MAA OMWATI DEGREE COLLEGE, HASSANPUR (PALWAL)

NOTES

SUBJECT : *FUNCTION & ALGEBRA*

CLASS : *BSc & BA 1st sem*

SESSION : *2024-2025*

☆ Unit - 1 ☆

(Sets, Relation and functions)

☆ **Topic Covered:**

Sets, Subsets, Equal Sets, Universal Sets, finite and Infinite Sets, operation on sets, Union and Intersection of Sets, Complements on Sets, Cartesian Product, Cardinality of set, Practical application of Set Theory.

☆ **Relation and functions:**

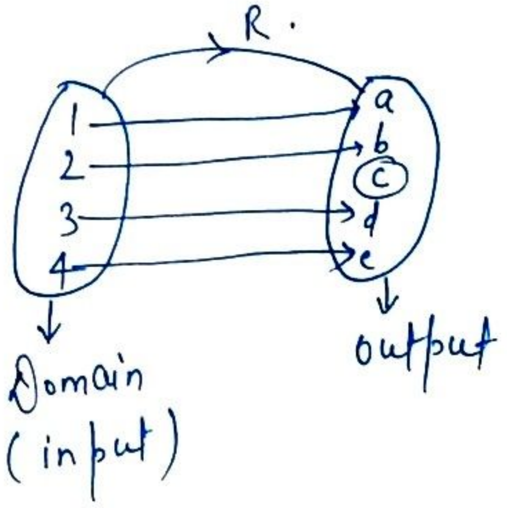
Properties of Relations, Equivalence Relation, Partial order Relation, function, Domain and Range, Onto, Into and one-one function, Composite and Inverse functions.

✯ Relation ✯

A relation R from set A to set B is a subset of Cartesian product of $A \times B$.

Every Relation has a property.

→ Domain: Set of all permissible input.
→ Range: Set of all permissible output.
→ Co-domain: All possible output.



Range = $\{a, b, d, e\}$
Co-domain = $\{a, b, c, d, e\}$

Note: The total number of relations that can be defined from set A to a set B is the number of possible subset of $A \times B$ if

$$n(A) = P\ (n(B)) = q$$

Total number of relations $= 2^{pq}$.

Ex- Let $A = \{1,2\}$ and $B = \{3,4\}$. Find the number of relations from A to B.

Sol^n:

$n(A) = 2$

$n(B) = 2$

No. of relations $= 2^{2 \times 2}$

$= \underline{\underline{2^4}}$.

Ex- Find the domain of the following real-valued function:

(i) $F(x) = \frac{1}{x+2}$ (ii) $f(x) = \frac{x-1}{x-3}$.

Sol^n:

(i) $f(x) = \frac{1}{x+2}$

$x + 2 \neq 0$

$x \neq -2$

Domain $\Rightarrow R - \{-2\}$.

(ii) $f(x) = \frac{x-1}{x-3}$

$x - 3 \neq 0$

$x \neq 3$

Domain $\Rightarrow R - \{3\}$.

Find domain of given function:

$$f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$$

sol^n:

$$f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$$

$4 - x \geq 0$

$4 \geq x$

$x \leq 4$

$x^2 - 1 > 0$

$(x-1)(x+1) > 0$

$x > 1$

$x < -1$

Domain $= (-\infty, -1) \cup (1, 4]$.

☆

Function	Domain	Range
1. Rational function $f(x) = \frac{P(x)}{q(x)}, q(x) \neq 0$	$R - \{0\}$	$R - \{0\}$
2. Modulus function $f(x) = \lvert x\rvert = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$	R	$[0, \infty)$
3. Greatest integer function $f(x) = [x]$	R	Z
4. Signum function $f(x) = \text{signum}(x) = \begin{cases} \frac{\lvert x\rvert}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$	R	$\{-1, 0, 1\}$.

Universal Set :-

A universal set is a set that contains all the elements of all related sets, without repeating any elements. It is denoted by "U".

Ex→ $A = \{1, 2, 3\}$

$B = \{1, a, b, c\}$

$\Rightarrow U = \{1, 2, 3, a, b, c\}$.

★ Finite Set :

A set having finite number of elements is called finite set.

Note : Cordinal number of a finite set → Number of elements in a set.

Ex→ $A = \{1, 2, 3, 4, 5\}$

$n(A) = \underline{5}$.

★ Infinite Set :-

A set having infinite number of elements in a set is called Infinite set.

Ex- A set of Natural numbers.

★ Algebra of functions:

$$(f+g)(x) = f(x) + g(x) \quad , x \in X$$

$$(f-g)(x) = f(x) - g(x) \quad , x \in X$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \quad , x \in X$$

for functions $f : X \rightarrow R$ and $g : X \rightarrow R$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad , x \in X \; , \; g(x) \neq 0.$$

* Sets :-

Set is a well-defined collection of digits.
Set is usually denoted by Capital letters.

Ex- A, B, C, D, X, Z.

Elements of set is denoted by small letters.

Ex- The set of Natural Numbers

$N = \{1, 2, 3, \dots\}$ → Roaster or Tabular form.

* Subset :-

A set is said to be a subset of a set B if all the elements of set A is also an element of set B.

It is denoted by $A \subseteq B$.
A is a subset of B.

Ex- $A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4\}$ $\underline{A \subset B}$.

* Equal set:

Two sets are said to be equal if they have exactly same element.

Ex- $A = \{1, 2, 3, 4\}$
$B = \{1, 2, 3, 4\}$.

Union of Sets :-

Let A and B two sets then union of A and B is the set which Consist of all the elements of A and all the elements of B, the Common elements being taken Only once. It is denoted by "∪".

Ex →
$$A = \{1, 2, 3\}$$
$$B = \{3, 4, 5, 6\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

☆ Intersection of sets:

The intersection of sets A and B is the set of all elements which are common to both A and B. It is denoted by ∩.

Ex →
$$A = \{1, 2, 3\}$$
$$B = \{3, 4, 5, 6\}$$
$$A \cap B = \{3\}.$$

☆ Complement of a set :

Let 'U' be the universal set and 'A' be the subset of 'U' then Complement of A is the set of all elements of U which are not the elements of A.

It is denoted by A' or $\bar{A}$.

$$\boxed{A' = U - A}$$

* <u>De-Morgan's Law</u> :-

1. $(A \cup B)' = A' \cap B'$.

2. $(A \cap B)' = A' \cup B'$.

* <u>Cartesian Product of sets</u> :- Let two non-empty sets 'P' and 'Q' then the Cartesian product $P \times Q$ is the set of all ordered pairs whose first elements is component of P's

$$P \times Q = \{(p, q) : p \in P, q \in Q\}$$

If P and Q is the null set then

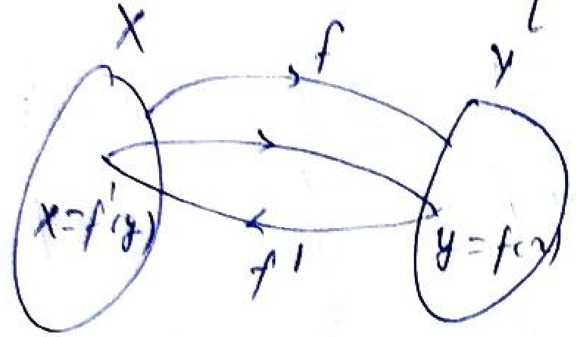
$$P \times Q = \phi.$$

<u>Note</u>:

$$A \times B \neq B \times A.$$

Inverse function:

If $f: X \to Y$ is a bijection then there always exist pre image $f^{-1}(y)$ of each element y of Y and this will be a unique element of X.



Let $f: R \to R$ s.t $f(x) = 2x-3$

We know that it is one-one & onto

So, $y = 2x-3$

$$x = \frac{y+3}{2}$$

$$f^{-1}(y) = \frac{y+3}{2}$$

UNIT 2 chapter 2

Relation between the roots and Coefficients of an eqn.

* Polynomial ⇝

When $n \geq 0$ is an integer, $a_0, a_1, a_2, \cdots a_n$ are constants and x is a variable is called Polynomial.
The constants $a_0, a_1, a_2, \cdots a_n$ are called co-efficients of the Polynomial.

\# If all the coefficient of a polynomial $f(x)$ are real then the polynomial $f(x)$ is said to be a real Polynomial.

\# If all the coefficient of a polynomial $f(x)$ are zero, the the polynomial $f(x)$ is said to be a zero Polynomial.

* Zero of a Polynomial ⇝

A number α is zero of a Polynomial $f(x)$ iff $f(\alpha) = 0$.

Eg. → $f(x) = x^2 - 5x + 6$

$$f(2) = f(3) = 0$$

$\therefore$ 2, 3 are zeros of polynomial $f(x)$.

* **General Eq^n.** → The relation $f(x)=0$ i.e., $a_0x^n+a_1x^{n-1}...+a_n=0$ $(a_0 \neq 0)$ is called the general equation.

* **Degree of an eq^n** → By this equation we means the highest power of the variable x occuring in the given equation, when it is expressed as a rational integral function of the variable i.e., when the eq^n is cleared of fractional indices or Fractions.

An eq^n of 1st degree, 2nd degree, 3rd degree and 4th degree are said to be linear, quadratic, cubic and biquadratic respectively.

Degree of constant Polynomial is zero.

* **Complete Eq^n.** → A complete eq^n. of the nth degree is one which contains all the powers of the variable from zero to n.

* **Incomplete eq^n** → An incomplete eq^n of nth degree is one in which some power or powers of the variable (other than 'n') are missing.

* **Roots of an eq^n.** → Any value of x which makes $f(x)$ vanish is called a root of the eq^n. $f(x)=0$.

The degree of $q(x)$ = degree of $f(x)$ − degree of $g(x)$ –

When $g(x) = x-h$ i.e, when the dividing Polynomial is of first degree, then remainder Polynomial $r(x)$ is either zero or constant polynomial b/z degree of $r(x)$ < degree of $g(x)$.

In such case we write $f(x) = (x-h).q(x) + R$ where R is constant.

