(5x) = e^{5x}\cdot 5 = 5e^{5x}.$$

$$\Rightarrow \frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)}\cdot \ln a \cdot f'(x).$$

$$\Rightarrow \frac{d}{dx}(a^x) = a^x \log a.$$

Q. If $y = x + \sqrt{x^2-1}$ then show that

$$(y-x)\frac{dy}{dx} - y = 0$$

Soln:- Given,

$$y = x + \sqrt{x^2-1}$$

Now, differentiate w.r.t 'x' both sides,

$$\frac{dy}{dx} = 1 + \frac{1}{2\sqrt{x^2-1}}\frac{d}{dx}(x^2-1) \qquad \left(\because \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{1}{2\sqrt{x^2-1}} \times 2x$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{1\cdot x}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{x}{\sqrt{x^2-1}}$$

$$\Rightarrow (y-x)\frac{dy}{dx} = (y-x)\left[1+\frac{x}{\sqrt{x^2-1}}\right]$$

$$\Rightarrow (y-x)\frac{dy}{dx} = (\cancel{x}+\sqrt{x^2-1}-\cancel{x})\left[1+\frac{x}{\sqrt{x^2-1}}\right] \quad \left(\because y = x+\sqrt{x^2-1}\right)$$

$$\Rightarrow (y-x)\frac{dy}{dx} = (\sqrt{x^2-1})\left[1+\frac{x}{\sqrt{x^2-1}}\right]$$

$$\Rightarrow (y-x)\frac{dy}{dx} = \sqrt{x^2-1} + x$$

$$\Rightarrow (y-x)\frac{dy}{dx} - y = (\sqrt{x^2-1}+x) - (\sqrt{x^2-1}+x)$$

$$\Rightarrow \boxed{(y-x)\frac{dy}{dx} - y = 0}$$

Hence Proved.

Q. Find $\frac{dy}{dx}$, if $y = \cot^{-1}(\operatorname{cosec} x + \cot x)$

Sol^n:- Given,

$$y = \cot^{-1}(\operatorname{cosec} x + \cot x)$$

$$y = \cot^{-1}\left[\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right]$$

$$y = \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right]$$

$$y = \cot^{-1}\left[\frac{2\cos^2 x/2}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right] \quad \left(\because 1+\cos x = 2\cos^2 x/2 \ \& \ \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$y = \cot^{-1}\left[\frac{\cos x/2}{\sin x/2}\right]$$

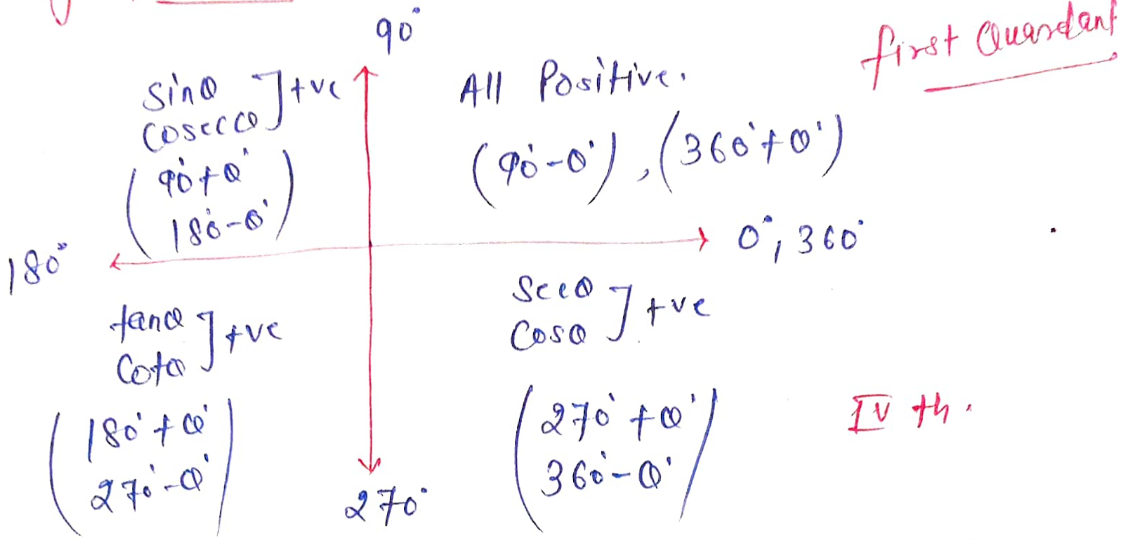
$y = \cot^{-1}\left(\cot \frac{x}{2}\right)$

$y = \frac{x}{2}$

Now, differentiate w.r.t 'x' both sides

$$\boxed{\frac{dy}{dx} = \frac{1}{2}}$$

* Angle change / Transform :



Change at 90°, 270°

No change at 180° and 360°.

