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Notes

Ordinary Differential equations.

B.Sc 2nd Sem (Mathematics)

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Exact Diff. Eqn

Types of Diff. eqn :-

(i) Ordinary diff eqn.

(ii) Partical diff. eqn

Solution of An Exact Diff. Eqn :-

$$M\,dx + N\,dy = d\left[u + \int f(y)\,dy\right]$$

But $M\,dx + N\,dy = 0$

$$d\left[u + \int f(y)\,dy\right] = 0$$

Integrating both sides, we get

$$u + \int f(y)\,dy = c$$

$$u = \int_{y\ \text{constant}} M\,dx$$

$$\int_{y\ \text{constant}} M\,dx + \int (\text{terms in } N \text{ not cont. } x)\,dy = c$$

which is required eqn

① Solve the diff eqn

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$$

Comparing eqn (i) with $M\,dx + N\,dy = 0$

we observe that

$M = x^4 - 2xy^2 + y^4$

$N = (-2x^2y + 4xy^3 - \sin y)$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3$$

$$\frac{\partial N}{\partial x} = -4xy + 4y^3$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ therefore the given eqn (i) is exact.

Hence the solution is

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = C$$

$$\int_{y \text{ constant}} (x^4 - 2xy^2 + y^4)\,dx + \int -\sin y\,dy = C$$

$$\frac{x^5}{5} - 2y^2 \cdot \frac{x^2}{2} + y^4 \cdot x + \cos y = C$$

$$\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = C$$

2. Prove that $\frac{1}{x^2y^2}$ is the integrating factor of the eqn.

$$(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$$

comparing it with $M\,dx + N\,dy = 0$

$M = (1+xy)\,y$ and

$N = (1-xy)\,x$

$\frac{\partial M}{\partial y} = 1+2xy \qquad \frac{\partial N}{\partial x} = 1-2xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ eqn (1) is not an exact eqn

Multiplying (1) by $\frac{1}{x^2y^2}$ we get,

$$\frac{1+xy}{x^2y}\,dx + \frac{1-xy}{xy^2}\,dy = 0 \qquad \text{---(2)}$$

$$\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

Comparing (2) with, $M\,dx + N\,dy = 0$

$M = \frac{1}{x^2y} + \frac{1}{x}$ and $N = \frac{1}{xy^2} - \frac{1}{y}$

$\frac{\partial M}{\partial y} = -\frac{1}{x^2y^2}$ and $\frac{\partial N}{\partial x} = -\frac{1}{x^2y^2}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Eqn (2) is an exact eqn.

Hence $\frac{1}{x^2y^2}$ is an integrating factor of eqn (1).

Chapter 2nd :- Eqn of First order but not of First degree.

1\. Solve the diff. eqn

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$$

putting $\frac{dy}{dx} = P$, we get

$$P^2 - 5P + 6 = 0$$

$$P^2 - 3P - 2P + 6 = 0$$

$$P(P-3) - 2(P-3) = 0$$

$$(P-3)(P-2) = 0$$

either $P - 3 = 0$

$P = 3$

$\frac{dy}{dx} = 3$

Integrating

$y = 3x + c$

$y - 3x - c = 0$

$P - 2 = 0$

$P = 2$

$\frac{dy}{dx} = 2$

Integrating

$y = 2x + c$

$y - 2x - c = 0$

Therefore, the most general solution is

$$(y - 3x - c)(y - 2x - c) = 0$$

Imp. Solve the diff. eqn

$$y = 3x + \log P$$

Diff w.r.t. x:-

$$\frac{dy}{dx} = 3 + \frac{1}{P}\frac{dP}{dx}$$

$$P = 3 + \frac{1}{P}\frac{dP}{dx}$$

$$dx = \frac{dP}{P(P-3)}$$

$$dx = \frac{1}{3}\left[\frac{1}{P-3} - \frac{1}{P}\right] dP$$

Integrating,

$$x = \frac{1}{3}\left[\log(P-3) - \log P\right] + \log c_1$$

$$3x = \log \frac{P-3}{P} + \log c_2$$

$$3x = \log \left(\frac{P-3}{P}\right) . c_2$$

$$e^{3x} = \frac{P-3}{P} . c_2 \Rightarrow \frac{P-3}{P} = c e^{3x}$$

$$1 - \frac{3}{P} = c\, e^{3x} \rightarrow P = \frac{3}{1-ce^{3x}}$$

Putting this value of P in (1), we, have

$$y = 3x + \log \frac{3}{1-ce^{3x}}.$$

Clairaut's eqn :-

Def. The diff. eqn of the type $y = Px + f(P)$ is known as Clairaut's eqn.

