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1 De Moivre's Theorem and its Applications

1.1. De Moivre's Theorem

Statement : (i) If n be any integer, positive or negative, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

and (ii) If n be a fraction, positive or negative, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

[K.U.2009, 08, 06]

Proof. Case-I. *Let n be a positive integer :*

By actual multiplication, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$$
$$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

Similarly, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$$= [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)][\cos\theta_3 + i\sin\theta_3]$$
$$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$

Proceeding in this way to n factors, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ \ldots\ldots\ (\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1+\theta_2+\ldots\ldots+\theta_n) + i(\sin\theta_1+\theta_2+\ldots\ldots+\theta_n)$$

Now, putting $\theta_1 = \theta_2 = \ldots\ldots = \theta_n = \theta$, on both sides, we have

$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\ \ldots\ldots \text{ to } n \text{ factors}$$
$$= \cos(\theta+\theta+\ldots\ldots \text{ to } n \text{ terms}) + i\sin(\theta+\theta+\ldots\ldots \text{ to } n \text{ terms})$$

i.e.,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Case-II. *Let n be a negative integer :*

Suppose $n = -m$, where m is a positive integer.

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$$ [By case I]

$$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$$

$$= \cos(-m\theta) + i\sin(-m\theta)$$

$$= \cos n\theta + i\sin n\theta$$ [$\because n = -m$]

which proves the theorem for a negative integer.

Case-III. *Let n be a fraction, positive or negative :*

Suppose $n = \frac{p}{q}$ (where $q \neq 0$, is a positive integer and p is any integer, positive or negative)

It follows from case I, that

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\left(q \cdot \frac{\theta}{q}\right) + i\sin\left(q \cdot \frac{\theta}{q}\right) = \cos\theta + i\sin\theta$$

Taking qth root of both sides, we get

$$\cos\frac{\theta}{q} + i\sin\frac{\theta}{q} \text{ is one of the values of } (\cos\theta + i\sin\theta)^{1/q}$$

Raising each of the quantities to the pth power,

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p \text{ is one of the values of } (\cos\theta + i\sin\theta)^{p/q}$$

or
$$\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right) \text{ is one of the values of } (\cos\theta + i\sin\theta)^{p/q}$$

Putting $\frac{p}{q} = n$, we get $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Note

1. **Total number of values in the case when n is a fraction [in lowest terms], positive or negative is equal to the denominator of the fraction n.**
2. **Even if n be irrational $(\cos\theta + i\sin\theta)^n$ has an infinite number of values, one of the infinite values being $\cos n\theta + i\sin n\theta$. The proof of this case is beyond the scope of this book.**

Cor. 1. $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$

$$= [\cos(-n\theta) + i\sin(-n\theta)] = \cos n\theta - i\sin n\theta.$$

Cor. 2. $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta.$

Cor. 3. $\dfrac{1}{\cos\theta \pm i\sin\theta} = (\cos\theta \pm i\sin\theta)^{-1} = (\cos\theta \mp i\sin\theta)$

Cor. 4. $(\cos\theta + i\sin\theta)^{mn} = (\cos m\theta + i\sin m\theta)^n = (\cos n\theta + i\sin n\theta)^m$.

Caution : It is to be noted that

(i) $(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta$, but

$$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right).$$

(ii) $(\cos\alpha + i\sin\beta)^m \neq \cos m\alpha + i\sin m\beta$

Angle must be the same with cosine and sine.

Remark : We generally denote $(\cos\alpha + i\sin\alpha)$ by cis α. It can be easily proved that

(i) $\text{cis}\,\alpha\,\text{cis}\,\beta = \text{cis}\,(\alpha+\beta)$ *(ii)* $\dfrac{\text{cis}\,\alpha}{\text{cis}\,\beta} = \text{cis}\,(\alpha - \beta)$

SOLVED EXAMPLES

Example 1.