$\sin\theta \longleftrightarrow \cos\theta$

$\sec\theta \longleftrightarrow \operatorname{cosec}\theta$

$\tan\theta \longleftrightarrow \cot\theta$

Ex→

$\sin(90-\theta) = \cos\theta$

$\sin(90+\theta) = \cos\theta$

$\sin(180-\theta) = \cancel{\cos\theta}\ \sin\theta$

$\sin(180+\theta) = -\sin\theta$

$\sin(270+\theta^\circ) = -\cos\theta$

$\sin(360-\theta^\circ) = -\sin\theta$

Q. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$, find $\frac{dy}{dx}$.

Soln:

Given,

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$$

Put $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

Now,

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}+1}{\tan\theta}\right)$$

$$y = \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}+1}{\tan\theta}\right) \qquad (\because \sec^2\theta - \tan^2\theta = 1)$$

$$y = \tan^{-1}\left(\frac{1+\sec\theta}{\tan\theta}\right)$$

$$y = \tan^{-1}\left(\frac{1+\frac{1}{\cos\theta}}{\sin\theta/\cos\theta}\right)$$

$$y = \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$y = \tan^{-1}\left(\frac{\cancel{2}\cos^{\cancel{2}}\theta/2}{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}\right) \qquad \left(\because \begin{array}{l} 1+\cos\theta = 2\cos^2\theta/2 \\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{array}\right)$$

$$y = \tan^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$$

$$y = \frac{\pi}{2}-\frac{\theta}{2}$$

$$y = \frac{\pi}{2}-\frac{\tan^{-1}x}{2} \qquad (\because \theta = \tan^{-1}x)$$

Now, diff. w.r.t 'x' both sides

$$\frac{dy}{dx} = 0 - \frac{1}{2(1+x^2)} = \frac{-1}{2(1+x^2)} \qquad \left(\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right)$$

Implicit functions :

$$f(x,y)=0$$

A function, written in terms of both dependent and independent variables is called, implicit function.

Ex- $y-3x^2+2x+5=0$

Q. If $x^p y^q = (x+y)^{p+q}$, Prove that $\frac{dy}{dx} = \frac{y}{x}$.

Sol^n: Given,

$$x^p y^q = (x+y)^{p+q}$$

Taking 'log' both sides

$$\Rightarrow \log[x^p y^q] = \log[(x+y)^{p+q}]$$

$$\Rightarrow \log x^p + \log y^q = \log[(x+y)^{p+q}] \quad (\because \log(ab) = \log a + \log b)$$

$$\Rightarrow p\log x + q\log y = (p+q)\log(x+y) \quad (\because \log m^n = n\log m)$$

Now, diff. w.r.t 'x' both sides,

$$\Rightarrow p\times\frac{1}{x} + q\times\frac{1}{y}\frac{dy}{dx} = (p+q)\times\frac{1}{(x+y)}\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y} + \frac{p+q}{x+y}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{q}{y} - \frac{p+q}{x+y}\right] = \frac{p+q}{x+y} - \frac{p}{x}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{qx+\cancel{qy}-py-\cancel{qy}}{(x+y)y}\right] = \frac{(px+qx-\cancel{px}-py)}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{\cancel{qx-py}}{\cancel{(x+y)}y}\right] = \frac{\cancel{(-py+qx)}}{\cancel{(x+y)}x}$$

$$\Rightarrow \frac{dy}{dx}\times\frac{1}{y} = \frac{1}{x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Hence Proved.