Exp. Solve the diff eqn $y = Px + \frac{a}{P}$ and obtain the singular Solution.

Solution:- The given eqn is $y = Px + \frac{a}{P}$ —(1)

Diff. w.r.t. x :-

$$\frac{dy}{dx} = P + x\frac{dP}{dx} - \frac{a}{P^2} . \frac{dP}{dx}$$

$$p = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx} = 0 \Rightarrow \left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$$

either $x - \frac{a}{p^2} = 0$ or $\frac{dp}{dx} = 0$

(i) when $\frac{dp}{dx} = 0$

Integrating both side, we get,

$$p = c$$

Putting $p = c$ in (i) we get

$$y = cx + \frac{a}{c}$$, which is the required solution.

(ii) when $x - \frac{a}{p^2} = 0$, $p^2 = \frac{a}{x} \Rightarrow p = \sqrt{\frac{a}{x}}$;

Putting $p = \sqrt{\frac{a}{x}}$ in (i), we have

$$y = \sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$y = \sqrt{ax} + \sqrt{ax}$$

$$= 2\sqrt{ax}$$

Chapter - 3

Orthogonal Trajectories

Exp. Find the orthogonal trajectories of the family parabolas $y = ax^2$

$y = ax^2$ — (1)

Diff (1) w.r.t x. we have

$$\frac{dy}{dx} = 2ax \quad — \quad (2)$$

eliminating a b/w (1) and (2) we get

$$y^2 = x^2 \cdot \frac{1}{2x} \frac{dy}{dx}$$

$$y = \frac{x}{2} \frac{dy}{dx}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3) we have

$$y = -\frac{x}{2} \frac{dx}{dy}$$

$$2y\,dy = -x\,dx$$

Integrating both side, we get

$$\int 2y\,dy = -\int x\,dx + c_1$$

$$y^2 = \frac{x^2}{2} + c_1$$

$$2y^2 = -x^2 + 2c_1$$

$$x^2 + 2y^2 = 2c_1$$

$$x^2 + 2y^2 = c^2 \qquad [\because 2c_1 = c^2$$

which is the required eqn.

Chapter 4th — Linear Diff. eqn with Constant co-efficients.

Exp. 1 — Solve the diff eqn

$$2\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$$

eqn in Symbolic form is

$$(2D^3-7D^2+7D-2)y=0$$

Auxiliary eqn is

$$2D^3-7D^2+7D-2=0$$

$$(D-1)(D-2)(2D-1)=0$$

$$\therefore D=1,2,\frac{1}{2}$$

Here, we have three real and diff. roots.

Hence the complete solution

$$y=c_1e^{x}+c_2e^{2x}+c_3e^{x/2}$$

7 Solve the Diff eqn :-

$$\frac{d^4y}{dx^4}-\frac{d^3y}{dx^3}-9\frac{d^2y}{dx^2}-11\frac{dy}{dx}-4y=0$$

The eqn is symbolic form is

$$(D^4-D^3-9D^2-11D-4)y=0$$

$$D^4-D^3-9D^2-11D-4=0$$

$$(D+1)^3(D-4)=0$$

$$D=-1,-1,-1,4.$$

Here, we have three equal roots and one diff. root.

Hence the complete solution is

$$y=(c_1+c_2x+c_3x^2)e^{-x}+c_4e^{4x}$$

Solve the diff eqn

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104e^{3x}$$

$$(D^2+6D+25)y = 104e^{3x}$$

$$D = \frac{-6 \pm \sqrt{36-100}}{2}$$

$$D = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$$C.F = e^{-3x}[c_1 \cos 4x + c_2 \sin 4x]$$

$$P.I = \frac{1}{D^2+6D+25} \cdot 104 \cdot e^{3x}$$

$$= \frac{1}{9+18+25} \cdot 104e^{3x} \quad [\text{Putting } D = 3]$$

$$\frac{104e^{3x}}{52} = 2e^{3x}.$$

$$y = CF + PI.$$

$$= e^{-3x}(c_1 \cos 4x + c_2 \sin 4x) + 2e^{3x}$$

Enf Solve the diff eqn

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$$

$$(D^2 - 6D + 9)y = e^{3x}$$

$$D^2 - 6D + 9 = 0$$

$$(D-3)^2 = 0$$

$$D = 3, 3$$

$C.F = (c_1 + c_2 x) e^{3x}$

case

$P.I = \frac{1}{(D-3)^2} \cdot e^{3x} = \frac{1}{(3-3)^2} \cdot e^{3x}$

$\frac{1}{0} \cdot e^{3x}$ which is a case of failure.