Simplify :

(i) $\dfrac{(\cos 2\theta + i\sin 2\theta)^3(\cos 4\theta - i\sin 4\theta)^{-2}}{(\cos 5\theta + i\sin 5\theta)^{-4}(\cos 8\theta - i\sin 8\theta)^2}$

(ii) $\left(\dfrac{1 + \cos\theta + i\sin\theta}{1 + \cos\theta - i\sin\theta}\right)^n$ (iii) $\left(\dfrac{\cos\alpha + i\sin\alpha}{\sin\beta + i\cos\beta}\right)^3$.

Solution. *(i)* $\dfrac{(\cos 2\theta + i\sin 2\theta)^3(\cos 4\theta - i\sin 4\theta)^{-2}}{(\cos 5\theta + i\sin 5\theta)^{-4}(\cos 8\theta - i\sin 8\theta)^2}$

$$= \frac{(\cos 2\theta + i\sin 2\theta)^3[\cos(-4\theta) + i\sin(-4\theta)]^{-2}}{(\cos 5\theta + i\sin 5\theta)^{-4}[\cos(-8\theta) + i\sin(-8\theta)]^2}$$

$$= \frac{(\cos 6\theta + i\sin 6\theta)(\cos 8\theta + i\sin 8\theta)}{[\cos(-20\theta) + i\sin(-20\theta)][\cos(-16\theta) + i\sin(-16\theta)]}$$

$$= \frac{\cos 14\theta + i\sin 14\theta}{\cos(-36\theta) + i\sin(-36\theta)}$$ *[Applying De Moivre'stheorem]*

$\ldots\beta) + i\sin(\alpha+\beta)]$

$= \cos [14 - (-36)\,\theta] + i \sin [14 - (-36)\,\theta]$

$\left[\because \dfrac{\cos \alpha + i \sin \alpha}{\cos \beta + i \sin \beta} = \cos(\alpha - \beta) + i \sin(\alpha -\right.$

$\ldots = \cos 50\theta + i \sin 50\theta$

Example 7.

If α, β be the roots of $x^2 - 2x + 4 = 0$, *prove that* $\alpha^n + \beta^n = 2^{n+1} \cos \frac{n\pi}{3}$.

[M.D.U. 2008]

Solution. The given equation is $x^2 - 2x + 4 = 0$

$$x = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm i\,2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

Since α, β are the roots of the given equation

Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 - i\sqrt{3}$

Put $1 = r\cos\theta$ and $\sqrt{3} = r\sin\theta$

Squaring and adding, $r^2 = 4 \Rightarrow r = 2$ [Taking + ve sign]

$$\cos\theta = \frac{1}{2} \text{ and } \sin\theta = \frac{\sqrt{3}}{2}, \text{ which gives } \theta = 60^\circ = \frac{\pi}{3}$$

Now,

$$\alpha^n + \beta^n = (1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n$$

$$= (r\cos\theta + ir\sin\theta)^n + (r\cos\theta - ir\sin\theta)^n$$

$$= r^n [(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n]$$

$$= r^n [\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta]$$

$$= r^n\, 2\cos n\theta = 2^n \left[2\cos n \cdot \frac{\pi}{3}\right] = 2^{n+1} \cos\frac{n\pi}{3}.$$

Example 1.

Expand

(i) $\cos 8\theta$ *in descending powers of* $\cos\theta$. *(ii)* $\dfrac{\sin 8\theta}{\cos\theta}$ *in ascending powers of* $\sin\theta$.

Solution. $\cos 8\theta + i\sin 8\theta = (\cos\theta + i\sin\theta)^8$

$$= \cos^8\theta + {}^8C_1\cos^7\theta\, i\sin\theta + {}^8C_2\cos^6\theta\,.\,i^2\sin^2\theta$$

$$+ {}^8C_3\cos^5\theta\,.\,i^3\sin^3\theta + {}^8C_4\cos^4\theta\,.\,i^4\sin^4\theta + {}^8C_5\cos^3\theta\,.\,i^5\sin^5\theta$$

$$+ {}^8C_6\cos^2\theta\,.\,i^6\sin^6\theta + {}^8C_7\cos\theta\,.\,i^7\sin^7\theta + i^8\sin^8\theta$$

