# MAA OMWATI DEGREE COLLEGE

# HASSANPUR

# NOTES

## Session 2020-21

## SUBJECT: - Mathematics (Vacter Calculus)

## CLASS:- B.Sc 2nd Sem

# {Multiple products of vectors}

Ex → 1.1

* **Determinant form of the scalar triple Product →**

let $a = a_1 i + a_2 j + a_3 k$

$b = b_1 i + b_2 j + b_3 k$

$c = c_1 i + c_2 j + c_3 k$

Now $b \times c = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$(b \times c) = (b_2 c_3 - b_3 c_2) i + (b_3 c_1 - b_1 c_3) j + (b_1 - c_2 - b_2 c_1) k$

$a \cdot (b \times c) = a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + c_3 (b_1 c_2 - b_2 c_1)$

$[a\, b\, c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

* The dot and cross in a scalar triple product can be interchanged

ie $a \cdot (b \times c) = (a \times b) \cdot c$

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let $a = a_1 i + a_2 j + a_3 k$
$b = b_1 i + b_2 j + b_3 k$
$c = c_1 i + c_2 j + c_3 k$

Now

$$b \times c = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$a.(b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$(a \times b) \cdot c = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

from (1) and (2) we have

$$a.(b \times c) = (a \times b) c$$

Volume of A Tetrahedron formula

$$\frac{1}{6}(\vec{AB} \ \vec{AC} \ \vec{AD})$$

$$\Rightarrow \frac{1}{6}(a \ b \ c)$$

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Q. Find the volume of parallelopiped whose edges are represented by

$a = 2\hat{i} - 3\hat{j} + 4k$
$b = \hat{i} + 2\hat{j} - k$
$c = 3\hat{i} - \hat{j} + 2k$

Sol. volume of parallelopiped $[a\ b\ c]$

$$\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$2(4-1) + 3(2+3) + 4(-1-6) = 6+15-28$
$-7 = 7$ cubic unit

. if $a, b, c$ be three non coplanar vector show that

$[a \times b \quad b \times c \quad c \times a] = [a\,b\,c]^2$

let $a = a_1\hat{i} + a_2\hat{j} + a_3k$
$b = b_1\hat{i} + b_2\hat{j} + b_3k$
$c = c_1\hat{i} + c_2\hat{j} + c_3k$

$a \times b = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j}$
$+ (a_1b_2 - a_2b_1)k$

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$$~~b \times c = (b_2c_3 - b_3c_2)i + (b_3c_1 - b_2c_3)j + (b_1c_2 - b_2c_1)~~$$

$$~~c \times a = (c_2a_3 - c_3a_2)i + (c_3a_1 - c_2a_3)j + (c_1a_2 - c_2a_1)~~$$

~~Now~~

$$~~[a\times b \;\; b\times c \;\; c\times a] = \begin{bmatrix} a_2b_3 - a_3b_2 \end{bmatrix}~~$$

$$[a\times b \;\; b\times c \;\; c\times a] = \begin{bmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & c_3a_1 - c_1a_3 & c_1a_2 - c_2a_1 \end{bmatrix}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Co-factors of the corresponding small letter in $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$[a\times b \;\; b\times c \;\; c\times a] = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = [abc]^2$$

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Q. Prove that the four points with position vectors

$4i+5j+k$, $-j-k$, $3i+9j+4k$ and $4(-i+j+k)$ are coplanar.

Sol. $\vec{AB} = (-j-k)-(4i+5j+k) = -4i-6j-2k$
$\vec{AC} = (3i+9j+4k)-(4i+5j+k) = -i+4j+3k$
$\vec{AD} = 4(-i+j+k)-(4i+5j+k) = -8i-j+3k$

$$[\vec{AB}\ \vec{AC}\ \vec{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$= -4(12+3)+6(-3+24)-2(1+32)$
$= -60+126-66 = 0$

Thus $\vec{AB}, \vec{AC}, AD$ are coplanar. Hence point A, B, C and D are coplanar.

Q. $A = i+aj+a^2k$
$B = i+bj+b^2k$
$C = i+cj+c^2k$ are non-coplanar

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$$

then show that $abc = -1$

Sol: Hence vector A, B, C are given to be non coplanar

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0 \quad - \text{(I)}$$

Now $$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (1+abc) = 0$$

$1 + abc = 0$

$\boxed{abc = -1}$ Ans

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## Ex → 1.2

Q. Show that $a\times(b\times c)+b\times(c\times a)+c\times(a\times b)=$

Sol We have

$$a\times(b\times c)=(a\cdot c)b-(a\cdot b)c \quad -(1)$$
$$b\times(c\times a)=(b\cdot a)c-(b\cdot c)a \quad -(2)$$
$$c\times(a\times b)=(c\cdot b)a-(c\cdot a)b \quad -(3)$$