* Important formulaes :

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
5. $\cos 2x = 2\cos^2 x - 1$
   $= 1 - 2\sin^2 x$
   $= \cos^2 x - \sin^2 x$
   $= \dfrac{1-\tan^2 x}{1+\tan^2 x}$
6. $\sin 2x = 2\sin x \cos x$
   $= \dfrac{2\tan x}{1+\tan^2 x}$
7. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$
8. $1+\cos 2x = 2\cos^2 x$
9. $1-\cos 2x = 2\sin^2 x$

Previous year Questions :

1. Find $\frac{dy}{dx}$, if $y = \frac{x^2+x+1}{x+5}$.

2. Find $\frac{dy}{dx}$, when $y = a^{3x-10}$.

3. Find $\frac{dy}{dx}$, if $y = \frac{\sqrt{x-1}}{\sqrt{x+1}}$.

4. If $y = x + \sqrt{x^2-1}$, prove that $(y-x)\frac{dy}{dx} - y = 0$

5. Find $\frac{dy}{dx}$, if $y = x^x + (\log x)^x$.

6. If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, find $\frac{dy}{dx}$.

7. If $y = \tan^{-1}\left[\frac{x}{\sqrt{a^2-x^2}}\right]$

8. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$

9. If $x^p y^q = (x+y)^{p+q}$

prove that $\frac{dy}{dx} = \frac{y}{x}$.

10. Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t $\tan^{-1}x$.

11. Find derivative of $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ w.r.t x.

12. If $y = \cos^{-1}\left[\frac{2\cos x + 3\sin x}{\sqrt{13}}\right]$ Prove that $\frac{dy}{dx} = -1$.

13. If $e^x + e^y = e^{x+y}$, ~~Prove that~~ find $\frac{dy}{dx}$.

Unit - 4

Matrix

* Topic Covered :

1. Definition
2. Types of Matrices
3. Addition, Substraction of Matrices.
4. Scalar Multiplication of Matrices.
5. Multiplication of Matrices.
6. Inverse, adjoint of a Matrix.
7. Determinat
8. Area of Triangle
9. Previous year Questions

* Matrix *

Unit - 4

A set of 'mn' numbers (real or imaginary) arranged in the form of a rectangular array of 'm' rows and 'n' columns is called an $m \times n$ matrix.

Ex →

$R_1 \rightarrow a_{11} \quad a_{12} \quad a_{13} \cdots\cdots a_{1j} \cdots a_{1n}$

$R_2 \rightarrow a_{21} \quad a_{22} \quad a_{23} \cdots\cdots a_{2j} \cdots\cdots a_{2n}$

$\vdots$

$R_m \rightarrow a_{1m} \cdots\cdots\cdots\cdots\cdots a_{mn}$

m = Rows
n = Columns

Ex.
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \qquad \begin{bmatrix} 1 & 2 & 5 \\ 6 & 7 & 0 \\ 9 & 8 & 3 \end{bmatrix}$$

$a_{11} = 1$, $a_{12} = 2$, $a_{13} = 5$ and so on.

* Types of Matrices :-

1. **Row Matrix :** A matrix having only one Row is called Row Matrix.

Ex – $[1]_{1\times 1}$, $[1 \; 2]_{1\times 2}$, $[1 \; 2 \; 3]_{1\times 3}$

2. **Column Matrix :** A matrix having only one column is called Column Matrix.

3. **Square Matrix :-**

A matrix in which the Number of Rows is equal to the number of Columns is called square Matrix

No. of m = No. of n.
(Rows) = (Columns)

Ex -

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3}$$

4. **Diagonal Matrix :-**

A square matrix $A = [a_{ij}]_{n \times n}$ is called leading diagonal Matrix if all the elements, except those in leading diagonal, are zero i.e $a_{ij} = 0$ for all $i \neq 0$.

Ex→

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}_{3 \times 3}$$

5. **Scalar Matrix :**

A diagonal Matrix in which all the diagonal elements are equal is called the Scalar Matrix,

6. **Identity or unit Matrix :-**

A square Matrix each of whose diagonal element is unity and each of whose non-diagonal elements is zero.

Ex-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

7. **Null Matrix :** A matrix whose all elements are zero is called Null Matrix or Zero Matrix.