$\frac{1}{(D-3)^2} e^{3x} = x \frac{1}{\frac{d}{dD}(D-3)^2} \cdot e^{3x}$

$= \frac{x}{2} \cdot \frac{1}{(D-3)} \cdot e^{3x}$

$= \frac{x}{2} \left[\frac{1}{3-3} e^{3x} \right]$

$= \frac{x}{2} \left[\frac{1}{0} e^{3x} \right]$

$= \frac{x}{2} \cdot x \frac{1}{\frac{d}{dD}(D-3)} e^{3x}$

$= \frac{x^2}{2} \cdot e^{3x}$

$PI = \frac{x^2}{2} \cdot e^{3x}$

Hence complete solution is

$y = C.F + PI$

$y = (c_1 + c_2 x) e^{3x} + \frac{x^2}{2} e^{3x}$.

Case II. (a) Show that $\frac{1}{F(D^2)} \sin ax = \frac{1}{F(-a^2)} \sin ax$ if $F(-a^2) \neq 0$

Proof. By successive diff.

$$\sin ax = \sin ax$$

$$D \sin ax = a \cos ax$$

$$D^2 \sin ax = -a^2 \sin ax$$

$$D^3 \sin ax = -a^3 \cos ax$$

$$D^4 \sin ax = a^4 \sin ax$$

$$(D^2)^2 \sin ax = (-a^2)^2 \sin ax$$

Similarly, $(D^2)^n \sin ax = (-a^2)^n \sin ax$

Operating $\frac{1}{F(D^2)}$ on both sides,

$$\frac{1}{F(D^2)} \cdot F(D^2) \sin ax = \frac{1}{F(D^2)} \cdot F(-a^2) \sin ax$$

$$\sin ax = F(-a^2) \cdot \frac{1}{F(D^2)} \sin ax$$

Dividing both side by $F(-a^2) \neq 0$

$$\frac{1}{F(-a^2)} \sin \alpha x = \frac{1}{F(D^2)} \sin \alpha x$$

$$\frac{1}{F(D^2)} \sin \alpha x = \frac{1}{F(-a^2)} \sin \alpha x \quad \text{if } F(-\alpha^2) \neq 0$$

Solve the diff. eqn

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y = \sin 2x$$

$$(D^2+D+1)\,y = \sin 2x$$

$$D^2+D+1 = 0$$

$$D = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$C.F = e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I = \frac{1}{D^2+D+1}\sin 2x$$

$$= \frac{1}{-2^2+D+1}\sin 2x$$

$$= \frac{1}{D-3}\sin 2x$$

$$= \frac{1}{D-3}\sin 2x$$

$$\frac{D+3}{(D-3)(D+3)}\sin 2x = \frac{D+3}{D^2-9}\sin 2x$$

$$\frac{D+3}{-2^2-9}\sin 2x$$

$$\frac{D+3}{-2^2-9}\sin 2x$$

$$= -\frac{1}{13}(D+3)\sin 2x = -\frac{1}{13}\left[D\sin 2x+3\sin 2x\right]$$

$-\frac{1}{13}\left[2\cos 2x + 3\sin 2x\right]$

Hence the complete solution is

$$y = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}\left(2\cos 2x + 3\sin 2x\right) \quad \underline{\underline{\text{Ans}}}$$

<u>Case III</u> To evaluate $\frac{1}{F(D)} x^m$ when m is a positive integer.

<u>Exp.</u> Solve the diff. eqn $\frac{d^2y}{dx^2} - 4y = x^2$

The Symbolic form of the eqn

$$(D^2-4)y = x^2$$

$\therefore$ Auxiliary eqn is $D^2 - 4 = 0$

$$D = \pm 2$$

$$C.F = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{D^2-4} x^2$$

$$= \frac{1}{-4\left(1-\frac{D^2}{4}\right)} \cdot x^2 = -\frac{1}{4}\left(1-\frac{D^2}{4}\right)^{-1} \cdot x^2$$

$$= -\frac{1}{4}\left[1 + \frac{D^2}{4} + \cdots\right] x^2$$

e^{x}

$$-\frac{1}{4}\left[x^2+\frac{1}{4}D^2(x^2)\right]$$

$$= -\frac{1}{4}\left[x^2+\frac{1}{4}\cdot\frac{2}{2}\right] = -\frac{1}{4}\left[x^2+\frac{1}{2}\right]$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2+\frac{1}{2}\right) \text{ Ans}$$

Case IV To Prove that $\frac{1}{F(D)}(e^{ax}V) = e^{ax}\frac{1}{F(D+a)}V$

where V is function of x.