Adding (1) (2) and (3)

$$a\times(b\times c)+b\times(c\times a)+c\times(a\times b)=0$$

Q. Show that $a\times(b\times c)=(a\times b)\times c$ iff $a$ and $c$ are collinear

Sol Condition is necessary

let $a\times(b\times c)=(a\times b)\times c$

$$(a\cdot c)b-(a\cdot b)c=-c\times(a\times b)$$

$$(a\cdot c)b-(a\cdot b)c=-[(c\cdot b)a-(c\cdot a)b]$$

$$(a\cdot c)b-(a\cdot b)c=-(c\cdot b)a+(c\cdot a)b$$

$$(a\cdot c)b-(a\cdot b)c+(b\cdot c)a-(a\cdot c)b=0$$
$$(b\cdot c)a-(b\cdot a)c=0$$
$$-b\times(c\times a)=0$$
$$(c\times a)\times b=0$$

$b=0$ or $c\times a=0$ But $b\neq 0$

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$$c \times a = 0$$

a and c are collinear vectors

Given a and c are collinear

~~$c \times$~~

$$c = \lambda a$$

$$(a \times b) \times c = c \times (a \times b) = -\lambda a \times (a \times b)$$
$$= \lambda [(a \times a) b - (a \cdot b) a]$$

$$a \times (b \times c) = a(b \times \lambda a) = (a \cdot \lambda a) b - (a \cdot b) \lambda a$$
$$\lambda [(a \cdot a) b - (a \cdot b) a]$$

form (1) and (2) we have $(a \times b) \times c = a \times (b \times$

Q. If a, b, c be three unit vector such that $a \times (b \times c) = \frac{1}{2} b$ find the angles which a makes with b and c, b and c being non parallel.

Sol Given $a \times (b \times c) = \frac{1}{2} b$

$$(a \cdot c) - (a \cdot b) c = \frac{1}{2} b$$

$$\left[ (a \cdot c) - \frac{1}{2} \right] b - (a \cdot b) c = 0$$

As b and c are non parallel vectors.

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$a \cdot c - \frac{1}{2} = 0$ ie. $a \cdot c = \frac{1}{2}$

$|a||c|\cos\theta = 1/2$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$

and also $a \cdot b = 0$

$|a||b| \cos\phi = 0$

$\cos\phi = 0$

$\phi = \pi/2$

Q. if $a, b, c$ are three vector such that $a \times b = c$ and $b \times c = a$ show that three vectors $a, b, c$ are orthogonal in pair and $|b| = 1$ $|c| = |a|$

Sol. (1) Given $a \times b = c$

$a \cdot (a \times b) = a \cdot c$

$[a\, a\, b] = a \cdot c \Rightarrow a \cdot c = 0$

similarly

$b \cdot (a \times b) = b \cdot c$

$\Rightarrow [b\, a\, b] = b \cdot c$

$\Rightarrow b \cdot c = 0$

Also given $b \times c = a$

$b \cdot (b \times c) = b \cdot a$

$\Rightarrow [b\, b\, c] = b \cdot a$

$\Rightarrow b \cdot a = 0$

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Thus we have

$a \cdot c = 0 \quad b \cdot c = 0 \quad b \cdot a = 0$

Hence $a, b, c$ are orthogonal in pairs

II $a \times b = c$

$c \cdot (a \times b) = c \cdot c = c^2$

$b \times c = a$

$a\,(b \times c) = a \cdot a = |a|^2$

from (1) and (2)

$|a|^2 = |c|^2 = |a| = |c|$

III Hence $a \times b = c$

$b \times (a \times b) = b \times c$

$(b \cdot b)\,a - (b \cdot a)\,b = b \times c$

$|b|^2 a - 0 = b \times c$

$|b|^2 a = a$

$|b| = 1.$

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# Product of four vectors.

* if a, b, c and d be any four vector that $(a\times b)\times(c\times d)$ is know as vector product of four vector

(1) $(a\times b)\times(c\times d) = [a\,b\,d]c - [a\,b\,c]d$

let us suppose that
$a\times b = p$

$$(a\times b)\times(c\times d) = p\times(c\times d$$
$$(p\cdot d)c - (p\cdot c)d$$
$$\Rightarrow [(a\times b)\cdot d]c - (a\times b)\cdot c]d$$

$$\Rightarrow [a\,b\,d]c - [a\,b\,c]d$$

II let us suppose that $c\times d = p$

$$(a\times b)\times(c\times d) = (a\times b)\times p$$
$$= (a\cdot p)b - (b\cdot p)a$$

$$[a\cdot(c\times d)]b - [b\cdot(c\times d)]a$$

$$[a\,c\,d]b - [b\,c\,d]a$$ Ans

Ex = 1.3



Q. Show that $(b \times c) \cdot (a \times d) + (c \times a) \cdot (b \times d) + \dots$
$\cdot (c \times d) = 0$

Sol $(b \times c) \cdot (a \times d) = (b \cdot a)(c \cdot d) - (b \cdot d)(c \cdot a)$