[Expanding R.H.S. by Binomial theorem]

$$= \cos^8\theta + i\,.\,8\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta - i\,.\,56\cos^5\theta\sin^3\theta$$

$$+ 70\cos^4\theta\sin^4\theta + i\,.\,56\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta - 8i\cos\theta\sin^7\theta +$$

Equating real and imaginary parts on both sides, we have

$$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$$

$$\sin 8\theta = 8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta + 56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta$$

(i) Putting $\sin^2\theta = 1 - \cos^2\theta$ in (1), we get

$$\cos 8\theta = \cos^8\theta - 28\cos^6\theta(1-\cos^2\theta) + 70\cos^4\theta(1-\cos^2\theta)^2$$

$$- 28\cos^2\theta(1-\cos^2\theta)^3 + (1-\cos^2\theta)^4$$

$$= \cos^8\theta - 28\cos^6\theta(1-\cos^2\theta) + 70\cos^4\theta(1-2\cos^2\theta+\cos^4\theta)$$

$$- 28\cos^2\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6$$

$$+ (1-4\cos^2\theta+6\cos^4\theta-4\cos^6\theta+\cos$$

$$= 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$$

(ii) Dividing (2) by $\cos\theta$, we get

$$\frac{\sin 8\theta}{\cos\theta} = 8\cos^6\sin\theta - 56\cos^4\theta\sin^3\theta + 56\cos^2\theta\sin^5\theta - 8\sin^7\theta$$

Putting $\cos^2\theta = 1 - \sin^2\theta$, we have

$$\frac{\sin 8\theta}{\cos\theta} = 8(1-\sin^2\theta)^3\sin\theta - 56(1-\sin^2\theta)^2\sin^3\theta + 56(1-\sin^2\theta)\sin^5\theta - 8\sin$$

$$= 8(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)\sin\theta - 56(1-2\sin^2\theta+\sin^4\theta)\sin^3\theta$$

$$+ 56(1-\sin^2\theta)\sin^5\theta - 8\sin^7\theta$$

$$= 8\sin\theta - 80\sin^3\theta + 192\sin^5\theta - 128\sin^7\theta\,.$$

Example 1.

Express the following in terms of cosines of multiples of θ.

(i) $\cos^6\theta$ (ii) $\cos^7\theta$

Solution. Let $x = \cos\theta + i\sin\theta$

$$\therefore \quad \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta} = \cos\theta - i\sin\theta$$

$$\therefore \quad x + \frac{1}{x} = 2\cos\theta \qquad ...(1)$$

Also $x^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$$\therefore \quad x^n + \frac{1}{x^n} = 2\cos n\theta \qquad ...(2)$$

(i)
$$\left(x + \frac{1}{x}\right)^6 = x^6 + {}^6C_1\, x^5 \cdot \frac{1}{x} + {}^6C_2\, x^4 \cdot \frac{1}{x^2} + {}^6C_3\, x^3 \cdot \frac{1}{x^3}$$

$$+ {}^6C_4\, x^2 \cdot \frac{1}{x^4} + {}^6C_5\, x \cdot \frac{1}{x^5} + {}^6C_6\, \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20$$

Using (1) and (2), we get $\quad (2\cos\theta)^6 = 2\cos 6\theta + 6\,(2\cos 4\theta) + 15\,(2\cos 2\theta) + 20$

Dividing by 2^6, we get
$$\cos^6\theta = \frac{2}{2^6}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$$

$$= \frac{1}{32}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$$

(ii) $$\left(x+\frac{1}{x}\right)^7 = x^7 + {}^7C_1\, x^6 \cdot \frac{1}{x} + {}^7C_2 \cdot x^5 \cdot \frac{1}{x^2} + {}^7C_3\, x^4 \cdot \frac{1}{x^3}$$

$$+ {}^7C_4\, x^3 \cdot \frac{1}{x^4} + {}^7C_5\, x^2 \cdot \frac{1}{x^5} + {}^7C_6\, x \cdot \frac{1}{x^6} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

From (1) and (2), we get

$$(2\cos\theta)^7 = 2\cos 7\theta + 7\,(2\cos 5\theta) + 21\,(2\cos 3\theta) + 35\,(2\cos\theta)$$

Dividing by 2^7, we get

$$\cos^7\theta = \frac{2}{2^7}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$$

$$= \frac{1}{64}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta].$$

Example 5.