Ex-

$$[0]_{1\times 1}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

* Addition of Matrices :-

Let A, B two matrices, each of order 'mn'. Then their sum A+B is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Ex-1. If $A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & -2 \\ -1 & 4 & -2 \end{bmatrix}$

then

$$A+B = \begin{bmatrix} -3+3 & 2+5 & -2+1 \\ -1+1 & -4+4 & 7-2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Ex-2.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then,

$$A+B = \begin{bmatrix} 1+1 & 2+0 & 3+0 \\ 4+0 & 5+1 & 6+0 \\ 7+0 & 8+0 & 9+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 3 \\ 4 & 6 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

* Substraction of Matrices :-

For two Matrices A and B of the same order the substraction of matrix 'B' from matrix 'A' is denoted by (A - B) and is defined as

$$\boxed{A - B = A + (-B)}$$

Ex→

If $A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & -2 \\ -1 & 4 & -2 \end{bmatrix}$

then

$$A - B = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -2 \\ -1 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} -3-3 & 2-5 & 1+2 \\ 1+1 & -4-4 & 7+2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -6 & -3 & 3 \\ 2 & -8 & 9 \end{bmatrix}$$

Important Questions :-

Q1. Simplify :

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} + \sin\theta\begin{bmatrix}\sin\theta & -\cos\theta \\ \cos\theta & \sin\theta\end{bmatrix}$$

Sol^n:- The given Matrices are:

$$\Rightarrow \cos\theta\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} + \sin\theta\begin{bmatrix}\sin\theta & -\cos\theta \\ \cos\theta & \sin\theta\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta\end{bmatrix} + \begin{bmatrix}\sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos^2\theta+\sin^2\theta & \sin\theta\cos\theta-\sin\theta\cos\theta \\ -\sin\theta\cos\theta+\sin\theta\cos\theta & \cos^2\theta+\sin^2\theta\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} \qquad \left\{ \because \text{As } \sin^2\theta+\cos^2\theta = 1. \right.$$

Q2 Find a Matrix A such that $2A-3B+5C=0$ where $B=\begin{bmatrix}-2 & 2 & 0 \\ 3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2 \\ 7 & 1 & 6\end{bmatrix}$

Sol^n:- Given that,

$$2A-3B+5C=0$$

$$\boxed{2A = 3B-5C}$$

$$2A = 3\begin{bmatrix}-2 & 2 & 0 \\ 3 & 1 & 4\end{bmatrix} - 5\begin{bmatrix}2 & 0 & -2 \\ 7 & 1 & 6\end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\Rightarrow \cancel{2}A = \cancel{2}\begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

Q3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

find $3A + 2B$.

Sol^n: Here, order of matrix $A = 2 \times 2$

order of Matrix $B = 2 \times 3$

Order is not same so, their sum i.e $3A + 2B$ is not Possible.

Multiplication of Matrices :-

Let us take a Row Matrix and a Column Matrix

$$A = [a_1 \quad a_2 \quad a_3 \cdots a_m] \Rightarrow \text{order} = m$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \text{order} = n$$

$$AB = [a_1 \quad a_2 \quad a_3 \cdots a_m] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1 b_1 + a_2 b_2 \cdots a_m b_n]$$

$\hookrightarrow$ order $= m \times n$

Two Matrices 'A' and 'B' are Conformable for the Product AB if the number of Columns in A (pre-multiplier) is same as the number of Rows in B (Post-multiplier)

Ex→

If $A = [1 \quad -2 \quad 3 \quad 4]$

$$B = \begin{bmatrix} 5 \\ -4 \\ 2 \\ 3 \end{bmatrix}$$

then

$$AB = [1 \quad -2 \quad 3 \quad 4] \begin{bmatrix} 5 \\ -4 \\ 2 \\ 3 \end{bmatrix} = [1 \times 5 + (-2) \times (-4) + 3 \times 2 + 4 \times 3]$$

* Previous Year Questions :-

Q1. If $A = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$, then find 'k' so that $kA^2 = 5A - 6I$, where I is unit Matrix.