Also
$(c \times a) \cdot (b \times d) = (c \cdot b)(a \cdot d) - (c \cdot d)(a \cdot b)$

$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d) \cdot (b \cdot c)$

Adding (1), (2) and (3)
we get

$(b \times c) \cdot (a \times d) + (c \times a) \cdot (b \times d) + (a \times b \dots$
$= 0$

Q. Prove that $(b \times c) \times (c \times a) = [a\ b\ c]\ c$
and hence deduce that
$[b \times c \quad c \times a \quad a \times b] = [a\ b\ c]^2$

Sol let $b \times c = p$
$(b \times c) \times (c \times a) = p \times (c \times a)$
$(p \cdot a) c - (p \cdot c) a$

$[(b \times c) \cdot a] c - [(b \times c) \cdot c] a$

$[b\ c\ a] c - [b\ c\ c] a$
$[a\ b\ c] c$

Deduction

$[b\times c \quad c\times a \quad a\times b] = [(b\times c)\times(c\times a)].(a\times b)$

$= [abc]c\cdot(a\times b)$

$= [abc)[cab]$

$= [abc][abc]$

$= [abc]^2$

Q. Show that $a\times(b\times c)$, $b\times(c\times a)$, $c\times(a\times b)$ are coplanar

Sol let $p = a\times(b\times c)$ $q = b\times(c\times a)$
and $r = c\times(a\times b)$

$q\times r = [b\times(c\times a)]\times[c\times(a\times b)]$

$\Rightarrow [(b\cdot a)c-(b\cdot c)a]\times[(c\cdot b)a-(c\cdot a)b]$

let $a\cdot b = x$
$b\cdot c = y$
$c\cdot a = z$

$q\times r = [(xc-ya)\times(ya-zb)]$

$p\cdot(q\cdot r) = [a\times(b\times c)].[xy(c\times a)-xz(c\times b)$
$+yz(a\times b)]$

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$[(a\cdot c)b - (a\cdot b)c] \cdot [xy(c\times a) - xz(c\times b) + yz(a\times b)]$

$(zb - xc) \cdot [xy(c\cdot a) - xz(c\times b) + yz(a\times b)$

$xyz(bca) - xz^2[bcb] + yz^2[bab] - x^2y[cc$

$+ x^2z[ccb] - xyz[cab]$

$\Rightarrow xyz[abc] - xyz[abc]$

$\Rightarrow 0$

Hence $p, q, r$ are coplanar

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* To show that $a.a' = b.b' = c.c' = 1$

Sol $a.a' = a.\frac{b \times c}{[abc]} = \frac{1}{[abc]}[abc] =$

Similarly we can show that

$b.b' = c.c' = 1$

* To show that

$a.b' = a.c' = b.a' = b.c' = c.a' = c.b' = 0$

Sol $a.b' = a.\frac{c \times a}{[abc]} = \frac{a.(c \times a)}{[abc]}$

$\frac{[aca]}{[abc]} = \frac{0}{[abc]} = 0$

Similarly we can show that other scalar products are zero

* To show that $a'(b' \times c')$

$= \frac{1}{a.(b \times c)}$

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Q. The orthonormal vector triad $i, j, k$ is self reciprocal

Sol $i' = j \times k / [i\ j\ k] = \frac{i}{1} = i$

Similarly we can show that $j' = j$ and $k' = k$

Ex - 1.4

Q. if $r$ is any vector show that

(1) $r = (r \cdot a')a + (r \cdot b')b + (r \cdot c')c$

(11) $r = (r \cdot a)a' + (r \cdot b)b' + (r \cdot c)c'$

Sol (I)

let $r = xa + yb + zc$

Multiplying both sides of (1)

$r \cdot (b \times c) = xa \cdot (b \times c) + yb \cdot (b \times c) + zc \cdot (b \times c)$