* **Remainder Theorem** →

If $f(x)$ is a polynomial, the $f(h)$ is the remainder when $f(x)$ is divided by $x-h$.

* **Factor Theorem** →

If α is root of the eq^n $f(x)=0$ then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$.

If $f(x)$ is divided by $(x-a)$, $(x-b)$ and $(x-c)$ successively The remainder R is given by:

R = first remainder + (Ist divisor) (2nd remainder) + (2nd divisor)·(3rd remainder)

* Synthetic division →

We shall therefore a very simple process known as synthetic division in order to find the quotient and remainder very convenienty and readily.

Suppose $x^3+5x^2+4x-15$ is divided by $(x-3)$

Ordinary Process

$$
\begin{array}{r|l}
 & x^2+8x+28 \\
\hline
x-3 & x^3+5x^2+4x-15 \\
 & \underline{x^3-3x^2} \\
 & \quad 8x^2+4x-15 \\
 & \quad \underline{8x^2-24x} \\
 & \qquad 28x-15 \\
 & \qquad \underline{28x-84} \\
 & \qquad\quad 69
\end{array}
$$

Synthetic Process

$$
\begin{array}{c|cccc}
3 & 1 & 5 & 4 & -15 \\
 & & 3 & 24 & 84 \\
\hline
 & 1 & 8 & 28 & \boxed{69 = R}
\end{array}
$$

The coefficient 1, 5, 4, −15 of divident in order in a straight line.

If in the divident any power of x are missing, their places are To be filled by zero coefficient in the synthetic division.

To find the quotient and remainder when $f(x)$ is divided by $(ax-b)$

The term containing x^2 is missing its cofficient has been taken as zero.

~~Tr~~ansposed conjugate a Matrix

If A is a matrix of type 'm×n' then the transpose of the conjugate matrix '$\bar{A}$' is called transposed conjugate

It is denoted by = A^{θ}

$$\boxed{A^{\theta} = (\bar{A})'}$$

Example :->

$$A = \begin{bmatrix} 3 & 2-i & 4i \\ 4i+6 & -7-i & 3i-2 \\ 1 & 4 & 3-3i \end{bmatrix}$$

$A^{\theta} = (\bar{A})'$

$$\bar{A} \Rightarrow \begin{bmatrix} 3 & 2+i & -4i \\ -4i+6 & -7+i & -3i-2 \\ 1 & 4 & 3+3i \end{bmatrix}$$

$$(\bar{A})' \Rightarrow \begin{bmatrix} 3 & -4i+6 & 1 \\ 2+i & -7+i & 4 \\ -4i & 3i-2 & 3-3i \end{bmatrix}$$

: $(\bar{A})' = (\bar{A'}) = A^{\theta}$

If A^{θ}, B^{θ} denote transposed conjugate (A, B)

i) $(A^{\theta})^{\theta} = A$

ii) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$ (A, B being conformable for addition

iii) $(KA)^{\theta} = \bar{K} \cdot A^{\theta}$ (where K is any complex number

iv) $(AB)^{\theta} \Rightarrow B^{\theta} \cdot A^{\theta}$ (A, B being conformable for multiplicati

v) $(A^n)^\theta = (A^\theta)^n$ (where 'A' is any square matrix and 'n' is Positive integer.)

K is any real number then $(KA)^\theta = KA^\theta$ in this case $\bar{K} = K$

Adjoint of a square-matrix

$$A \Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$$

$$1(-12-12) - 1(-4-6) + 3(-4+6)$$

$$-24 + 10 + 6 = -8$$

Co-factors 1st row

$$A_{11} = \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} = -12-12 = -24, \quad A_{12} = \begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} = -(-4-6) = 10$$

$$A_{13} = \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -4+6 \Rightarrow 2$$

Co-factor's 2nd row

$$A_{21} = \begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -(-4+12) = -8 \quad A_{22} \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -4+6 = 2$$

$$A_{23} \begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = (-4+2) = 2$$

Co factors of 3rd row

$A_{31} = \begin{vmatrix} 1 & 3 \\ 3 & -3 \end{vmatrix} = -3-9 = -12$, $A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = -(-3-3) = 6$

$A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 3-1 = 2$

$$\operatorname{adj} A = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}' \Rightarrow \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

A be a square matrix type $n \times n$

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| I_n$$

where I_n denoted n-rowed unit matrix

A and B are two matrix same order

$$\operatorname{adj}(AB) = (\operatorname{adj} B)(\operatorname{adj} A)$$

Inverse (or Reciprocal) of a Square matrix

If 'A' a square matrix of order 'n' and there exists another matrix 'B' of order n

$$AB = BA = I_n$$

A is said <u>invertible</u> and B is called <u>inverse of A</u>

<u>written as</u> $AA^{-1} = A^{-1}A = I_n$

irrational roots occur in conjugate pairs.

$(A^{-1})^{-1} = A$

Inverse of every square matrix if it exists is unique.

A square matrix A to be invertible is that A is non - singular (i,e $[|A| \neq 0]$

Singular matrice cannot inverse $[A = 0]$

$A^{-1} = \frac{adj.A}{|A|}$

A and B are non singular matric same order
$(AB)^{-1} = B^{-1}A^{-1}$

A is a non - singular matrix
$\det(A^{-1}) = (\det A)^{-1}$

adjoint of non - singular matrix is non singular.

[illegible]

A is a non-singular matrix order 'n'

$|\operatorname{adj} A| = |A|^{n-1}$

$\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A$

A is non singular

$(A')^{-1} = (A^{-1})'$

Relation between the Roots and cofficients of an equation

Fundamental theorem of Algebra →

Every Polynomial equation $F(x) = 0$ of degree $n \geq 1$ has atleast one root, real or imaginary.

* Theorem →

Every equation $F(x) = 0$ of the n^{th} degree has n roots and no more.

* In an equation with real co-efficients, imaginary roots occur in conjugate pairs.

cor. →

* Every eq^n of odd degree and real co-efficients has atleast one real root.

Since imaginary roots occur in conjugate pairs and the eq^n being of odd degree has an odd number of roots therefore atleast one of the roots must be real.

* If $\alpha + i\beta$ is a multiple root of the equation $F(x) = 0$ occuring r times, then the conjugate root $(\alpha + i\beta)$ will also occur r times.

#

In an equation with Rational co-efficients, irrational roots occur in Conjugate pairs.

Cor.1

* If an equation $f(x)=0$ with rational co-efficients, has an irrational root $(\alpha+\sqrt{\beta})$ repeated r times, then it has also $(\alpha-\sqrt{\beta})$ as a root repeated r times.

* If the irrational root of the equation $f(x)=0$ is of the type $(\sqrt{\alpha}+\sqrt{\beta})$ where α and β are both not perfect squares, then $(\sqrt{\alpha}-\sqrt{\beta})$, $(-\sqrt{\alpha}+\sqrt{\beta})$ and $(-\sqrt{\alpha}-\sqrt{\beta})$ will also be the roots of the equation.

* If one root is of the form $(\sqrt{\alpha}+i\beta)$, then $(\sqrt{\alpha}-i\beta)$, $(-\sqrt{\alpha}+i\beta)$ and $(-\sqrt{\alpha}-i\beta)$ will also be the roots of the equation.

Example →

Form an equation whose one root is $2-3i$.

Solution:-

Let

$$x = 2-3i$$
$$x-2 = -3i$$

Squaring both sides, we get

$$x^2-4x+4 = 9i^2$$
$$x^2-4x+4 = -9 \qquad [\because i^2 = -1]$$
$$x^2-4x+13 = 0$$

which is the Required equation.

* Relation between the Roots and the co-efficients of an equation.

* In Particular Cases :-

(i) for the cubic $a_0x^3+a_1x^2+a_2x+a_3=0$, we have

$$\sum \alpha_1 = \alpha_1+\alpha_2+\alpha_3 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = \frac{a_2}{a_0}$$

$$\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

(ii) for the quartic $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$, we have

$$\sum \alpha_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4 = \frac{-a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4 = \frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 = \alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4 = \frac{-a_3}{a_0}$$

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \frac{a_4}{a_0}$$

* Solutions of Polynomial equations having condition on Roots.

* In case of cubic equations:

(i) Three roots given in A.P. can be taken as $\cancel{\frac{\alpha}{\beta}}$ $\alpha-\beta, \alpha, \alpha+\beta$.

(ii) Three roots given in G.P can be taken as $\frac{\alpha}{\beta}, \alpha, \alpha\beta$.

* In Case of biquadratic equations:

(i) four roots given in A.P. can be taken as $\alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$

i.e; Sum of 1st and 4th root = Sum of 2nd and 3rd root.

(ii) four roots given in G.P. Can be taken as $\frac{\alpha}{\beta^3}, \frac{\alpha}{\beta}, \alpha\beta\ \alpha\beta^3$

i.e; Product of 1st and 4th root = Product of 2nd and 3rd root.

Let us illustrate the method with the help of examples.

example :-

The given eqn is = $4x^3+16x^2-9x-36=0$

Let the root be α, β, γ such that $\alpha+\beta=0$

$$\alpha+\beta+\gamma = -\frac{16}{4} = -4$$

$$\alpha+\beta=0 \rightarrow \gamma = -4$$

Thus one root of (1) is -4 and hence $x+4$ is a factor of (1)

Dividing (1) by $x+4$, we have

-4	4	16	-9	-36
	.....	-16	0	36
	4	0	-9	$\boxed{0}$

$\therefore$ Depressed eqn is $4x^2-9=0 \rightarrow x^2 = \frac{9}{4} \rightarrow x = \pm\frac{3}{2}$

Hence the Roots are $-\frac{3}{2}, \frac{3}{2}, -4$.

CHAPTER 3

Transformation of equations.

▶ ROOTS WITH SIGNS CHANGED

☆ To transform an equation into another whose roots shall be equal in magnitude but opposite in sign to those of the given equation.

● Roots Multiplied By A given Number

☆ To transform an equation into another whose roots are m times those of the given equation.

▶ Reciprocal Roots.

☆ To transform an equation into another whose roots are the reciprocal roots.

● Reciprocal Equation.