Sol^n:- Given,

$$A = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$$

Now,
$$A^2 = A \cdot A = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 0\times 0 + 3\times(-2) & 0\times 3 + 5\times 3 \\ -2\times 0 + 5\times(-2) & 3\times(-2) + 5\times 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix}$$

Now,
$$kA^2 = 5A - 6I$$

$$\Rightarrow k\begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix} = 5\begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6k & 15k \\ -10k & 19k \end{bmatrix} = \begin{bmatrix} 0 & 15 \\ -10 & 25 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6k & 15k \\ -10k & 19k \end{bmatrix} = \begin{bmatrix} 0 & 15 \\ -10 & 25 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6k & 15k \\ -10k & 19k \end{bmatrix} = \begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix}$$

$$\Rightarrow -6k = -6$$

$$\boxed{k = 1}$$ Ans

* Previous Year Questions :-

Q1. If $A = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$, then find 'k' so that $kA^2 = 5A - 6I$, where I is unit Matrix.

Sol^n:- Given,

$$A = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$$

Now,

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 0\times0+3\times(-2) & 0\times3+5\times3 \\ -2\times0+5\times(-2) & 3\times(-2)+5\times5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix}$$

Now,

$$kA^2 = 5A - 6I$$

$$\Rightarrow k\begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix} = 5\begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6k & 15k \\ -10k & 19k \end{bmatrix} = \begin{bmatrix} 0 & 15 \\ -10 & 25 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6k & 15k \\ -10k & 19k \end{bmatrix} = \begin{bmatrix} 0 & 15 \\ -10 & 25 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6k & 15k \\ -10k & 19k \end{bmatrix} = \begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix}$$

$$\Rightarrow -6k = -6$$

$$\boxed{k = 1}$$ Ans

Symmetric Matrix :-

A matrix is said to be Symmetric if transpose of given matrix is equal to that Matrix

i.e

$$\boxed{A^T = A}$$

Ex→

If $A = \begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 2 & 0 \\ -1 & -3 \end{bmatrix}$

* Skew-Symmetric Matrix :-

A matrix is said to be Skew-symmetric if transpose of given matrix is equal to negative of that matrix.

$$\boxed{A^T = -A}$$

or

$$\boxed{A' = -A}$$

Ex. If $A = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$

$A' = \begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & -2 \\ 5 & 2 & 0 \end{bmatrix}$

$A' = -\begin{bmatrix} 0 & 3 & 5 \\ 3 & 0 & 2 \\ 5 & 2 & 0 \end{bmatrix} = -A \Rightarrow \boxed{A' = -A}$ ∀

Q2. Let $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that

$$F(x) \cdot F(y) = F(x+y).$$

Sol^n: Given that,

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$$F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(x) \cdot F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y & 0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= F(x+y)$$

Hence Proved.

* Determinants *

Every square matrix can be associated to an expression or a number which is known as its determinant.

If $A = [a_{ij}]$ is a square matrix of order 'n' then the determinant of A is denoted by det A or $|A|$.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Rightarrow a_{11}a_{22} - a_{12}a_{21}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\Rightarrow |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Q. Evaluate the following determinant:

$$\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$$

Sol^n: The given determinant is

$$\text{let } |A| = \begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$$

$\Rightarrow |A| = \cos 15^\circ \cos 75^\circ - \sin 15^\circ \sin 75^\circ$

$|A| = \cos(75^\circ + 15^\circ) = \cos 90^\circ$ $\quad \{ \because \cos(x+y) = \cos x \cos y - \sin x \sin y$

$|A| = \cos 90^\circ = 0$

$$\boxed{|A| = 0}$$

Q. Evaluate $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix}$

Sol^n: The given determinant is:

$$\text{let } |A| = \begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix}$$

$|A| = \log_3 512 \times \log_4 9 - \log_4 3 \times \log_3 8$

$|A| = \dfrac{\log 512}{\log 3} \times \dfrac{\log 9}{\log 4} - \dfrac{\log 3}{\log 4} \times \dfrac{\log 8}{\log 3}$ $\quad \left\{ \because \log_a b = \dfrac{\log b}{\log a} \right.$, $\log m^n = n \log m$.