$\Rightarrow x[a\ b\ c]$

$$x = \frac{[r\ b\ c]}{[a\ b\ c]}$$

Similarly we have $y \frac{[r\ c\ a]}{[a\ b\ c]}$

$z = [r\,a\,b] / [a\,b\,c]$

II Similarly we can prove that

$r = (r \cdot a)a' + (r \cdot b)b' + (r \cdot c)c'$

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Ex = 4.1

1. Let $u, v, w$ be orthogonal co-ordinate. Prove that

(i) $|\Delta u| = \frac{1}{h_1}$, $|\nabla v| = \frac{1}{h^2}$

$|\nabla w| = \frac{1}{h^3}$

II $e_1 = E_1$ $e_2 = E_2$ $e_3 = E_3$

I We know that

$\nabla u = e_1/h_1$, $\nabla v = \frac{e_2}{h^2}$

$\nabla w = e_3/h_3$

$|\nabla u| = \frac{1}{h_1}$, $|\nabla v| = \frac{1}{h_2}$

$|\nabla w| = \frac{1}{h_3}$

II. We know that

$E_1 = \nabla u / |\nabla u|$, $E_2 = \frac{\nabla v}{|\nabla v|}$

$E_3 = \frac{\nabla w}{|\nabla w|}$

$E_1 = \frac{\nabla u}{|\nabla u|} = h_1 \nabla u = e_1$

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$E_2 = e_2 \qquad E_3 = e_3$

Q. Prove that $e_1 = h_2 h_3 \nabla v \times \nabla w$ with similar results for $e_2$ and $e_3$ where $u, v, w$ are otheregonal co-ordinates

Sol We know that

$\nabla u = e_1/h_1 \qquad \nabla v = e_2/h_2 \qquad \nabla w = \frac{e_3}{h_3}$

$$\nabla v \times \nabla w = \frac{e_2 \times e_3}{h_2 h_3} = \frac{e_1}{h_2 h_3}$$

$$e_1 = h_2 h_3 \nabla v \times \nabla w$$
$$e_2 = h_3 h_1 \nabla w \times \nabla u$$
$$e_3 = h_1 h_2 \nabla u \times \nabla v$$

Q. Determine the transformation from cylindrical to rectangluar co-ordinates.

Sol

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

Squaring (1) and (2) and adding

We get

$$x^2+y^2 = \rho^2$$

$$\rho = \sqrt{x^2+y^2}$$

Dividing (2) by (1) we get

$$y/x = \tan\phi \qquad \phi = \tan^{-1} y/x$$

Hence the required transformation

$$\rho = \sqrt{x^2+y^2}$$
$$\phi = \tan^{-1} y/x$$
$$z = z$$

Q. Transform the function $f = \rho e_\rho + \rho e_\phi$ from cylindrical to cartesian co-ordinates.

Sol We have $f = \rho e_\rho + \rho e_\phi$

Relation b/w cylindrical and cartesian co-ordinates are

$$x = \rho\cos\phi \qquad y = \rho\sin\phi$$
$$z = z$$
$$e_\rho = \cos\phi\, i + \sin\phi\, j$$
$$e_\phi = -\sin\phi\, i + \cos\phi\, j$$

From $f = \rho(\cos\phi\, \hat{i} + \sin\phi\, \hat{j}) + \rho$
$(-\sin\phi\, \hat{i} + \cos\phi\, \hat{j})$

$x\hat{i} + y\hat{j} - y\hat{i} + x\hat{j}$
$(x-y)\hat{i} + (x+y)\hat{j}$

Q. Find the volume element $dv$ in a) cylindrical (b) spherical polar co-ordinates.

Sol We know that $dv = h_1 h_2 h_3\, du\, dv\, dw$

a) In cylindrical co-ordinates
$u = \rho \quad v = \phi \quad w = z \quad h_1 = 1$
$h_2 = \rho \quad h_3 = 1$
$dv = 1 \times \rho \times 1 \; d\rho\, d\phi\, dz$
$\rho\, d\rho\, d\phi\, dz$

(b) In spherical polar co-ordinates

$u = r \quad v = \theta \quad w = \phi \quad h_1 = 1$
$h_2 = r \quad h_3 = r\sin\theta$
$dv = 1 \times r \times r\sin\theta\; dr\, d\theta\, d\phi$
$r^2 \sin\theta\, dr\, d\theta\, d\phi$

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Q. if $\psi = xyz$ evaluate $\Delta\psi$ in spherical co-ordinates

Sol We have $\psi = xyz$

$(r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta)$

$r^3\sin^2\theta\cos\theta\sin\phi\cos\phi$

$$\nabla\psi = \frac{e_r}{h_1}\frac{\partial\psi}{\partial r} + \frac{e_\theta}{h_2}\frac{\partial\psi}{\partial\theta} + \frac{e_\phi}{h_3}\frac{\partial\psi}{\partial\phi}$$

$\frac{e_r}{1}(3r^2\sin^2\theta\cos\theta\sin\phi\cos\phi) + \frac{e_\theta}{r}$

$[r^3\sin\phi\cos\phi(2\sin\theta\cos^2\theta - \sin^3\theta)]$

$+ \frac{e_\phi}{r\sin\theta}(r^3\sin^2\theta\cos\theta(\cos^2\phi - \sin^2\phi))$