☆ Definition → If an equation is not altered when x is changed into its reciprocal $\frac{1}{x}$, it is called a reciprocal Equation.

▶ Roots Diminished By A Given number.

☆ To transform a given equation into another whose roots are the roots of the given equation diminished by h.

▶ PRACTICAL METHOD →

☆ To diminish the roots of a given equation by a given quantity h.

▶ TRANSFORMATION of the CUBIC.

Solution

☆ To Reduce the cubic $a_0x^3+3a_1x^2+3a_2x+a_3=0$ to the form in which second term may be wanting and the Co-efficient of the leading term be unity, all other Co-efficient being integers.

▶ Transformation of the Biquadratic.

☆ To reduce the biquadratic $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ to the form in which Second term is wanting and the Co-efficient of the leading term is unity, all other Co-efficients being integers.

▶ Equation of Squared Differences of A CUBIC

☆ To find an equation whose roots are the Squares of the differences of the roots of the equation $x^3+qx+r=0$

Example → find an equation whose roots are four times the roots of the equation $x^3+2x^2+3x-5=0$

Solution → given $x^3+2x^2+3x+5=0$

given equation is Complete, therefore, multiplying the Successive terms by $4^0, 4^1, 4^2, 4^3$ respectively, we have

$$4^0 \cdot x^3+4^1 \cdot 2x^2+4^2 \cdot 3x-4^3 \cdot 5=0$$

$$x^3+8x^3+48x-320=0$$

which is the required equation.

CHAPTER 4 (Descarte's Rule of Signs)

✯ DEFINITION ⟶

(a) Continuation or Permanence of Signs →

In a polynomial f(x) in which its terms are arranged in descending power of x if two Successive terms have the Same Sign, a Continuation or permanence of Signs is Said to occur.

Thus in $x^6 - 2x^5 - 3x^4 - 5x^3 + 4x^2 - 9x + 1$, there are two Continuations of Signs occurring each at $-3x^4$, $-5x^3$

(B) Variation of change of Signs →

In a polynomial f(x) in which its terms are arranged in descending power of (x) if two Successive term have different Sign's, a variation or change of Signs is Said to occur.

Thus in $x^4 - 2x^3 - 4x^2 + 3x - 7$, there are three changes or Variations of Signs occurring at $-2x^3$, $3x$ and -7

(C) Complete Equation → An equation is said to be Compete when none of its Coefficient is Zero.

In a Complete equation

(i) the sum of the number of Continuations and the Variations of Signs is equal to the degree of the equation

(ii) if x is changed into $-x$, a Continuation of Sign becomes a variation of Sign and viceversa.

(D) Ambiguity of Sign → An ambiguity of Sign is said to occur when any term of a polynomial $f(x)$ has the double sign $\pm$ or $\mp$.

★ DESCARTE'S RULE of SIGNS →

a) The number of positive roots in any polynomial equation $f(x)=0$ with real Coefficents Cannot exceed the number of changes of Signs of the Coefficients in $f(x)$.

The number of negative roots in any polynomial equation $f(x)=0$ with real Coefficients Cannot exceed the number of changes of signs of the Coefficients in $f(-x)$.

☆ Complex Roots →

the equation $f(x)=0$ is Complete and of nth degree, then the sum of the number of changes of signs if $f(x)$, positive roots and number of changes of sign in $f(-x)$ negative roots is equal to the degree of equation and as such no Conclusion Can be draw about the existence of the Complex Roots.

In an incomplete equation of degree n. if p and q be the number of change of signs in $f(x)$ and $f(-x)$ respectively. then the equation $f(x)=0$ has at the most p positive and q is negative roots.

Example → Show that the equation $x^{2n}-1=0$ has only two real roots.

Let $f(x) = x^{2n}-1$

$f(-x) = x^{2n}-1$

The number of variations of signs in $f(x)$ and $f(-x)$ are one and one respectively. Hence the equation $f(x)=0$ has atmost two real roots.

But $f(x)=0$ is satisfied by $x = \pm 1$.

Hence the equation has exactly two real roots.

CHAPTER 5

Solution of Cubic and Biquadratic Equation.

9.2. Cardan's Method of Solving a Cubic Equation :-

Let the General Cubic equation be

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

After removing the Second term and multiplying the roots by a_0, Equation (1) reduces to the form.

$$z^3 + 3Hz + G = 0$$

where

$$z = a_0x + a_1,\ H = a_0a_2 - a_1^2 \text{ and } G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$$

let $\quad z = u + v$

Cubing both sides, we have.

$$z^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvz.$$

$$z^3 - 3uvz - (u^3+v^3) = 0$$

Comparing the co-efficients of like terms in equations (2) and (3), we get.

$$-3uv = 3H \Rightarrow uv = -H$$

$$u^3v^3 = -H^3$$

$$u^3 + v^3 = -G$$

Equation with roots u^3, v^3 is $t^2 - (u^3+v^3)t + u^3v^3 = 0$

$$t^2 + Gt - H^3 = 0$$

$$t = \frac{-G \pm \sqrt{G^2+4H^3}}{2}$$

Let $u^3 = \dfrac{-G + \sqrt{G^2+4H^3}}{2}$ and $v^3 = \dfrac{-G - \sqrt{G^2+4H^3}}{2}$

Thus to obtain the values of $z = u+v$, the values of u and v are so chosen that their product $uv = -H$ is

Hence the three values of z are $u+v$, $u\omega + v\omega^2$ and $u\omega^2 + v\omega$.

The three values of z being known, the corresponding values of x can be obtained from the relation $z = a_0 x + a_1$.

☆ To Discuss the nature of the Roots of the Cubic $z^3+3Hz+G=0$

1. If G^2+4H^3 is positive :- Then the roots of (3) are real.
2. If G^2+4H^3 is negative :- Then the roots of (3) are conjugate complex.
3. If $G^2+4H^3=0$, Then the roots of (3) are equal.
4. If $G=H=0$, The cubic becomes $z^3=0$. Then the three roots are equal, each being equal to zero.

☆ Descarte's Solution of the Biquadratic

(a). To Explain Descarte's method of solving the biquadratic $x^4+qx^2+rx+s=0$.

The given biquadratic is $x^4+qx^2+rx+s=0$

Suppose that equation (1) can be split up into two quadratic factors,

$$x^2+lx+m \text{ and } x^2-lx+n.$$

then,
$$x^4+qx^2+rx+s \equiv (x^2+lx+m)(x^2-lx+n)$$

Comparing the co-efficients of like powers of x in (2).

we get,
$$m+n-l^2=q \Rightarrow n+m = l^2+q$$
$$ln-lm=r \Rightarrow n-m=\frac{r}{l}$$
$$mn=s$$

To eliminate m and n, we use the identity.

$$(n+m)^2-(n-m)^2=4mn$$

$\therefore$ from (3), (4) and (5), we have.

$$(l^2+q^2)-\frac{r^2}{l^2}=4s \Rightarrow (l^4+2l^2q+q^2)-\frac{r^2}{l^2}=4s$$

$$l^6+2ql^4+(q^2-4s)l^2-r^2=0$$

which is a cubic equation in l^2. Solving by trail method, l^2 and thus l can be determined.

Thus m and n can be found from (3) and (4).

Hence,

The L.H.S of (1) reduces to two quadartic factors x^2+lx+m and x^2-lx+n which when equated to zero give four values of x, which are the roots of the given biquadartic.

☆. To Solve the biquadratic $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ by Descart's method

The given equation is $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$.

Transforming the given equation into one in which the second is missing and the co-efficient of the leading term is unity by putting $z=a_0x+a_1$ in (1), we get.

$z^4+6Hz^2+4Gz+(a_0^2I-3H^2)=0$

where

$H=a_0a_2-a_1^2,\ G=a_0^2a_3-3a_0a_1a_2+2a_1^3$

$I=a_0a_4-4a_1a_3+3a_2^2$

Suppose that equation (1) can be split up into two quardartic factors.

(z^2+pz+q) and (z^2-pz+q')

then, $z^4+6Hz^2+4Gz+(a_0^2I-3H^2)\equiv(z^2+pz+q)(z^2-pz+q')$

Comparing the co-efficient of like power's of z in (3), we get.

$q+q'-p^2=6H \Rightarrow q'+q=p^2+6H$

$pq'-pq=4G \Rightarrow q'-q=\frac{4G}{p}$

$qq'=a_0^2I-3H^2$

To eliminate q and q', we use the identity.

$(q'+q)^2-(q'-q)^2=4qq'$

$(p^2+6H)^2-\frac{16G^2}{p^2}=4(a_0^2I-3H^2)$

$(p^4+12Hp^2+36H^2)-\frac{16G^2}{p^2}=4(a_0^2I-3H^2)$

$p^6+12Hp^4+4(12H^2-a_0^2I)p^2-16G^2=0$

Equation (7) is a cubic equation in p^2. Solving it by trial method, p^2 and thus p can be determined.

$\therefore$ q and q' can be found from (4) and (5)

Thus, the L.H.S of biquadratic (3) reduces to product of two quadratic factors which when equated to zero given four values of z which are the roots of the equation (2), and the four values of x are then obtained from the relation.

$z=a_0x+a_1$

These four values of x are the required roots.

☆ 9.4 Ferrari's Method of Solving a Biquadratic :-

To Solve the biquadratic $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ by Splitting it into product of two quadratic factor's by ferrari's Method.

⇒ Solve $x^4+2x^3-7x^2-8x+12=0$ by ferrari's Method.

Solution :→ The given equation is $x^4+2x^3-7x^2-8x+12=0$ (1)

we may write equation (1) as $x^4+2x^3=7x^2+8x-12$.