If $\tan(\theta + i\phi) = \sin(x + iy)$, *prove that* $\coth y \,.\, \sinh 2\phi = \cot x \,.\, \sin 2\theta$.

[M.D.U. 2011, 07, 06]

Solution. Given $\quad \tan(\theta + i\phi) = \sin(x + iy)$ ...(1)

Changing i into $-i$, we have $\quad \tan(\theta - i\phi) = \sin(x - iy)$ ...(2)

Adding (1) and (2), we have

$$\tan(\theta + i\phi) + \tan(\theta - i\phi) = \sin(x + iy) + \sin(x - iy)$$

or
$$\frac{\sin(\theta + i\phi)}{\cos(\theta + i\phi)} + \frac{\sin(\theta - i\phi)}{\cos(\theta - i\phi)} = 2\sin x \cos iy$$

or
$$\frac{\sin(\theta + i\phi)\cos(\theta - i\phi) + \cos(\theta + i\phi)\sin(\theta - i\phi)}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2\sin x \cosh y$$

i.e.,
$$\frac{\sin(\theta + i\phi + \theta - i\phi)}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2\sin x \cosh y$$

i.e.,
$$\frac{\sin 2\theta}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2\sin x \cosh y \qquad ...(3)$$

Again, subtracting (2) from (1), we have

$$\tan(\theta + i\phi) - \tan(\theta - i\phi) = \sin(x + iy) - \sin(x - iy)$$

or
$$\frac{\sin(\theta + i\phi)}{\cos(\theta + i\phi)} - \frac{\sin(\theta - i\phi)}{\cos(\theta - i\phi)} = 2\cos x \sin iy$$

or
$$\frac{\sin(\theta + i\phi)\cos(\theta - i\phi) - \cos(\theta + i\phi)\sin(\theta - i\phi)}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2\cos x \sin iy$$

or
$$\frac{\sin(\theta + i\phi - \theta + i\phi)}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2i\cos x \sinh y$$

or
$$\frac{\sin(2i\phi)}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2i\cos x \sinh y$$

$$\frac{i\sinh 2\phi}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2i\cos x \sinh y$$

$\therefore$
$$\frac{\sinh 2\phi}{\cos(\theta + i\phi)\cos(\theta - i\phi)} = 2\cos x \sinh y$$

$\coth y \sinh 2\phi = \cos x \sin 2\theta$

Example 6.

If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$, *where the letters denote real quantities, prove that*

(i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$, *where n is an integer* (ii) $e^{2\phi} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$.

[K.U. 2007; M.D.U. 2005] [K.U. 2007]

Solution. (i) If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$ ...(1)

Then $\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$ ...(2)

Now,
$$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$
$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$$
$$= \frac{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$ [*Using (1) and (2)*]
$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0}$$
$$\tan 2\theta = \infty = \tan\frac{\pi}{2}$$
$$2\theta = n\pi + \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{n\pi}{2} + \frac{\pi}{4}.$$

(ii)
$$\tan(2i\phi) = \tan[(\theta + i\phi) - (\theta - i\phi)]$$
$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$
$$= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$ [*Using (1) and (2)*]
$$i\tanh 2\phi = \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = \frac{2i\sin\alpha}{2} = i\sin\alpha$$
$$\tanh 2\phi = \sin\alpha \quad \textit{i.e.,} \quad \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

Applying componendo and dividendo, we have
$$\frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{1 + \sin\alpha}{1 - \sin\alpha} = \frac{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

Example 2.