$\Rightarrow r^2\sin\theta[(3\sin\theta\cos\theta\sin\phi\cos\phi)e_r$

$+ [\sin\phi\cos\phi(\cos^2\theta + \cos2\theta)e_\theta + \cos\theta$

$\cos 2\phi e_\phi)$

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Q. Ex 4.2

I. Interpret the symbol $a \cdot \nabla$

II. Show that $(a \cdot \nabla)\phi = a \cdot \nabla\phi$

III. Show that $(a \cdot \nabla) r = a$

Sol (i) Let $a = a_1 i + a_2 j + a_3 k$

Then

$$a \cdot \nabla = (a_1 i + a_2 j + a_3 k) \cdot \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + \right.$$

$$= a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$$

Symbol $a \cdot \nabla$ stands for the operator $a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$

II) $$(a \cdot \nabla)\phi = \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right)\phi$$

$$= a_1 \frac{\partial \phi}{\partial x} + a_2 \frac{\partial \phi}{\partial y} + a_3 \frac{\partial \phi}{\partial z}$$

$$a \cdot \nabla\phi = (a_1 i + a_2 j + a_3 k) \cdot \left( i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + \right.$$

$$= a_1 \frac{\partial \phi}{\partial x} + a_2 \frac{\partial \phi}{\partial y} + a_3 \frac{\partial \phi}{\partial z}$$

from (1) and (2)

$$(a.\nabla)\phi = a.\nabla\phi$$

iii Sol $(a.\nabla)r = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)r$

$$a_1\frac{\partial r}{\partial x} + a_2\frac{\partial r}{\partial y} + a_3\frac{\partial r}{\partial z}$$

But $r = xi + yj + zk$

$$\frac{\partial r}{\partial x} = i \quad \frac{\partial r}{\partial y} = j \quad \frac{\partial r}{\partial z} = k$$

Putting these values in (3) we get

$$(a.\nabla)r = a_1 i + a_2 j + a_3 k = a$$

Q. Prove that $\nabla\phi . dr = d\phi$

Sol let $xi + yj + zk$

$$\nabla\phi = i\frac{d\phi}{dx} + j\frac{d\phi}{dy} + k\frac{d\phi}{dz}$$

$$dr = dxi + dyj + dzk$$

$$\nabla\phi . dr = \left[i\frac{d\phi}{dx} + j\frac{d\phi}{dy} + k\frac{d\phi}{dz}\right)$$

$$(dxi + dyj + dzk)$$

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi$$

Q. Represent the vector $A = zi - 2xj + yk$ in cylindrical co-ordinates. Hence determine $A_\rho$, $A_\phi$ and $A_z$

Sol We know that

$$e_\rho = \cos\phi i + \sin\phi j$$

$$e_\phi = -\sin\phi i + \cos\phi j \quad e_z = k$$

solving the Ist two equations for $i$ and $j$ we have

$$i = \cos\phi e_\rho - \sin\phi e_\phi$$
$$j = \sin\phi e_\rho + \cos\phi e_\phi$$

$$A = zi + 2xj + yk \quad \text{Given}$$

$$z(\cos\phi e_\rho - \sin\phi e_\phi) - 2x(\sin\phi e_\rho + \cos\phi e_\phi) + \ldots$$

$$= (z\cos\phi - 2x\sin\phi)e_\rho - (z\sin\phi + 2x\cos\phi)e_\phi + ye_z$$

Since $x = \rho\cos\phi$ $y = \rho\sin\phi$
therefore

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$A = (z\cos\phi - 2\rho\cos\phi\sin\phi)e_\rho - (z\sin\phi + 2\rho\cos^2\phi) + \rho\sin\phi e_z$

$\Rightarrow A_\rho e_\rho + A_\phi e_\phi + A_z e^z$

Here $A_\rho, A_\phi, A_z$ stands for the components of A when expressed in $e_\rho, e_\phi$ and $e_z$

$A_\rho = (z\cos\phi - 2\rho\cos\phi\sin\phi)$
$A_\phi = -(z\sin\phi + 2\rho\cos^2\phi)$
$A_z = \rho\sin\phi$

Q. If $\psi = xyz$ evaluate $\nabla\psi$ in spherical co-ordinates

Sol We have $\psi = xyz$

$\Rightarrow (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta)$

$\Rightarrow (r^3\sin^2\theta\cos\theta\sin\phi\cos\phi)$

$\nabla\psi = \frac{e_r}{h_1}\frac{\partial\psi}{\partial r} + \frac{e_\theta}{h_2}\frac{\partial\psi}{\partial\theta} + \frac{e_\phi}{h_3}\frac{\partial\psi}{\partial\phi}$

$\Rightarrow r^2\sin\theta(3\sin\theta\cos\theta\sin\phi\cos\phi)e_r + [\sin\phi\cos\phi(\cos^2\theta + \cos 2\theta)]e_\theta + \cos\theta\cos 2\phi e_\phi]$

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# Vector integration.