Adding x^2 on both Sides to makes L.H.S a perfect Square, we get

$$x^4+2x^3+x^2=8x^2+8x-12$$

$$(x^2+x)^2=8x^2+8x-12$$

$$[(x^2+x)+\lambda]^2=(x^2+x)^2+2(x^2+x)\lambda+\lambda^2$$

Putting the value of (x^2+x^2) from (2) in (3), we get.

$$[(x^2+x)+\lambda]^2=(8x^2+8x-12)+2(x^2+x)\lambda+\lambda^2$$

$$\Rightarrow (8+2\lambda)x^2+(8+2\lambda)x+(\lambda^2-12)$$

Let us choose λ in Such a way that R.H.S of (4) is a perfect Square.

for this purpose, discriminant of R.H.S = 0

$$(8+2\lambda)^2-4(8+2\lambda)(\lambda^2-12)=0$$

$$(8+2\lambda)[(8+2\lambda)-4(\lambda^2-12)]=0$$

$\Rightarrow$ Either

$8+2\lambda=0$
$\lambda=-4$

$2[4+\lambda-2\lambda^2+24]=0$
$-2\lambda^2+\lambda+28=0$
$2\lambda^2-\lambda-28=0$

$\Rightarrow \lambda=\dfrac{1\pm\sqrt{1+224}}{4}=\dfrac{1\pm 15}{4}=4,-\dfrac{7}{2}$

Let us take $\lambda=-4$. then equation (4) becomes.

$(x^2+x-4)^2=4$
$(x^2+x-4)^2-2^2=0$
$(x^2+x-4+2)(x^2+x-4-2]=0$
$[x^2+x-2][x^2+x-6]=0$
$[x+2][x-1][x+3][x-2]=0 \Rightarrow x=-2,1,-3,2$

Hence the roots are $\underline{\underline{-3,-2,1,2}}$.

UNIT 3

chapter 6

★ **Important Notes :-**

Matrix :- A set of mn numbers, arranged in the form of an ordered set of m rows, each row consisting of n numbers and enclosed in the bracket [ ] is called an m×n Matrix.

★ **Type of Matrix →**

Rectangular Matrix :- An m×n matrix is said to be rectangular matrix if $m \neq n$

for example -
$$\begin{bmatrix} 3 & 0 & -4 & 5 \\ 0 & -3 & 2 & 1 \\ 5 & 2 & 1 & 9 \end{bmatrix}$$

Square Matrix :- A Matrix is a square matrix if the number of rows = number of columns.

for example -
$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 9 \\ 3 & 0 & 4 \end{bmatrix}$$

Row Matrix :- A matrix A is said to be a row matrix if it has only one row and

and any number of columns.

for example - $[2, 3]$, $[4, 9, 5]$

Column Matrix :-

A Matrix A is said to be a column matrix if it has only one column and any number of rows.

for ex :-

$$\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Diagonal Matrix :-

A square matrix said to be a diagonal Matrix. These diagonal entries may or may not be zero.

for ex :-

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Scalar Matrix :-

A diagonal Matrix in which all of the diagonal elements are equal is called a scalar Matrix.

For ex :- Diag. $[2, 2, 2] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Unit Matrix :-

A diagonal matrix is said to be

unit Matrix or Identity Matrix.

for ex:-

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Triangular Matrix :-

A square matrix is said to be a triangular matrix.

Two types of Triangular Matrix →

Upper-triangular Matrix :-

A square matrix is said to be an upper-triangular matrix if all the elements below the principal diagonal are zero.

For ex:-

$$\begin{bmatrix} 4 & 7 & 9 & 3 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Lower-triangular Matrix :-

A square matrix is said be an lower-triangular matrix if all the elements above the principal diagonal are zero.

for ex:-

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 7 & 3 & 2 \end{bmatrix}$$

Singular and Non-singular Matrics :-

A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$

for ex:- $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = |A| = 12 - 12 = 0$

is a singular matrix

$A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} = |A| = 12 - 10 \neq 0$

is a non-singular matrix.

Transpose of Matrix :-

A matrix is interchanging its rows and columns is called the transpose of A matrix.

It is denoted by A' or A^T

for ex:-

$$A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 4 & 3 & 5 & 1 \\ 2 & 4 & 3 & 6 \end{bmatrix}$$

$$A' = \begin{bmatrix} 5 & 4 & 2 \\ 6 & 3 & 4 \\ 7 & 5 & 3 \\ 8 & 1 & 6 \end{bmatrix}$$

Theorm of Transpose of a Matrix :-

If A', B' denote the transpose of A, B respectively –

(a) $(A')' = A$

(b) $(A+B)' = A' + B'$, A, B being conformable for additio

(c) $(kA)' = kA'$, where k is a scalar,

(d) $(AB)' = B'A'$, A, B being conformable for multiplicatio

(e) $(A^n)' = (A')^n$, where A is a square matrix and n is a positive Inetger.

Conjugate of a Matrix :-

If A is a matrix of the type $m \times n$ over the complex number, then matrix of the same type $m \times n$ obtained from A by replacing each of its elements by their corresponding complex conjugates is called the conjugate of A Matrix.

It is denoted by $\bar{A}$

for exp–

$$A = \begin{bmatrix} 3+i & 3 & 4i \\ 2-i & -3i & 4 \\ 3 & 6+i & 4+3i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 3-i & 3 & -4i \\ 2+i & 3i & 4 \\ 3 & 6-i & 4-3i \end{bmatrix}$$

* Theorm of a conjugate of Matrix :-

If $\bar{A}$, $\bar{B}$ denote the conjugate matrixs of A , B

$\overline{(\bar{A})} = A$

$\overline{(A+B)} = \bar{A} + \bar{B}$, A , B being conformable for addition.

$\overline{(kA)} = \bar{k} \cdot \bar{A}$, where k is any complex number.

$\overline{(AB)} = \bar{A} \cdot \bar{B}$, A , B being conformable for multiplication.

$\overline{(A^n)} = (\bar{A})^n$, where A is a square matrix and n is a positive Inetger.

"Symmetric Matrix": -

"Definition :"

A square matrix A is said to be symmetric if $A' = A$ i,e, If the Transpose of the matrix is equal to matrix itself.

"Examples :-"

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 4+i & 5i \\ 4+i & 2-i & -4 \\ 5i & -4 & 7 \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

"The Diagonal Element of a symmetric matrix are arbitrary."

"As a result :"

For a symmetric matrix $a_{ij} = a_{ji}$ for all i and j.

Diagonal entries, We have $i = j$.

$a_{ij} = a_{ij}$ for all i and j Which is always true.

Skew-symmetric matrix :

Definition :-

A square matrix A is said to be skew-symmetric if $A' = -A$ i,e, If the transpose of the matrix is equal to the negative of the matrix itself.

Examples :-

$$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3i & i-2 \\ -3i & 0 & 4-i \\ -i+2 & -4+i & 0 \end{bmatrix}, \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

The Diagonal Elements of a skew-symmetric matrix are all zero.

As a result :

for a skew-symmetric matrix

$$a_{ij} = -a_{ji} \text{ for all } i \text{ and } j$$

Diagonal entries $i = j$

$$a_{ii} = -a_{ii} \text{ for all } i$$

$$2a_{ii} = 0 \text{ for all } i \quad \text{i,e } a_{ii} = 0 \text{ for all } i.$$

Hermitian Matrix:

Definition:

A square matrix A over the complex numbers is said to be Hermitian If $A^{\theta} = A$, i.e., If the transposed conjugate of the matrix is equal to the Matrix itself.

Examples:

$$\begin{bmatrix} 2 & 3-2i \\ 3+2i & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3-i & 5+2i \\ 3-i & 9 & 4i \\ 5-2i & -4i & 0 \end{bmatrix}$$

The Diagonal Elements of a Hermitian matrix are all real.

As a Result:

For a Hermitian matrix $a_{ij} = \overline{a_{ji}}$ for all i and j

Diagonal Entries $i=j$

$$a_{ii} = \overline{a_{ii}} \text{ for all } i$$

$$\alpha + i\beta = \alpha - i\beta$$

$$2i\beta = 0 \text{ i.e } \beta = 0$$

$$a_{ii} = \alpha + i \times 0 = \alpha.$$

$$\left[\because a_{ii} = \alpha + i\beta \Rightarrow \overline{a_{ii}} = \alpha - i\beta\right].$$

"Skew- Hermitian Matrix:"

"Definition:"

A square matrix A over the Complex number is said to be Skew-Harmitian if $A^{\theta} = -A$ i,e, if transposed Conjugate of the matrix is equal to negative of the Matrix itself.

"Examples:"

$$\begin{bmatrix} 3i & 3i+5 \\ 3i-5 & 0 \end{bmatrix}, \quad \begin{bmatrix} -4i & 1+i & 3-2i \\ -1+i & 0 & 4+i \\ -3-2i & -4+i & 5i \end{bmatrix}$$

The Diagonal element of a skew- Hermitian Matrix are either purely imaginary or zero.

"As a Result:"

for skew- Harmitian matri are either:

$a_{ij} = -\overline{a_{ji}}$ for all i and j

Diogonal Entries $i = j$

$a_{ii} = -\overline{a_{ii}}$ (for all i

$\alpha + i\beta = -(\alpha - i\beta)$

$\alpha + i\beta = -\alpha + i\beta \Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$

$a_{ii} = \alpha + i\beta = i\beta$

= { Purely imaginary, if $\beta \neq 0$

if $\beta = 0$.

Example :

(ii) If A is Hermitian, then show that iA is skew-Harmitian.

Solution :

Since A is Hermitian

$$A^{\theta} = A$$

We have to show that iA is skew-Hermitian

i.e $(iA)^{\theta} = -(iA)$

Now

$$(iA)^{\theta} = \bar{i}\, A^{\theta}$$

$$= -iA$$

$[\because \bar{i} = -i \text{ and } A^{\theta} = A, \text{ from } (1)]$

$$(iA)^{\theta} = -(iA)$$

Hence iA is skew-Harmitian.

ii)

"If A is skew - Hermitian, then show that iA is Hermitian."

"Solution:"

Since A is skew- Hermitian

$$A^\theta = -A$$

We have to show that iA is Hermitian i,e, $(iA)^\theta = iA$

Now

$$(iA)^\theta = \bar{i} \cdot A^\theta$$

$$= -i - (A)$$

$[\because \bar{i} = -i$ and $A^\theta = -A$, from 2$]$

$$(iA)^\theta = iA$$

" Hence iA is Hermitian."

chapter 7

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RANK of A MATRIX

* **Sub matrix of a matrix** — Matrix B obtained by deleting some rows or some colums or both of a matrix A of the type $m \times n$ is called a sub-matrix of matrix A.