Solve the diff eqn

$$\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = x^2 e^{x}$$

$$(D^3 - 3D^2 + 2)y = x^2 e^{x}$$

$$D^3 - 3D + 2 = 0$$

$$(D-1)(D^2+D-2) = 0$$

$$(D-1)(D+2)(D-1) = 0$$

$$(D-1)^2(D+2) = 0$$

$$D = 1, 1, -2.$$

$$C.F = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$$

$$P.I = \frac{1}{D^3-3D+2}(x^2 e^{x}) = \frac{1}{D^3-3D+2}(e^{x} x^2)$$

$$e^x \cdot \frac{1}{(D+1)^3 - 3(D+1) + 2} \cdot x^2$$

$$= e^x \frac{1}{D^3+3D^2} \cdot x^2$$

$$e^x \cdot \frac{1}{3D^2\left(1+\frac{D}{3}\right)} \cdot x^2$$

$$= \frac{1}{3} e^x \frac{1}{D^2} \left(1+\frac{D}{3}\right)^{-1} \cdot x^2$$

$$= \frac{1}{3} e^x \frac{1}{D^2} \left[1 - \frac{D}{3} + \frac{D^2}{9} + \dots\right] x^2$$

$$= \frac{1}{3} e^x \frac{1}{D^2} \left[x^2 - \frac{1}{3} D(x^2) + \frac{1}{9} D^2(x^2)\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D^2} x^2 - \frac{1}{3} \cdot \frac{1}{D}(x^2) + \frac{1}{9} x^2\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D} \int x^2 dx - \frac{1}{3} \int x^2 dx + \frac{1}{9} x^2\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D}\left(\frac{x^3}{3}\right) - \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{3} \int x^3 dx - \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$$= \frac{1}{3} e^x \left[\frac{x^4}{12} - \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$$\frac{1}{108} x^2 e^x \left[3x^2 - 4x + 4\right]$$

$$y = (c_1 + c_2 x) e^x + c_3 e^{-2x} + \frac{1}{108} x^2 e^x (3x^2 - 4x + 4)$$

Case V:

To evaluate $\frac{1}{F(D)}(xV)$ where V is any function of x. Show that $\frac{1}{F(D)}(x \cdot V)$

$$= x \cdot \frac{1}{F(D)} \cdot V + \frac{d}{dD}\left[\frac{1}{F(D)}\right] \cdot V$$

Exp

Solve the diff eqn

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$$

Symbolic form:

$$(D^2 - 2D + 1)y = x e^x \sin x$$

$$D^2 - 2D + 1 = 0$$

$$(D-1)^2 = 0 \qquad D = 1, 1$$

$$C.F = (C_1 + C_2 x)e^x$$

$$P.I = \frac{1}{D^2 - 2D + 1}(x e^x \sin x)$$

$$= \frac{1}{(D-1)^2}(x e^x \sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \sin x)$$

$$= \frac{e^x}{D^2}(x \sin x)$$

$$= e^x\left[x \frac{1}{D^2} \sin x + \frac{1}{dD}\left(\frac{1}{D^2}\right) \sin x\right]$$

$$= e^x\left[x \cdot \frac{1}{D^2} \sin x - \frac{2}{D^3} \sin x\right]$$

$$e^{x}\left[x \cdot \frac{1}{-1}\sin x - \frac{2}{D}\left(\frac{1}{D^2}\sin x\right)\right]$$

$$= e^{x}\left[-x\sin x - \frac{2}{D}\cdot\frac{1}{(-1)}\sin x\right]$$

$$= e^{x}\left[-x\sin x + \frac{2}{D}\sin x\right]$$

$$= e^{x}\left[-x\sin x + 2\int \sin x\,dx\right]$$

$$= e^{x}\left[-x\sin x + 2(-\cos x)\right]$$

$$= -e^{x}\left[x\sin x + 2\cos x\right]$$

Hence the complete solution is

$$y = (C_1 + C_2 x)e^{x} - e^{x}(x\sin x + 2\cos x)$$

Chapter: 5th Homogeneous Linear Eqn

Homogeneous Linear eqn

Def. An eqn of the form.