$|A| = \dfrac{9 \cancel{\log 2}}{\cancel{\log 3}} \times \dfrac{2 \cancel{\log 3}}{2 \cancel{\log 2}} - \dfrac{\cancel{\log 3}}{2 \cancel{\log 2}} \times \dfrac{3 \cancel{\log 2}}{\cancel{\log 3}} \Rightarrow 9 - \dfrac{3}{2} = \boxed{\dfrac{15}{2}}$

Minors :-

let $A = [a_{ij}]$ be a square matrix of order n. The minor M_{ij} of ij in A is the determinant of the square sub-matrix of order $(n-1)$ obtained by leaving ith row and jth column of A.

Ex- let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \Rightarrow |A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Minors

$a_{11} = 4 \qquad a_{21} = 2$

$a_{12} = 3 \qquad a_{22} = 1$

let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$

Minors

$a_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \Rightarrow 45-48 \Rightarrow -3$

$a_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} \Rightarrow 36-42 \Rightarrow -6$

$a_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \Rightarrow 32-35 \Rightarrow -3$

$a_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} \Rightarrow 18-24 \Rightarrow -6$

$a_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} \Rightarrow 9-21 \Rightarrow -12$

$a_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \Rightarrow 8-14 \Rightarrow -6$

$a_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \Rightarrow 12-15 \Rightarrow -3$

$a_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \Rightarrow 6-12 \Rightarrow -6$

$a_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \Rightarrow 5-8 \Rightarrow -3.$

* Co-factor :-

C_{ij} = Cofactor of a_{ij} in $A = (-1)^{i+j} M_{ij}$

where M_{ij} = Minor of a_{ij} in A.

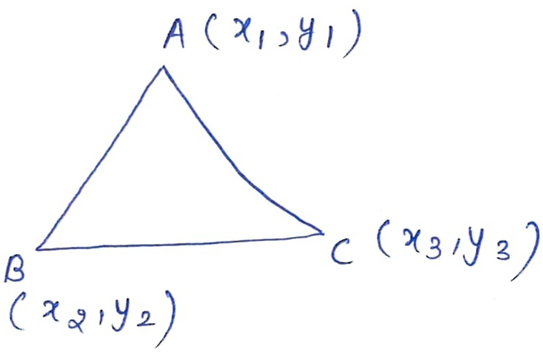
Ex → Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$a_{11} = (-1)^{1+1} \times 4 \Rightarrow 4$ $\quad$ $a_{21} = (-1)^{2+1} \times 2 \Rightarrow -2$

$a_{12} = (-1)^{1+2} \times 3 \Rightarrow -3$ $\quad$ $a_{22} = (-1)^{2+2} \times 1$

$\Rightarrow 1$.

* Area of Triangle :-



$$\text{Area of } \Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

OR

$$\Delta = \frac{1}{2} \left[x_1 (y_2 - y_3) - y_1 (x_2 - x_3) + 1 (x_2 y_3 - x_3 y_2) \right]$$

Note: If A, B, C are Collinear then area $\Delta = 0$.

If the points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear prove that $a + b = ab$.

Sol^n:

Area $= 0$

$$\begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad a[b-1] - 0 + 1[0-b] = 0$

$\Rightarrow \quad ab - a - b = 0$

$\Rightarrow \quad \boxed{ab = a + b}$ Hence, Proved.

* Adjoint of a Matrix :-

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2}$

Cofactor $= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$a_{11} = (-1)^{1+1} \times d = d$

$a_{12} = (-1)^{1+2} \times c = -c$

$a_{21} = (-1)^{2+1} \times b = -b$

$a_{22} = (-1)^{2+2} \times a = a$

$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= (\text{Cofactors Matrix})^T$

* Inverse of a Matrix :-

$$\boxed{A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)}$$

★ Important Previous Year Questions :-

1. Find Inverse of Matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

2. Prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$$

3. Find Inverse of Matrix:

$$A = \begin{bmatrix} 1 & -5 & 0 \\ 3 & 6 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

4. Prove that

$$\begin{bmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

Solution of System of Linear Equations:-

Let

$$a_1 x_1 + a_1 x_2 + \cdots + a_1 x_n = b_1$$

$$a_2 x_1 + a_2 x_2 + \cdots + a_2 x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \; b_n$$

$$A = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{bmatrix}_{3 \times 3} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Method:

$$AX = B \qquad A^{-1} = \frac{1}{|A|} \cdot adj(A).$$

$$\boxed{X = A^{-1} B}$$