For Example :—

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\text{Sub matrix of } A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 5 & 8 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

* **Minors of a matrix** — If A be a matrix of the type $m \times n$, then the determinant of every square sub-matrix of A is called a minor of the matrix A.

The order of the square sub-matrix is called order of minor.

For Example —

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}$$

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The third order minors of matrix A –

$$|A_1| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad |A_2| = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}$$

$$|A_3| = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix}, \quad |A_4| = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}$$

Minors of order 2 = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}$,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}, \quad \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$$

★ **Rank of a matrix** – Let $A = (a_{ij})_{m \times n}$ be an $m \times n$ non-zero matrix.

Then non-negative integer r is called rank of A if,

(i) A has atleast one non-zero minor of order r.

(ii) Every minor of order $r+1$ of A, if any, is zero.

In other words, Rank of a matrix is the highest order of a non-zero minor of the matrix.

The rank of a matrix A is denoted as $\rho(A)$.

★ Tips to find Rank of a matrix –

(1). If all the elements of a matrix A are zero, then $\rho(A) = 0$. If A is non-zero matrix, then it has atleast one non-zero element in it and hence has atleast a non-zero minor of order 1.
Thus $\rho(A) \geq 1$.

(2). If I is an identity matrix of order n, then $\rho(A) = n$.

(3) The rank of a diagonal matrix is equal to the number of non-zero elements in the diagonal.

(4). When A is a non-zero square matrix of order n:
In this case, if $|A| \neq 0$, then $\rho(A) = n$ otherwise Find all the minors of order

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$n-1$ and the procedure used in step (4) is adopted to find the rank.

(5). When A is a non-zero matrix of order $m \times n$:

Let $P = \min(m, n)$. Find all the minors of order p. If any minor of order p is non-zero, then rank of matrix A is p. If all the minors of order p are zero, then $\rho(A) < P$. Next, Find all the minors of order $p-1$. Otherwise Find all the minors of $P-2$. Continuing like this we shall get atleast one minor of order r which is non-zero and minors of order $r+1$ are zero. Thus the rank of matrix A is r. Hence rank of $A \leq \min(m, n)$

Example :- $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Here, A is a square matrix and

$|A| = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \neq 0$

Hence, the rank of the given matrix is 3.

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★ Elemementary Operations – The following six operations applied on rows or colums of a matrix are called elementary operations. Among these six operations, first three are called row operations and other three are called elementary colums operations.

(i) The interchange of any two distinct rows of a matrix is a elementary row operation. Interchanging the ith row and jth column is denoted by

$R_i \leftrightarrow R_j$

(ii). Multiplying each element of any row by a non-zero scaler is a elementary row operation.

$R_i \to kR_i$ or $R_i(k)$.

(iii). The addition of a multiple of one row to another row is a elementary row operation. $R_i \to R_i + kR_j$

(iv). The interchanging of any two distinct colums of a matrix is a elementary column operation. $C_i \leftrightarrow C_j$

(v) Multiplying each element of any column by a non - zero scaler is a elementary column operation. $C_i \to kC_i$

(vi) The addition of a multiple of one column is a elementary column operation. $C_i \to C_i + kC_j$

★ Row Equivalent Matrix – If A and B are two matrix, then A is to be row equivalent to the matrix B if A can be obtained by applying in succession a finite sequence of elementary row operations on B and is written as $A \overset{R}{\sim} B$.

★ Column Equivalent Matrix – If A and B are two matrix, then A is said to be column equivalent to the matrix B. If A can be obtained by applying in succession a finite sequence of elementary columns operations on B and is written as $A \overset{C}{\sim} B$.

All equivalent Matrices have the same rank.

ORTHOGONAL AND UNITARY MATRICES :- 2

ORTHOGONAL MATRICES *:-

Definition :- A square matrix A over the field of reals of rals is said to be orthogonal if

$$AA' = A'A = I$$

from the definition, it is clear that matrix A is invertible and $A' = A^{-1}$ Also, $|A| \neq 0$

An orthogonal matrix A is said to be proper if $|A| = 1$ and improper if $|A| = -1$

Illustrations. 1. if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, than $A' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore$ $AA' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

also $A'A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Hence, A an orthogonal matrix.

5.3 PROPERTIES of ORthogonal matrix

Theorem 1. the determinant of an orthogonal matrix $\neq 1$.

OR

Every orthogonal matrix is non-singular.

Proof Theorem 2 The product of two orthogonal matrices of the same order is orthogonal

Theorem 3. The inverse and transpose of an orthogonal matrix are orthogonal.

Theorem 4 5.4 UNITARY Matrix

Definition. A square matrix A over the field A is of complex numbers is said to be unitary if

$$A \cdot A^{\theta} = A^{\theta} \cdot A = I$$

from the definition, it is clear that matrix A is invertible and $A^{-1} = A^{\theta}$

Illustrations i) let $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})' = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

So. $AA^{\theta} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$= \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Similarly, $A^{\theta}A = I$

Hence A is an unitary matrix

5.5 Properties of unitary matrix

Theorem 1. A real matrix is unitary iff it is orthogonal.

II Theorem — The transpose of a unitary matrix is unitary.

Theorem 3. The Conjugate of a unitary matrix is unitary.

Theorem 4 — The transposed conjugate of a unitary matrix is unitary.

Theorem 5 Inverse of a unitary matrix is unitary.

Theorem 6 :- Product of two unitary matrices is unitary

Theorem 7. The determinant of a unitary matrix has absolute value 1. $5.$

Theorem 8 Prove that the characteristic roots of a unitary matrix are of unit modulus.

Cor. Prove that the characteristic roots of an orthogonal matrix are of unit modulus.

EXAMPLE = 8

Show that if A is Hermitian and P is unitary, then $P^{-1}AP$ is Hermitian.

Solution $A^{\theta} = A$

Also, since P is unitary

$PP^{\theta} = I$

$P^{-1} = P^{\theta}$

Now, $(P^{-1}AP)^{\theta} = (P^{\theta}AP)^{\theta}$

$= P^{\theta}A^{\theta}(P^{\theta})^{\theta}$

$= P^{\theta}AP$

$= P^{-1}AP$

$[\because A^{\theta} = A$ (by 1) and $(P^{\theta})^{\theta} = P]$

Thus, $(P^{-1}AP)^{\theta} = P^{-1}AP$

Hence, $P^{-1}AP$ is Hermitian.

NORMAL FORM of A MATRIX (CANONICAL FORM):-

Every non-zero matrix of order $m \times n$ can be reduced by means of elementary row and column operations into equivalent matrix of any of the following form. For e.x:-

(i) $\begin{bmatrix} I_r & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} I_r \\ \cdots \\ 0 \end{bmatrix}$ (iii) $[I_r : 0]$

(iv) $[I_r]$.

Where I_r is the identity matrix of order r and O represents zero matrix of any order which is called its normal from or cononical form

ELEMENTARY MATRICES

Definition:- A matrix obtained from an identity matrix I_n, by a single elementary transformotion is called an elementary matrix (or E-matrix).

For e.x:- $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying $R_1 \rightarrow R_1 + R_2 + 3R_3$, we get.

$$I_3 \sim \begin{bmatrix} 1+0+8.0 & 0+1+3.0 & 0+0+3.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is an elementary matrix.

All the elementary matrices are non-singular

Let A and B be two matrices over the field F of type $(m\times n)$ and $(n\times P)$ respectively. Then application of any elementary row (column) operation to A(B) result in the application of the same to the product matrix AB and vice versa.

Every Non-singular matrix can be expressed as a product of elementary matrices.

If A be an $m\times n$ matrix of rank r, then there exist non-singular matrix P such that $PA = \begin{bmatrix} B \\ O \end{bmatrix}$, where B is an $r\times n$ matrix of rank r and O is $(m-r)\times n$ matrix.

If A be an $m\times n$ matrix of rank r, then there exists a non-singular matrix Q such that $AQ = [C \cdot O]$, where C is an $m\times r$ and is $m\times(n-r)$.

The rank of the transpose of a matrix is the same as that of the original matrix.

i.e., $\rho(A) = \rho(A')$.

RANK OF THE PRODUCT OF TWO MATRICES:-

Theorem. The rank of the product of two matrices cannot exceed the rank of either matrix i.e.,

$$\rho(AB) \le \rho(A)$$
$$\rho(AB) \le \rho(B).$$

Ex:- Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

by reducing it to normal form.

Sol:- $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

Operating $C_2 \to C_2 - 2C_1$; $C_3 \to C_3 + C_1$; $C_4 \to C_4 - 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 2 & -1 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

operating $C_3 \to C_3 + 2C_4$; $C_4 \to (-1)C_4$,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

operating $R_2 \to R_2 + 2R_1$; $R_3 \to R_3 - R_1$,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_2 \leftrightarrow C_4$.,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$$

which is the required normal form.

Rank of the matrix i.e., $\rho(A) = 2$.

inverse of Matrix by using elementary operations:-

we know that if A is non-singular matrix A^{-1} exist and $AA^{-1} = A^{-1}A = I$.

Working Rule:- To find A^{-1}, we write $A = IA$, where I is a unit matrix of same order and go on performing suitable elementary row (column) operations on the matrix A on the left hand side and same operations on right hand side till we reach the result $I = BA$

- Then $A^{-1} = B$.

TO CALCULATE P AND Q WHERE $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

if A is a matrix of order $m \times n$, there write $A = I_m A I_n$, where I_m and I_n are mth and nth order unit matrices respectively. The elementary ~~doc~~ row operation on A can be effected by pre-multiplication with corresponding elementary matrix.

Similarly, application of an elementary column operation to A is equivalent to application of the same to I_n or the matrix obtained from I_n in subsequent steps. In the end when we get $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in place of A on left hand side, we have P in place of I_m and Q in place of I_n on the right hand side.