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots P_n(y) = X$$

where $P_0, P_1, P_2, \ldots P_n$ are constant and X is a function of x, is called a homogeneous linear diff eqn.

Solve the diff eqn

$$x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$

Put $x = e^z$ so that

$$z = \log x$$

$D = x\frac{d}{dx} = \frac{d}{dz}$ then given

eqn (i);

$$[D(D-1)-2]y = e^{2z} + e^{-z}$$

$$[D^2 - D - 2]y = e^{2z} + e^{-z}$$

Auxiliary eqn is $D^2 - D - 2 = 0$

$$(D-2)(D+1) = 0$$

$$D = 2, -1$$

$$C.F = c_1 e^{2z} + c_2 e^{-z}$$

$$P.I = \frac{1}{D^2-D-2}(e^{2z} + e^{-z})$$

$$= \frac{1}{D^2-D-2} e^{2z} + \frac{1}{D^2-D-2} e^{-z}$$

$$= z . \frac{1}{2D-1} e^{2z} + z . \frac{1}{2D-1} e^{-z}$$

$$= z . \frac{1}{2(2)-1} . e^{2z} + \frac{z \cdot 1}{2(-1)-1} . e^{-z}$$

$$= \frac{1}{3} z e^{2z} - \frac{1}{3} z e^{-z}$$

$$= \frac{1}{3} . z (e^{2z} - e^{-z})$$ Ans

Hence the complete solution is

$$y = c_1 e^{2z} + c_2 e^{-z} + \frac{1}{3}(e^{2z} - e^{-z}) z$$

$$y = c_1 e^{x^2} + \frac{c_2}{x} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right) \log x$$

$(\because x = e^z)$

* Eqn Reducible to Homogeneous Linear form.

Exp Solve the diff eqn:—

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$$

put $x+1 = e^z$ so that

$$z = \log(x+1)$$

$$D = \frac{d}{dz}$$

$$[D(D-1) + D]y = (2e^z+1)(2e^z+2)$$

$$D^2 y = (2e^z+1)(2e^z+2)$$

$$D^2 = 0, \quad D = 0, 0$$

$$C.F = (c_1 + c_2 z) e^{0z}$$

$$= c_1 + c_2 z$$

$$P.I = \frac{1}{D^2}(2e^z+1)(2e^z+2)$$

$$= \frac{1}{D^2}(4e^{2z} + 6e^z + 2)$$

$$\frac{1}{D}\int \left(4e^{2z}+6e^{z}+2\right)dz$$

$$\frac{1}{D}\left[\frac{4e^{2z}}{2}+6e^{z}+2z\right] = \frac{1}{D}\left[2e^{2z}+6e^{z}+2z\right]$$

$$= \int\left(2e^{2z}+6e^{z}+2z\right)dz = 2\cdot\frac{e^{2z}}{2}+6e^{z}+\frac{2\cdot z^2}{2}$$

$$= e^{2z}+6e^{z}+z^2$$

Hence the complete or general solution

$$y = C_1 + C_2 z + e^{2z} + 6e^{z} + z^2$$

$$y = C_1 + C_2 \log(x+1) + (x+1)^2 + 6(x+1) + \left(\log(x+1)\right)^2$$

$$y = C_1 + C_2 \log(x+1) + x^2 + 8x + \left[\log(x+1)\right]^2$$

$\Rightarrow$ Ans

Chapter 6th

Linear Diff eqn of second Order.

Working rule:-

Step I. Find a solution of the eqn $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ as .

Change the dependent variable y to v by putting $y = uv$ the reduced eqn is of the form $\frac{d^2v}{dx^2} + P' \frac{dv}{dx} = Q'$

Step III Put $\frac{dv}{dx} = p$ so that transformed eqn is the form $\frac{dp}{dx} + P'p = Q'$ where P', Q' are function of x only.

IV Solve the eqn $\frac{dp}{dx} + P'p = Q'$ and $p = \frac{dv}{dx}$ integrate it and find v.

V Find the solution $y = uv$, which is general solution of the given eqn.