Q. Evaluate $\int f \cdot dr$ where $f = xyi + yzj + zxk$ the curve $c$ is $r = ti + t^2 j + t^3 k$ and $t$ varies from $-1$ to $1$

Sol Curve $c$ is $r = xi + yj + zk$

$= ti + t^2 j + t^3 k$

$x = t \quad y = t^2 \quad z = t^3$

from (1) $\quad dr/dt = i + 2tj + 3t^2 k$

$f = xyi + yzj + zxk$

$t^3 i + t^5 j + t^4 k$

Now

$$\int_c f \cdot dr = \int_c \left( f \cdot \frac{dr}{dt} \right) dt$$

$$\int_{-1}^{1} (t^3 i + t^5 j + t^4 k) \cdot (i + 2tj + 3t^2 k) dt$$

$$= \int_{-1}^{1} (t^3 + 2t^6 + 3t^6) dt$$

$$= \int_{-1}^{1} (t^3 + 5t^6) dt$$

$$\left[ \frac{t^4}{4} + \frac{5t^7}{7} \right]_{-1}^{1} = \left( \frac{1}{4} + \frac{5}{7} - \frac{1}{4} + \frac{5}{7} \right)$$

$= 10/7$

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Q. if $f = 3xy\hat{i} - y^3\hat{j}$ evaluate $\int_c f \cdot dr$

where c is the curve in xy plane $y = 2x^2$ from

Sol Let $x = t$ $y = 2t^2$

Parametric equations of the curve C are $x = t$ $y = 2t^2$

Since the integration is to be performed in xy plane

$$r = x\hat{i} + y\hat{j} = t\hat{i} + 2t^2\hat{j}$$

$$dr/dt = \hat{i} + 4t\hat{j}$$

At $(0,0)$ $t = 0$ and at $(1,2)$ $t = 1$

$$\int_c f \cdot dr = \int_c \left(f \cdot \frac{dr}{dt}\right) dt =$$

$$\int_c (3xy\hat{i} + y^2\hat{j})(\hat{i} + 4t\hat{j})\, dt$$

$$= \int_0^1 (3xy - 4y^2 t)\, dt$$

$$\int_0^1 (6t^3 - 16t^5)\, dt$$

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$$\left[\frac{6}{4}t^4 - \frac{16}{4}t^6\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{-14}{12} = \frac{-7}{6}$$

Q. Find the circulation of $f$ around the curve $C$ when $f = yi + zj + xk$ and $C$ is the circle $x^2 + y^2 = 1$, $z = 0$

Circulation of $f$ along the curve $C$ is $\oint_C f \cdot dr$

Equation of circle is $x^2 + y^2 = 1$, $z = 0$

Its parametric equations are

$x = \cos\theta$ $y = \sin\theta$ $z = 0$

Now $r = xi + yj + zk = \cos\theta i + \sin\theta j + 0k$

$dr = (-\sin\theta i + \cos\theta j + 0k)d\theta$

$f = yi + zj + xk = \sin\theta i + 0j + \cos\theta k$

$\oint f \cdot dr = \oint_C (yi + zj + xk)$

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$(-\sin\theta\, i + \cos\theta\, j + 0k)\, d\theta$

$\Rightarrow \oint_c (\sin\theta\, i + \theta j + \cos\theta\, k)\cdot(-\sin\theta\, i + \cos\theta\, j + 0k)\, d\theta$

$\Rightarrow \oint_0^{2\pi} -\sin^2\theta\, d\theta$

$\Rightarrow -\int_0^{2\pi} \left[\frac{1-\cos 2\theta}{2}\right] d\theta.$

$\Rightarrow -\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$-\left[\frac{2\pi}{2} - 0\ (0-0)\right] = -\pi$

If $f = (2x+y)i + (3y-x)j$ evaluate $\int_c f\cdot dr$ where $C$ is the curve in $xy$ plane consisting of the straight lines from $(0,0)$ to $(2,0)$ and then to $(3,2)$

$\int_c f\cdot dr = \int_c [(2x+y)i + (3y-x)j]\cdot(dx\, i + dy\, j)$

$\int_c f\cdot dr = \int (2x+y)dx + (3y-x)dy)$

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On OA $y=0$ and $dy=0$ and $x$ varies from 0 to 2