* LINEAR DEPENDENCE AND INDEPENDENCE *

*1) Linear Dependence :-

The vector $x_1, x_2, \ldots x_n$ are said to be Linearly dependent if there exists n numbers $c_1, c_2, c_3 \ldots c_n$ not all zero such that $c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = 0$

* Linear Independence :-

The vector $x_1, x_2, x_3 \ldots x_n$ are said to be linearly independence if there exists of n number $c_1, c_2, c_3 \ldots c_n$ such that the relation

$$c_1x_1 + c_2x_2 + c_3x_3 + \ldots c_nx_n = 0$$
$$c_1 = 0,\ c_2 = 0,\ c_3 = 0 \ \ldots \ldots c_n = 0$$

* Has :- if two vectors are linearly dependent, then one of them is the scalar multiple of the other.

* Has :-
i) A set containing only the zero vector is linearly dependent.
ii) A set containing only a non-zero is linearly dependent

* Has :- A set which contain the null vector '0' is linearly dependent.

Let $x_i = 0$

Then:- $0.x_1 + 0.x_2 + \dots + 0x_{i-1} + 1.x_i + 0x_{i+1} + \dots + 0.x_r = 0$

In this linear Combination, all scalars are not zero.

* Has :-

Every super set of a linearly dependent set is linearly dependent.

$[x_1, x_2 \dots x_p, X_p, X_{p+1} \dots X_r]$ is a Linearly dependent "

* Every sub-set of a linearly independent set of vector is linearly Independent.

$[x_1, X_2 \dots X_t]$ is a linearly dependent set.

* If vector $X_1, X_2 \dots X_P$ are lineary dependent then atleast one of them may be written as a Linear Combination of the rest. "

Has :-

if one of the vector $X_1, X_2 \dots X_P$ can be expressed as a linear Combination of the rest, then the P vectors are Linearly dependent.

$$a_1x_1 + a_2X_2 + \dots + a_{i-1}X_{i-1} + (-1)X_i + a_{i+1}X_{i+1} + \dots + a_pX_p = 0$$

$[x_1, X_2 \dots X_P]$ are linearly dependent.

Has :-
If the set $[X_1, X_2 \cdots X_p]$ is Linearly dependent and the set $[X_1, X_2 \cdots X_p, Y]$ is linearly dependent, then y is a linear combination of the vector $X_1, X_2 \cdots X_p$.

* Has :-
The k n-vectors $A_1, A_2, A_3 \cdots A_k$ are linearly dependent if and only if the rank of the matrix $A = [A_1 A_2 \cdots A_k]$ with given vector as columns is less than k.

* Has :-
working rules to determine whether vector $X_1, X_2 \cdots X_n$ are linearly dependent or linearly independent.

working Rule :-

i) construct a matrix A whose n column are gives vector $X_1, X_2, \cdots X_n$.

ii) Reduce matrix A to row echelon from by using elementary row operations.

iii) find rank of A which is equal to number of non-zero rows.

iv) If $\rho(A) = n$ then the given n vectors $X_1, X_2, \cdots X_n$ are linearly (indepent) Independent.

v) If $\rho(A) < n$ then the given n vector $X_1, X_2 \cdots X_n$ are linearly dependent.

Example:- show that the vector.

i) $(1,2,3)$ $(2,2,0)$ from a linearly independent set.

ii) $(3,1,-4)$ $(2,2-3)$ from a linearly independent set.

i) for two scalar:-

$$c_1,(1,2,3)+c_2(2,20)=(0,0,0)$$

$$(c_1,2c_1,3c_1)+(2c_2,2c_2,0)=(0,0,0)$$

$$[c_1+2c_2,\ 2c_1+2c_2,\ 3c_1+0]=(0,0,0)$$

$c_1+2c_2=0$ $\quad$ $2c_1+2c_2=0$

put:-

$c_1=0$	$c_1=0$
$2c_2=0$	$2c_2=0$
$c_2=0$	$c_2=0$

$3c_1=0$

$c_1=0$

ii)

$$a(3,1,-4)+b(2,2-3)=0$$

$$(3a,a,-4a)+(2b,2b,-3b)=(0,0,0)$$

$$(3a+2b,a+2b,-4a-3b)=(0,0,0)$$

$3a+2b=0$

$a+2b=0$

$-4a-3b=0$

substracting ② from ①

$2a=0 \Rightarrow a=0$

$2b=0 \Rightarrow b=0$

UNIT 3

CHAPTER 8

APPLICATION OF MATRICES TO A SYSTEM OF LINEAR EQUATION

System of non-Homogeneous linear Equation :-

Consider the non-homogeneous (m) linear equation in n unknowns.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

It Can be written in matrix form as $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A is called matrix of Co-efficients and the matrix $[A:B]$ is called Augmented Matrix.

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

Some theorem :-

(1) The System of linear equations $AX = B$ is Consistent if the Co-efficient matrix A and the augmeted matrix $[A:B]$ have the same rank i.e., $\rho(A) = \rho[A:B]$

(2) If A is a Square matrix of order n and X and B are matrices of order $(n \times 1)$ then the system $AX = B$

Process a unique solution if A is non-singulal i.e., if A^{-1} exists.

Working rule for Solution of 'm' Equations in 'n' variables :-

Let $AX = B$ be the matrix form of the system of m equation in n variables.

(i) $\rho(A) \neq \rho[A:B]$, then the equation are inconsistent i.e., there is no Solution.

(ii) If $\rho(A) = \rho[A:B] = n$, then the equation are Consistent and there is a unique Solution.

(iii) If $\rho(A) = \rho[A:B] < n$, then the equations are Consistent and there are infinite number of solution. $\rho(A) = r$. Giving arbitrary values to $n-r$ of the unknows, we get values of the r unknown.

Solution of System of linear Homogeneous Equation :-

Consider the homogeneous linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

The above system of equations can be written in matrix form as:-

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{i.e., } AX = 0$$

first we find the rank.

(1) If $\rho(A) = n$ then there is only a trivial solution which is $x_1 = 0, x_2 = 0, x_3 = 0, \ldots, x_n = 0$

(2) If $\rho(A) < n$ then A matrix which has r non zero and $(n-r)$ zero rows. So system is consistent & infinite sol.

(3) If $m = n$ then for non-trivial sol. $\rho(A) < n$.
necessary and sufficient condition for non-trivial sol. is $|A| = 0$.

for example :-

Solve the following system of equations:-

$$x - y + z = 0$$
$$x + 2y - z = 0$$
$$2x + y + 3z = 0$$

Sol. writing the given equation in the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

Operating $R_2 \to R_2 + (-R_1)$, $R_3 \to R_3 + (-2)R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix}$$

operating $R_3 \to R_3 + (-R_3)$

$$A \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

operating $R_3 \rightarrow R_2 \times \left(\frac{1}{3}\right)$,

$R_3 \rightarrow R_3 \times \left(\frac{1}{3}\right)$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\because \rho(A) = 3 = n$

Hence the given system of equations has only trivial solution viz; $x = y = z = 0$.

Bilinear AND QUADRATIC FORMS :-

Linear Transformation:- Any function 'f' from the non-empty set S to itself is called linear transformation $f(Y) = AY = X$, where X and Y are n-vectors given by.

i.e $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

And A is a square matrix of order n. Here Y is transformed by the square matrix A on X.

$|A| \neq 0$ non singular transformation

$|A| = 0$ Singular transformation.

Bilinear form:- An expression of the form $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$ is called billinear form. if

i) Degree of each term is 2.

ii) Each term contains exactly one variable out of $[x_1, x_2 --- x_n]$ and exactly one variable out of $(y_1, y_2 ..... y_n)$

MATRIX NOTATION of BILINEAR FORM:-

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = [x_1, x_2 ------ x_m] \begin{bmatrix} a_{11} & a_{12} & ---- & a_{1n} \\ a_{21} & a_{22} & ---- & a_{2n} \\ ---- & -- & -- & --- \\ a_{m1} & a_{m2} & ---- & a_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

$\Rightarrow X'AY$ matrix $A = [a_{ij}]_{m \times n}$

Types of Bilinear forms:-

i) Symmetric if the Matrix A is Symmetric

ii) Alternate if the matrix A is skew-sym.

iii) Hermitian if the matrix A is Hermitian.

iv) Alternate if the matrix A is skew Hermi.

Methods to write Matrix of Bilinear form:-

In first row.

coeff. of x_1y_1, coeff. of x_1y_2, coeff of x_1y_3 ..... so on.

In 2nd row

coeff. of x_2y_1, coeff. of x_2y_2, coeff. of x_2y_3 ...... so. on.

- -

In mth row

coeff. of x_my_1, coeff. of x_my_2, coeff. x_my_3 ------ so on.

A bilinear form remains a bilinear form when subjected to linear transformation.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j = X'AY$$

if a matrix ~~X'AY=~~ .

$X = BU$ of order $m\times1, m\times m$ & $m\times1$

And $Y = CV$ of order $n\times1, n\times n$ & $n\times1$

$$X'AY = (BU)'A(CV)$$
$$= (U'B')A(CV)$$
$$= U'(B'AC)V$$
$$= U'DV, \quad \text{Where } D = B'AC$$

The Rank of a bilinear form remain invariant under non-singular transformation.

Bilinear form $X'AY$

Let $X = BU$ and $Y = CV$ are Non-singular matrices.

$$X'AY = (BU)'A(CV)$$
$$= U'B'ACV$$
$$= U'DV, [D = B'AC]$$

$\rho(U'DV) = \rho(D)$ [$\because$ U' & V are non-singular matrices]

$= \rho(B'AC)$ & does not alter the Rank.

$$= \rho(AC) \quad \left[\because B' \text{ \& } C \text{ also non-singular matrix}\right]$$
$$= \rho(A)$$
$$= \rho(X'AY)$$

CANONICAL FORM OF A BILINEAR FORM:→

Bilinear form ~~in~~ the $X'AY$ is said to be in the canonical form if A is of type $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where I_r denotes the identity matrix of order r.

Any bilinear form $X'AY$ of rank r over field F can be reduced by non-singular linear transformation $X = P'U$ and $Y = QV$ over F to the form.