Exp. Solve the diff eqn

$$x^2 \frac{d^2y}{dx^2} - (x^2 + 2x)\frac{dy}{dx} + (x+2)y = x^3 e^x$$

$$\frac{d^2y}{dx^2} - \left(1 + \frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = xe^x$$

which is the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

When $P = -\left(1 + \frac{2}{x}\right)$

$Q = \frac{1}{x} + \frac{2}{x^2}$ and $R = xe^x$

$P + Qx = -1 - \frac{2}{x} + \frac{x}{x} + \frac{2}{x} = 0$

$y = x$ is a part of C.F

$y = vx$

$\frac{dy}{dx} = \frac{dv}{dx} x + v$

$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} x + 2 \frac{dv}{dx}$

Putting these value in (1); we get

$\frac{d^2v}{dx^2} + 2 \frac{dv}{dx} - \left(1 + \frac{2}{x}\right)\left(\frac{dv}{dx} x + v\right)$
$+ \left(\frac{1}{x} + \frac{2}{x^2}\right) v \cdot x = x e^x$

$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x$

$\frac{dP}{dx} - P = e^x$ where $P = \frac{dv}{dx}$

which is a linear eqn of type $-$ (2)

$\frac{dP}{dx} + P'P = Q_1$

$I.F = e^{\int P' dx} = e^{\int - dx} = e^{-x}$

Thus the solution of 2 is

$P e^{-x} = \int e^x e^{-x} dx + c_1$

$P e^{-x} = x + c_1$

$$P = x e^x + C_1 e^x$$

$$\frac{dv}{dx} = x e^x + C_1 e^x$$

$$v = x e^x - e^x + C_1 e^x + C_2$$

Hence the complete solution is

$$y = vx$$

$$y = x^2 e^x - x e^x + C_1 x e^x + C_2 x$$

* Solve a linear diff eqn of second order by Removing the First derivative.

Working Rule:-

1. Find $P' = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ ...

2. Find $e^{-\frac{1}{2}\int P\,dx}$ and call this as u.

3. After Putting $y = uv$, the given eqn transforms to the form $\frac{d^2v}{dx^2} + P'v = Q'$

where $Q' = \frac{R}{u}$

4. Solve the eqn $\frac{d^2v}{dx^2} + P'v = Q'$

5. The general solution is $y = uv$ and it contains two arbitrary constant

1 Solve $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0$ by removing the first derivative.

The given eqn is :

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0 \quad -(1)$$

Comparing (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$P = -2\tan x \quad Q = 5 \quad R = 0$$

Putting $y = uv$ in the given eqn, we get

$$\frac{d^2v}{dx^2} + P'v = Q_1 \quad -- \quad (2)$$

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\tan x \, dx}$$

$$e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

$$P' = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$5 - \frac{1}{2}\frac{d}{dx}(-2\tan x) - \frac{1}{4}(4\tan^2 x)$$

$$= 5 + \sec^2 x - \tan^2 x$$

$$= 5 + (1 + \tan^2 x) - \tan^2 x = 6$$

$$Q_1 = \frac{R}{u} = 0$$

Eq (2) becomes $\frac{d^2v}{dx^2} + 6v = 0$

Auxiliary eqn is

$$D^2+6=0$$
$$D^2=-6$$
$$D=\pm\sqrt{6}\,i$$
$$v = c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x$$

the solution of the eqn is $y = uv$

$$y = \sec x\left[c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x\right.$$

* To solve a linear Diff. eqn of second order by the Method of variation of Parameters.

Working rule :-

Step 1. Find two independent solutions of the eqn $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ and call them y_1 and y_2

Step 2. Put $C.F = Ay_1 + By_2$ where A and B are arbitrary constants.

Step 3. Find $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} =$ $\quad u = -\int \frac{y_2 R}{W}dx + c_1$

$$v = \int \frac{y_1 R}{W}dx + c_2$$

Step 4. Replace arbitrary constants A and B in $C.F = Ay_1 + By_2$ by functions u and v so that the complete solution is $y = uy_1 + vy_2$

Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$

Sol. The given eqn $\frac{d^2y}{dx^2} + 4y = \tan 2x$ — ①

$(D^2+4)y = \tan 2x$

where $D = \frac{d}{dx}$

∴ Auxiliary eqn is $D^2+4=0$

$D^2 = -4 \Rightarrow D = \pm 2i$

$C.F = A\cos 2x + B\sin 2x$

Let the complete solution of ① be

$y = u\cos 2x + v\sin 2x$

where u, v are unknown function of x

$y = uy_1 + vy_2$ where $y_1 = \cos 2x$
$y_2 = \sin 2x$

Let $|W| = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$|W| = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x$