Now equation of line AB is

$$y-0=\frac{2-0}{3-2}(x-2)$$

$$y=2(x-2)=2x-4$$

$$y=2x-4$$

$dy=2dx$ and $x$ varies from

from

$$\int_C F\cdot dr=\int_0^2 2x\,dx+\int_2^3 (2x+2x-4)dx$$

$$+(6x-12-x)2dx$$

$$[x^2]_0^2+\int_2^3(4x-4)dx+2\int_2^3(5x-12)dx$$

$$4+\left[\frac{4x^2}{2}-4x\right]_2^3+2\left[\frac{5x^2}{2}-12x\right]_2^3$$

$$4+[(18-12)-(8-8)]+2\left[\left(\frac{45}{2}-36\right)-(10-24)\right]$$

$$4+6+2\left[-\frac{27}{2}+14\right]=4+6+1$$

$$=11$$

Ex: 5.3

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mple

Evaluate $\iint_S f \cdot n \, ds$ where $f = zi + xj + 3y^2 z k$ and $S$ is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant b/w $z = 0$ and $z = 5$

Sol: A vector normal to the surface $S$ is given by

$$\nabla(x^2+y^2) = 2xi + 2yj$$

$n$ = unit vector

$$\frac{2xi + 2yj}{\sqrt{4x^2+4y^2}} = \frac{2xi + 2yj}{2\sqrt{16}}$$

$$\frac{1}{4}xi + \frac{1}{4}yj$$

$$\iint_S f \cdot n \, ds = \iint_R f \cdot n \frac{dx\,dz}{|n \cdot j|}$$

$$= \iint_R \left(\frac{xz}{4} + \frac{xy}{4}\right) \cdot \frac{dx\,dz}{1/4\, y}$$

$$= \iint_R \left(\frac{xz + xy}{y}\right) dx\,dz$$

$$= \iint_R \left(\frac{xz + xy}{y}\right) dx\,dz$$

$$= \iint_R \left(\frac{xz}{y} + x\right) \cdot dx\,dz$$

$$= \int_{z=0}^{5} \int_{x=0}^{4} \left(\frac{xz}{\sqrt{16-x^2}} + x\right) dx\,dz$$

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$$\int_0^5 \left[ \left(-\tfrac{1}{2}2\right) \frac{\sqrt{16-x^2}}{1/2} + \frac{x^2}{2} \right]_0^4 dz$$

$$\int_0^5 (4z+8)\,dz = \left[\frac{4z^2}{2} + 8z\right]_0^5$$

$= 90$

Q. Evaluate $\iint_S F\cdot n\,ds$ where $F = (x+y^2)i - 2xj + 2yzk$ and $S$ is the surface of the plane $2x+y+2z=6$ in the first octant.

Sol Vector normal to surface $S$ is given by

$$\nabla(2x+y+2z) = 2i + j + 2k$$

$n =$ unit vector normal

$$\frac{2i+j+2k}{\sqrt{4+1+4}} = \frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k$$

$$F\cdot n = \left[(x+y^2)k - 2xj + 2yzk\right]\cdot\left[\frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k\right]$$

$$\frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}zy$$

$$= \frac{2}{3}y^2 + \frac{4}{3}yz$$

Taking projection on xy plane

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$$\iint_S f\cdot n\, ds = \iint_R f\cdot n \frac{dx\,dy}{|n\cdot k|}$$

$$= \iint_R f\cdot n \frac{dx\,dy}{2/3}$$

from (1) $\iint_S f\cdot n\, ds = \frac{3}{2}\int_0^6 \int_0^{\frac{6-y}{2}}$

$$\left[\frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)\right] dx\,dy$$

$$= \left(\frac{3}{2}\right)\left(\frac{1}{6}\right)\int_0^6 \int_0^{\frac{6-y}{2}} (4y^2 + 24y - 8xy - 4y^2)\, dx\, dy$$

$$\frac{1}{4}\int_0^6 \int_0^{6-y/2} 8y(3-x)\, dx\, dy$$

$$\frac{2}{8}\int_0^6 y(6-y)\,(12-6+y)\, dy$$

$$\frac{1}{4}\int_0^6 (36y - y^3)\, dy$$

$$\frac{1}{4}\left[18y^2 - \frac{y^4}{4}\right]_0^6$$

$$= \frac{1}{4}\left[18\times 36 - \frac{(36)^2}{4}\right]$$

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$$\frac{1}{4} \times 18 \times 36 \left[1 - \frac{1}{9}\right] = 81$$

Ex: 54

Q. If $f = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x$ evaluate $\iiint \nabla \times f \, dv$ where $v$ is region bounded by the co-ord planes and the plane $2x + 2y$