- $u_1v_1 + u_2v_2 + \ldots\ldots\ldots u_rv_r$

OR

Every bilinear form can be reduced to canonical form:-

$$X'AY = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j$$

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$X'AY = (P'U)'A(QV)$$
$$= (U'P)A(QV) = U'(PAQ)V$$
$$= U'\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}V$$

FACTORIZABLE BILINEAR FORM:-

A bilinear form is said to be factorizable. As a product of two bilinear factors.

A non zero bilinear form is factorizable if and only if its rank is 1.

$$\rho(A) = 1$$

The Rank of a quadratic form is invariant under a non-singular linear transformation. Let $q(x) = x'Ax$

$$q(x) = (BY)'A(BY) \qquad L.T \quad X = BY$$

$$= (Y'B')A(BY) = Y'(B'AB)Y$$

$$= Y'CY, \text{ where } C = B'AB$$

$$\rho(Y'CY) = \rho(C)$$
$$= \rho(B'AB) = \rho(AB)$$
$$= \rho(A)$$
$$= \rho(x'Ax)$$

* DIAGONALIZATION OF A QUADRATIC FORM:-

* Every quadratic form can be reduced to a diagonal form

quadratic form $x'Ax$ rank $= r$

$$\rho(A) = r$$

$$P'AP = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}, \quad D_r = \text{diag}(K_1, K_2 \text{---} K_r)$$

$$x'Ax = (PY)'A(PY) = (Y'P')A(PY)$$

$$= Y'(P'AP)Y = Y'\begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}Y$$

$$= K_1Y_1^2 + K_2Y_2^2 + \text{-----} K_rY_r^2$$

$$= \sum_{i=1}^{r} K_iY_i^2$$

Diagonal form:- A quadratic form $q = x'Ax$ is called a diagonal form if matrix A is of the type $\begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}$ the quadratic form reduces to the form

$$a_1Y_1^2 + a_2Y_2^2 + \text{-----} + a_rY_r^2.$$

Quadratic forms :-

A homogeneous polynomial having every term of degree two in any number of variables is called a quadratic form.

Ex. $ax^2 + 2hxy + by^2$

MATRIX OF QUADRATIC FORM :-

Let $q(x)$ be the quadratic form arranged in the form $x'Ay$. Then A is called matrix of the quadratic form if A is symmetric.

METHOD TO WRITE MATRIX NOTATION OF QUADRATIC FORM :-

In first row

- coeff. of x_1^2, $\frac{1}{2}$ coeff. of x_1x_2, $\frac{1}{2}$ coeff. of x_1x_3 ------- so on.

In 2^{nd} row

$\frac{1}{2}$ coeff. of x_2x_1, coeff. of x_2^2, $\frac{1}{2}$ coeff. of x_2x_3 -------- so on.

In nth row

$\frac{1}{2}$ coeff. of x_nx_1, $\frac{1}{2}$ coeff. of x_nx_2 ------- coeff. of x_n^2.

LINEAR TRANSFORMATION OF A QUADRATIC FORM :-

Let $q(x) = x'Ax$ be a quadratic form. $X = BY$ in it is called a linear transformation of a quadratic form.

- A quadratic form remains quadratic form when subjected to linear transformation.

$$X = PY$$

$$X'AX = (PY)'A(PY) = (Y'P')A(PY)$$

$$= Y'(P'AP)Y = Y'CY, \text{ where } C = P'AP$$

$$C' = (P'AP)' = P'A(P')' = P'AP = C$$

Canonical form:- A quadratic form $q = x'Ax$ is said to be canonical form if it is in diagonal form and all the diagonal entries 0, 1 or, -1

Canonical form over Complex field:-
quadratic form $\Rightarrow q = x'Ax$ is said to be canonical if is in the diagonal form and all non-zero diagonal entries 1.

Rank:- The number of non-zero terms in the diagonal form of a quadratic form is called its Rank.
Denoted by r.

Index:- The number of positive terms in the diagonal form of a quadratic form is called index.
Denoted by p.

Signature:- In a quadratic form if r is the rank and p is the index, then $2p-r$ is the signature.
Denoted by s.

LAGRANGE'S METHOD OF DIAGONALIZATIONS:-

i) Arrange all the terms containing x_1 in one bracket and take the coeff. of x_1^2 common outside the bracket to make it unity.

ii) Complete the square by adding and subtracting $\left(\frac{1}{2} \text{coeff } x_1\right)^2$

iii) Repeating the above procedure for x_2, x_3 ---- and so on.
the quadratic form reduces to the form
$a_1y_1^2 + a_2y_2^2 + \cdots\cdots + a_ry_r^2$.

FACTORABLE QUADRATIC FORMS:-

. A quadratic form $x'Ax \neq 0$ with complex coefficients is the product of two linear factors if and only if its rank ≤ 2.

$$x'Ax = (a_1x_1 + a_2x_2 + \cdots + a_nx_n)(b_1x_1 + b_2x_2 + \cdots + b_nx_n)$$

I. If factors are linearly dependent.

$$b_1x_1 + b_2x_2 + \cdots b_nx_n = k(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$$

$$x'Ax = k(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2$$

$$= ky_1^2$$

It show that

- rank of the $x'Ax$ is 1

II. If factors are linearly independent,

$$x'Ax = RS \quad (\text{say})$$

$$R = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$$S = b_1x_1 + b_2x_2 + \cdots + b_nx_n$$

$$= \frac{1}{4}[R+S]^2 - \frac{1}{4}\{R-S\}^2$$

$$x'Ax = \frac{1}{4}y_1^2 - \frac{1}{4}y_2^2$$

- It shows that rank of $x'Ax$ is 2.

- Suppose the rank of quadratic form be either 1 or 2.

I. If rank of $x'Ax$ is 1.

$$x'Ax = ay_1^2 = ay \cdot y_1 = \text{Product of two linear factors.}$$

II. If rank of $x'Ax$ is 2, then.

$$x'Ax = ay_1^2 + by_2^2$$

$$= (\sqrt{a}\,y_1 + i\sqrt{b}\,y_2)(\sqrt{a}\,y_1 - i\sqrt{b}\,y_2)$$

$$= \text{Product of two linear factors.}$$

$\therefore$ Quadratic form cannot have zero rank.

SOME IMPORTANT DEFINITIONS:-

a) positive definite quadratic form :- A real non-singular quadratic form $q = x'Ax$ is called positive definite if q is always positive for all real values of x and is zero only if $x = 0$

(b) positive semi-definite quadratic form:- A real singular quadratic form $q = x'Ax$ $(|A| = 0)$ is called positive semi-definite if q is always ≥ 0 for all real values of x including 0

(c) Negative definite quadratic form:- A Real non-singular quadratic form $q = x'Ax$ $(|A| \neq 0)$ is called negative definite if q is always negative for all real values of x and is zero only when $x = 0$

(d) Negative Semi-definite quadratic form:- A Real singular quadratic form $q = x'Ax$ is called negative semi-definite if q is always ≤ 0 for all values of x including $x = 0$

Canonical form	Rank	Index	Types of quadratic form
$\sum_{i=1}^{n} y_i^2$	$r = n$	$p = n$	positive definite
$-\sum_{i=1}^{n} y_i^2$	$r = n$	$p = 0$	Negative definite
$\sum_{i=1}^{r} y_i^2$	$r < n$	$p = r$	positive semi-definite
$\sum_{i=1}^{r} y_i^2$	$r < n$	$p = 0$	Negative semi-definite

UNIT-4

Characteristic Matrix

Let $A = [a_{ij}]_{n\times n}$ be square Matrix of order n; λ be some scalar

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{n-2}-\lambda \end{bmatrix}$$ → called characteristic Matrix

Determinant of characteristic Matrix is called characteristic Polynomial or function $\boxed{|A-\lambda I|}$

The eq^n $|A-\lambda I|=0$ is called characteristic Eq^n of Matrix A

char Eq of 2×2 Matrix

$$\boxed{\lambda^2 - (\text{trace of } A).\lambda + |A| = 0}$$

char. Eq of 3×3 Matrix

$$\boxed{\lambda^3 - (\text{trace of } A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0}$$

Note :- trace of A = Sum of diagonal element of A

$A_{11}, A_{22}, A_{33} \Rightarrow$ Co-factor of the diagonal element of A.

UNIT-4

Characteristic Matrix

let $A = [a_{ij}]_{n\times n}$ be squar Matrix of order n; λ be some scalar

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{n-2}-\lambda \end{bmatrix} \rightarrow \text{called characteristix Matrix}$$

Determinant of characteristic Matrix is called characteristic Polynomial or function $\boxed{|A-\lambda I|}$

The eqn $|A-\lambda I| = 0$ is called charachertistic Eqn of Matrix A

\# char Eq of 2x2 Matrix

$$\boxed{\lambda^2 - (\text{trace of } A).\lambda + |A| = 0}$$

char. Eq of 3x3 Matrix

$$\boxed{\lambda^3 - (\text{trace of } A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0}$$

Note :- trace of A = Sum of diagonal element of A

$A_{11}, A_{22}, A_{33} \Rightarrow$ Co-factor of the diagonal element of A.

Eigen Value

Root of a characteristic Eq $|A - \lambda I| = 0$

say $\lambda_1, \lambda_2, \ldots \lambda_n$ called Eigen value / characteristic Roots / characteristic value

Eigen Vector

$\lambda \to$ Eigen value of A

then a non zero vector ~~of A~~ X which satisfying the Eq^n $AX = \lambda X$ is called characteristic vector.

Question find the Characteristic Equation, Eigen value & Eigen vector of matrix.