$2(\cos^2 2x + \sin^2 2x) = 2$

$u = -\int \frac{y_2 R}{|W|} dx + C_1$ $R = \tan 2x$

$= -\int \frac{(\sin 2x)\tan 2x}{2} dx + C_1 = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$

$$-\frac{1}{2}\int\left(\frac{1-\cos^2 2x}{\cos 2x}\right)dx + c_1$$

$$= -\frac{1}{2}\int(\sec 2x - \cos 2x)\,dx + c_1$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + c_1$$

$$v = \int \frac{y_1 R}{W}\,dx + c_2 = \int \frac{\cos 2x \cdot \tan 2x}{2}\,dx + c_2$$

$$= -\frac{1}{2}\int \sin 2x\,dx + c_2 = -\frac{\cos 2x}{4} + c_2$$

$$y = u\cos 2x + v\sin 2x$$

$$= -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right]\cos 2x$$

$$-\frac{\cos 2x}{4}\sin 2x + c_1\cos 2x + c_2\sin 2x$$

$$y = -\frac{\cos 2x}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x + \sin 2x\right] + c_1\cos 2x + c_2\sin 2x$$

Hence the complete solution of 1.

$$y = c_1\cos 2x + c_2\sin 2x - \frac{\cos 2x}{4}\left[\log(\sec 2x + \tan 2x)\right]$$

Chapter 7th Ordinary Simultaneous Diff. eqn

(1) Solve the simultaneous eqn

$$\frac{dy}{dt} + 5x + y = e^{t}$$

$$\frac{dy}{dt} - x + 3y = e^{2t}$$

Sol. the given eqn

$$\frac{dx}{dt} + 5x + y = e^{t} \quad - (1)$$

$$\frac{dy}{dt} - x + 3y = e^{2t} \quad - (2)$$

putting $\frac{d}{dt} = D$ in eqn (1) and (2) we have

$$(D+5)x + y = e^{t} \quad - (3)$$

$$-x + (D+3)y = e^{2t} \quad - (4)$$

In order to eliminate y, multiplying (3) by $(D+3)$ and subtracting (4) from it, we get

$$(D+5)(D+3)x + x = (D+3)e^{t} - e^{2t}$$

$$(D^2+8D+16)x = D.e^{t} + 3e^{t} - e^{2t}$$

$$(D^2+8D+16)x = e^{t} + 3e^{t} - e^{2t}$$

$$(D^2+8D+16)x = 4e^{t} - e^{2t}$$

$$D^2+8D+16 = 0$$

$$(D+4)^2 = 0 \;, \quad D = -4, -4$$

$C.F = (c_1 + c_2 t)\, e^{-4t}$

$P.I = \frac{1}{(D+4)^2}(4e^t - e^{2t})$

$= 4 \cdot \frac{1}{(D+4)} e^t - \frac{1}{(D+4)^2} e^{2t}$

$= 4 \cdot \frac{e^t}{25} - \frac{e^{2t}}{36}$

$x = (c_1 + c_2 t)\, e^{-4t} + \frac{4e^t}{25} - \frac{e^{2t}}{36}$ — (5)

Diff. (5) w.r.t t we have

$\frac{dx}{dt} = -4(c_1 + c_2 t)\, e^{-4t} + c_2 e^{-4t} + \frac{4e^t}{25} - \frac{e^{2t}}{18}$

From (1) we have

$y = e^t - \frac{dx}{dt} - 5x$

$= e^t + 4(c_1 + c_2 t)\, e^{-4t} - c_2 e^{-4t} - \frac{4}{25} e^t$

$+ \frac{e^{2t}}{18} - 5(c_1 + c_2 t)\, e^{-4t} - \frac{4}{5} e^t + \frac{5}{36} e^{2t}$

$y = -(c_1 + 2c_2 t)\, e^{-4t} + \frac{1}{25} e^t + \frac{7}{36} e^{2t}$

Hence the required solution is

$x = (c_1 + c_2 t)\, e^{-4t} + \frac{4}{25} e^t - \frac{e^{2t}}{36};$

$y = -(c_1 + 2c_2 t)\, e^{-4t} + \frac{1}{25} e^t + \frac{7}{36} e^{2t}$ Ans