Prove that :

(*i*) $\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$ (*ii*) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right) = \dfrac{1}{2}\tan^{-1}x$

(*iii*) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2$

Solution. (*i*) Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\dfrac{x}{a}\right)$

Now, $$\frac{x}{\sqrt{a^2-x^2}} = \frac{a\sin\theta}{\sqrt{a^2-a^2\sin\theta}} = \frac{a\sin\theta}{a\cos\theta} = \tan\theta$$

$\therefore$ $$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

(*ii*) Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

Now, $$\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} = \frac{\sqrt{\sec^2\theta}-1}{\tan\theta}$$

$$= \frac{\sec\theta-1}{\tan\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$\therefore$ $$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x.$$

(*iii*) Let $x^2 = \cos 2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x^2$

Example 3.

Prove that $\log \tan\left(\frac{\pi}{4}+\frac{x}{2}i\right) = i \tan^{-1}(\sinh x)$

[K.U. 2011, 08, 04; M.D.U. 2011]

Solution. $\log \tan\left(\frac{\pi}{4}+\frac{x}{2}i\right) = \log\left(\dfrac{1+\tan\frac{x}{2}i}{1-\tan\frac{x}{2}i}\right)$

$$= \log\left(\dfrac{1+i\tanh\frac{x}{2}}{1-i\tanh\frac{x}{2}}\right) = \log\left(\dfrac{\cosh\frac{x}{2}+i\sinh\frac{x}{2}}{\cosh\frac{x}{2}-i\sinh\frac{x}{2}}\right)$$

$$= \log\left(\cosh\frac{x}{2}+i\sinh\frac{x}{2}\right) - \log\left(\cosh\frac{x}{2}-i\sinh\frac{x}{2}\right)$$

$$= \frac{1}{2}\log\left(\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}\right) + i\tan^{-1}\left(\tanh\frac{x}{2}\right)$$

$$-\frac{1}{2}\log\left(\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}\right) + i\tan^{-1}\left(\tan\right.$$

$$= 2i\tan^{-1}\left(\tanh\frac{x}{2}\right) = i\tan^{-1}\left(\dfrac{2\tanh\frac{x}{2}}{1-\tanh^2\frac{x}{2}}\right) = i\tan^{-1}(\sinh x).$$

Example 3.

Find the sum of the series :

sec α sec 2α + sec 2α sec 3α + sec 3α sec 4α + ... to n terms.

Solution. We have $T_1 = \sec\alpha \,.\, \sec 2\alpha = \dfrac{1}{\cos\alpha\cos 2\alpha}$

$$= \frac{\sin\alpha}{\cos\alpha\cos 2\alpha\sin\alpha} = \frac{\sin(2\alpha-\alpha)}{\cos\alpha\cos 2\alpha\sin\alpha}$$

$$= \frac{1}{\sin\alpha}\left[\frac{\sin 2\alpha\cos\alpha - \cos 2\alpha\sin\alpha}{\cos\alpha\cos 2\alpha}\right]$$

$$= \frac{1}{\sin\alpha}[\tan 2\alpha - \tan\alpha]$$

Similarly,

$$T_2 = \sec 2\alpha\sec 3\alpha = \frac{1}{\sin\alpha}[\tan 3\alpha - \tan 2\alpha]$$

$$T_3 = \sec 3\alpha\sec 4\alpha = \frac{1}{\sin\alpha}[\tan 4\alpha - \tan 3\alpha]$$

....

....

$$T_n = \sec n\alpha \,.\, \sec(n+1)\alpha = \frac{1}{\sin\alpha}[\tan(n+1)\alpha - \tan n\alpha]$$

Adding vertically, we get

$$S = \frac{1}{\sin\alpha}[\tan(n+1)\alpha - \tan\alpha]$$

$$= \frac{1}{\sin\alpha}\left[\frac{\sin(n+1)\alpha}{\cos(n+1)\alpha} - \frac{\sin\alpha}{\cos\alpha}\right]$$

$$= \frac{1}{\sin\alpha}\left[\frac{\sin(n+1)\alpha\cos\alpha - \cos(n+1)\alpha\sin\alpha}{\cos(n+1)\alpha\cos\alpha}\right]$$

$$= \frac{\sin[(n+1)\alpha - \alpha]}{\sin\alpha\cos\alpha\cos(n+1)\alpha}.$$

Example 2.