Solution

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

character. Eq^n of A is given by $|A - \lambda I| = 0$

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = \boxed{(1-\lambda)(-4-\lambda)(7-\lambda)} = 0$$

$\hookrightarrow$ char. Eq^n ~~Polynomial~~

$$\boxed{\lambda = 1, -4, 7} \to \text{char. Root}$$

characteristic Vector

sol^n of $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Put $\lambda=1$ $\begin{bmatrix} 1-1 & 2 & 3 \\ 0 & -4-1 & 2 \\ 6 & 0 & 7-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_2+3x_3=0 \qquad -5x_2+2x_2=0 \qquad 6x_3=0$

$\Rightarrow \boxed{x_3=0}$

$\Rightarrow \boxed{x_2=0}$

x_1 is arbitrary. let $x_1=k$

for Eigen value $\lambda=1$. $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$

Put $\lambda=-4$ in $(A-\lambda I)X=0$

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1+2x_2+3x_3=0 \qquad 2x_3=0 \qquad 11x_3=0$

$\boxed{x_3=0}$

$5x_1+2x_2=0$

$x_1=-\frac{2}{5}x_2 \qquad \boxed{x_2=k}$

$\boxed{x_1=-\frac{2}{5}k}$

for $\lambda=-4$

$X=\begin{bmatrix} -2/5\,k \\ k \\ 0 \end{bmatrix}$

Put $\lambda=7$ in $(A-\lambda I)X=0$

$$\begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-6x_1+2x_2+3x_3=0$

$-11x_2+2x_3=0$

$x_3=\frac{11}{2}x_2 \qquad \boxed{\cancel{x_2=k}} \qquad \boxed{\cancel{x_3=\frac{11}{2}k}}$

$-6x_1 + 2x_2 + \frac{33}{2}x_2 = 0$ {By putting value of x_3}

$-6x_1 + \frac{37}{2}x_2 = 0$

$6x_1 = \frac{37}{2}x_2$

$x_1 = \frac{37}{12}x_2$

let $\boxed{x_2 = 12k}$ then $\boxed{x_1 = 37k}$

$x_3 = \frac{11}{2}x_2 = \frac{11}{2} \times 12k = \boxed{66k = x_3}$

for $\lambda = 7$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 37k \\ 12k \\ 66k \end{bmatrix}$

Cayley Hamilton Theorem

Every Square Matrix Satisfies its characteristic eqn

Question Verify cayley Hamilton theorem for the Matrix A and Compute A^{-1}.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

Solution

char. Eqn. $\Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda[(1-\lambda)(4-\lambda)] + [3+2(1-\lambda)] = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0 \quad \text{—(1)}$$

To verify Cayley Hamilton theorem, we have to prove that

$$A^3 - 5A^2 + 6A - 5I = 0 \quad \text{—(2)}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix}$$

$$\therefore A^3 - 5A^2 + 6A - 5I = \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix} - 5\begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence cayley Hamilton theorem is verified.

To find A^{-1}

Pre Multiply by A^{-1} in eq^n A^{-1}

$$A^{-1}A^3 - 5A^{-1}A^2 + 6A^{-1}A - 5A^{-1}I = 0$$

$$\Rightarrow A^2 - 5A + 6I - 5A^{-1} = 0$$

$$A^{-1} = \frac{1}{5}A^2 - A + \frac{6}{5}I \quad \text{——— (3)}$$

Put value of A^2 & A in eq^n (3)

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + \frac{6}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$$

Verification

$$AA^{-1} = \frac{1}{5}\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question for Practice

Varify Cayley Hamilton theorm.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 2 & 0 & 3 \end{bmatrix}$$

Verify Cayley Hamilton theorm, & compute A^{-1}.

$$A = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$$

Answer = $A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$

Note (i) Eigen value of a diagonal Matrix are the diagonal element of the matrix

(ii) Eigen value of a triangular Matrix are the diagonal element of the matrix

Theorem Prove that the characteristic roots of a real Symmetric matrix | Hermition matrix are all real.

Proof Let A be a symmetric matrix

$$A' = A \quad \& \quad \bar{A} = A$$

$$A^{\theta} = A \text{ [Hermition Matrix]}$$

λ be the characteristic Root of A & X be eigen value

$$AX = \lambda X$$

$$(AX)^{\theta} = (\lambda X)^{\theta}$$

$$X^{\theta} A^{\theta} = \bar{\lambda} X^{\theta}$$

$$X^{\theta} A = \bar{\lambda} X^{\theta} \qquad \{A^{\theta} = A\}$$

Post Multiply by X

we get $\quad X^{\theta} A X = \bar{\lambda} X^{\theta} X$

$$X^{\theta} \lambda X = \bar{\lambda} X^{\theta} X$$

$$\lambda X^{\theta} \cancel{X} = \bar{\lambda} X^{\theta} X$$

$$\lambda X^{\theta} X - \bar{\lambda} X^{\theta} X = 0$$

$$(\lambda - \bar{\lambda}) X^{\theta} \cancel{X} = 0$$

$$X^{\theta} X \neq 0$$

$$(\lambda - \bar{\lambda}) = 0$$

$$\boxed{\lambda = \bar{\lambda}}$$

$\therefore$ λ is real.

for H.W

Prove that the characteristic root of a Skew-Hermition matrix are either zero or imaginary [Pure imaginary]

Minimal Polynomial

If $f(x)$ is a polynomial of the lowest degree with leading coefficient unity such that $f(A)=0$, then $f(x)$ is called Minimal polynomial of the ~~A f(x) is called~~ ~~Minimal Equation~~ ~~to~~ Matrix A. & the eq^n $f(x)=0$ is called the minimal Eq^n of the matrix A

Derogatory Matrix :- The n-square matrix is said to be a derogatory matrix if degree of its minimal Eq^n is less than 'n'

Non-Derogatary Matrix :- The n-square matrix is said to be a Non-derogatary matrix if degree of its minimal eq^n is equal to n

Question find the minimal polynomial of the Matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

& show that it is derogatory

Solution

$$A = \begin{bmatrix} +5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix}$$

$R_3 \to R_3 + R_2$
$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 2 & -2-\lambda & -2-\lambda \end{vmatrix}$$

$= C_2 \to C_2 - C_3$
$$\begin{vmatrix} +5-\lambda & 0 & -6 \\ -1 & 2-\lambda & 2 \\ 2 & 0 & -2-\lambda \end{vmatrix}$$

$$= -(2-\lambda)\begin{vmatrix} 5-\lambda & -6 \\ 2 & -2-\lambda \end{vmatrix} \quad \text{Expanding by } C_2$$

$$= -(2-\lambda)\left[\lambda^2 - 3\lambda + 2\right]$$
$$= (\lambda-2)^2(\lambda-1)$$

$\boxed{\text{Characteristic Eq}^n = (\lambda-2)^2(\lambda-1) = 0}$

$\boxed{\lambda = 2, 2, 1}$ $\Rightarrow$ Eigenroot of A

$m(x)$ be the minimal polynomial, then both $(x-2)$ & $(x-1)$ must be factors of $m(x)$.

first we check, whether $(x-2)(x-1) = x^2 - 3x + 2$ is the minimal polynomial of A or not

~~$A^2 - 3A + 2I = 0$~~

$$A^2 = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -18 & -16 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix}$$

$$A^2 - 3A + 2I = \begin{bmatrix} 15 & -18 & -18 \\ -5 & 10 & 6 \\ 7 & -18 & -14 \end{bmatrix} + \begin{bmatrix} -15 & 18 & 18 \\ 5 & -10 & -6 \\ -7 & 18 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

thus matrix A satisfies the eq^n $x^2 - 3x + 2 = 0$

Hence $m(x) = x^2 - 3x + 2$ is the minimal polynomial of A

Since its degree is less than the order of A,

$\Rightarrow$ A is derogatory.

<u>Question for Practice</u>

① find the characteristic polynomial & minimal polynomial of

(i)
$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Also check Derogatory & Non Derogatory of Matrix

Sol^n char. poly $= -\lambda^3 + 12\lambda^2 - 36\lambda + 32$

Minimal poly $= x^2 - 10x + 16$

Derogatory Matrix.

(ii)
$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

char. poly $= \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$

Minimal poly $= x^3 - 6x^2 - 3x + 18 = 0$

Non-Derogatory Matrix

Diagonalization of a Square Matrix

A square Matrix A of the order $n \times n$ of is said to be diagonalizable if there exist an invertible matrix 'P' of the same order $n \times n$ such that $P^{-1}AP$ is a diagonal matrix

Ques Diagonalize, if possible the Matrix $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & -4 \\ 9 & 1 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & -4 \\ 9 & 1 & 3 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 6-\lambda & 0 & 0 \\ 0 & 7-\lambda & -4 \\ 9 & 1 & 3-\lambda \end{bmatrix}$$

Char. eq. $\Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & 0 & 0 \\ 0 & 7-\lambda & -4 \\ 9 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda) \begin{vmatrix} 7-\lambda & -4 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(\lambda^2 - 10\lambda + 25) = 0$$

$$(6-\lambda)(\lambda-5)^2 = 0$$

$$\boxed{\lambda = 6, 5, 5}$$

Case I $\lambda = 6$ eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$

$AX = \lambda X$ or $(A - 6I)X = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -4 \\ 9 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y - 4z = 0$	$9x + y - 3z = 0$
$y = 4z$	$9x + 4z - 3z = 0$
	$9x = -z$

let $z = 9$ $x = -1$, $y = 36$

$$\boxed{X = \begin{bmatrix} -1 \\ 36 \\ 9 \end{bmatrix} \text{ for } \lambda = 6}$$

Case II $\lambda = 5$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$

$AX - \lambda X = 0 \Rightarrow (A - 5I)X = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -4 \\ 9 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 9R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{x=0}$

$$2y-4z=0$$
$$y=2z$$

let $z=1 \Rightarrow x=0,\ y=2$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{for } \lambda=5$$

As there are only two linearly independent eigen vector of the matrix A of order 3

$\Rightarrow$ A is not diagonalizable.

Imp

Note $A \to$ Matrix of order $n\times n$ over $\mathbb{R}$

A is diagonalizable iff A matrix has n linearly independent eigen vector

Question for Practice

Diagonalize the following matrix if possible

① $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ ② $\begin{bmatrix} 2 & 1 \\ -7 & 6 \end{bmatrix}$

③ $\begin{bmatrix} 1 & 3 & 6 \\ -3 & -5 & 6 \\ 3 & 3 & 4 \end{bmatrix}$ ④ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}$

Sol^n ① $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ ② $\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ ③ $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ④ Not Diagonaliza