Find the sum of the series : $\sin\alpha + \frac{1}{2}\sin 2\alpha + \left(\frac{1}{2}\right)^2 \sin 3\alpha + \ldots\text{to } \infty.$

[M.D.U. 2011, 06]

Solution. Let $C = \cos\alpha + \frac{1}{2}\cos 2\alpha + \left(\frac{1}{2}\right)^2 \cos 3\alpha + \ldots\ldots \text{ to } \infty$

and
$$S = \sin\alpha + \frac{1}{2}\sin 2\alpha + \left(\frac{1}{2}\right)^2 \sin 3\alpha + \ldots\ldots \text{ to } \infty$$

$$\therefore \quad C + iS = (\cos\alpha + i\sin\alpha) + \frac{1}{2}(\cos 2\alpha + i\sin 2\alpha) + \left(\frac{1}{2}\right)^2 (\cos 3\alpha + i\sin 3\alpha) + \ldots\text{ to } \infty$$

$$= e^{i\alpha} + \frac{1}{2} e^{2i\alpha} + \left(\frac{1}{2}\right)^2 e^{3i\alpha} + \ldots \text{ to } \infty$$

$$= \frac{e^{i\alpha}}{1 - \frac{1}{2} e^{i\alpha}} \qquad \left[\because S = \frac{a}{1-r}\right]$$

$$= \frac{e^{i\alpha}}{1 - \frac{1}{2} e^{i\alpha}} \times \frac{1 - \frac{1}{2} e^{-i\alpha}}{1 - \frac{1}{2} e^{-i\alpha}} = \frac{e^{i\alpha}\left[1 - \frac{1}{2} e^{-i\alpha}\right]}{1 - \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) + \frac{1}{4}}$$

$$= \frac{e^{i\alpha} - \frac{1}{2}}{1 - \cos\alpha + \frac{1}{4}} = \frac{\cos\alpha + i\sin\alpha - \frac{1}{2}}{1 - \cos\alpha + \frac{1}{4}}$$

Equating the imaginary parts, we have

$$S = \frac{\sin\alpha}{1 - \cos\alpha + \frac{1}{4}} = \frac{4\sin\alpha}{5 - 4\cos\alpha}.$$

Example 1.

Sum the series : $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots\ldots \infty$.

[M.D.U. 2006]

Solution. Let $C = \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots\ldots \infty$

and $$S = \sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \ldots\ldots \infty$$

$$\therefore \quad C + iS = (\cos\theta + i\sin\theta) - \frac{1}{2}(\cos 2\theta + i\sin 2\theta) + \frac{1}{3}(\cos 3\theta + i\sin 3\theta) - \ldots\ldots \infty$$

$$= e^{i\theta} - \frac{1}{2}e^{2i\theta} + \frac{1}{3}e^{3i\theta} - \ldots\ldots \infty$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\ldots \infty, \text{ where } z = e^{i\theta}$$

$$= \log(1+z) = \log(1+e^{i\theta}) = \log(1+\cos\theta + i\sin\theta)$$

$$= \log \sqrt{(1+\cos\theta)^2 + \sin^2\theta} + i \tan^{-1} \frac{\sin\theta}{1+\cos\theta}$$

$$\left[\because \log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}\right]$$

$$= \log \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} + i\tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)$$

$$= \log\sqrt{2(1+\cos\theta)} + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \log\sqrt{4\cos^2\frac{\theta}{2}} + i \cdot \frac{\theta}{2}$$

$$= \log\left(2\cos\frac{\theta}{2}\right) + i \cdot \frac{\theta}{2}$$

Equating real parts, we get

$$C = \log\left(2\cos\frac{\theta}{2}\right).$$

Example 2