Sol

$$\nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x^2 - 3z) & -2xy & -4x \end{vmatrix}$$

$$\nabla \times f = (0-0)\hat{i} + (-3+4)\hat{j} + (-2y - 0)k$$
$$0\hat{i} + \hat{j} - 2yk$$

$$\iiint_V \nabla \times f \, dv = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (\hat{j} - 2yk) \, dz \, dy$$

$$\int_{x=0}^{2} \int_{y=0}^{2-x} \left[z\hat{j} - 2yzk\right]_0^{4-2x-2y} dy \, dx$$

$$\int_{x=0}^{2} \int_{y=0}^{2-x} \left[(4-2x-2y)\hat{j} - 2y(4-2x-2y)k \, dy\right.$$

$$\int_{x=0}^{2} \left[4y - 2xy - y^2\right]\hat{j} - \left(4y^2 - 2xy^2 - \frac{4}{3}y^3\right]k$$

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$$\frac{1}{4} \times 18 \times 36 \left[1 - \frac{1}{9}\right] = 81$$

$\Sigma x = 54$

Q. If $f = (2x^2 - 3z)i - 2xyj - 4xk$ evaluate $\iiint \nabla \times f \, dv$ where $V$ is the region bounded by the co-ordinate planes and the plane $2x + 2y + z = 4$

Sol

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x^2 - 3z) & -2xy & -4x \end{vmatrix}$$

$$\nabla \times f = (0-0)i + (-3+4)j + (-2y - 0)k$$

$$0i + j - 2yk$$

$$\iiint_V \nabla \times f \, dv = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (j - 2yk) \, dz \, dy \, dx$$

$$\int_{x=0}^{2} \int_{y=0}^{2-x} \left[zj - 2yzk\right]_0^{4-2x-2y} dy \, dx$$

$$\int_{x=0}^{2} \int_{y=0}^{2-x} \left[(4 - 2x - 2y)j - 2y(4 - 2x - 2y)k\right] dy \, dx$$

$$\int_{x=0}^{2} \left[(4y - 2xy - y^2)j - \left(4y^2 - 2xy^2 - \frac{4}{3}y^3\right)k\right]_0^{2-x}$$

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$$\int_{x=0}^{2}\left[(4-4x+x^2)j - \left(-\frac{2}{3}x^3+4x^2-8x+\frac{16}{3}\right)k\right] dx$$

$$\frac{8}{3}j - \frac{8}{3}k = \frac{8}{3}(j-k)$$

Q. If $f = 2xzi - xj - y^2k$ then evaluate $\iiint_V f\,dv$ where $V$ is the region bounded by the surface $x=0, y=0, x=2, y=6$, $z=x^2, z=4$

Sol: $$\iiint_V f\,dv = \int_0^2\int_0^6\int_{x^2}^4 (2zx\,i - xj + y^2k)\,dz\,dy\,dx$$

$$\int_0^2\int_0^6 \left[(z^2x)i - zxj + y^2zk\right]_{x^2}^4 dy\,dx$$

$$= \int_0^2\int_0^6 \left[(16xi - 4xj + 4y^2k) - (x^5i - x^3j + y^2x^2k)\right]$$

$$\int_0^2 \left[(16x-x^5)yi + (x^3-4x)yj + \frac{y^3}{3}(4-x^2)k\right]_0^6 dx$$

$$\int_0^2 \left[(16x-x^5)yi + (x^3-4x)yj + \frac{y^3}{3}(4-x^2)k\right]_0^6$$

$$\left(32-\frac{64}{6}\right)6i + (4-8)6j + 72\left(8-\frac{8}{3}\right)k$$

$$= 128i - 24j + 384k.$$

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from (1) and (2)

$$(a.\nabla)\phi = a.\nabla\phi$$

iii) Sol

$$(a.\nabla)r = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)$$

$$a_1\frac{\partial r}{\partial x} + a_2\frac{\partial r}{\partial y} + a_3\frac{\partial r}{\partial z}$$

But $r = x\hat{i} + y\hat{j} + z k$

$$\frac{\partial r}{\partial x} = \hat{i} \quad \frac{\partial r}{\partial y} = \hat{j} \quad \frac{\partial r}{\partial z} = k$$

Putting these values in (5) we get

$$(a.\nabla)r = a_1\hat{i} + a_2\hat{j} + a_3 k = a$$

Q. Prove that $\nabla\phi . dr = d\phi$

Sol let $x\hat{i} + y\hat{j} + z k$

$$\nabla\phi = i\frac{d\phi}{dx} + j\frac{d\phi}{dy} + k\frac{d\phi}{dz}$$

$$dr = dx i + dy j + dy k$$

$$\nabla\phi . dr = \left[i\frac{d\phi}{dx} + j\frac{d\phi}{dy} + k\frac{d\phi}{dz}\right]$$

$$(dx\hat{i} + dy\hat{j} + dz k)$$

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$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi$$

Q. Represent the vector $A = zi - 2xj + yk$ in cylindrical co-ordinates. Hence determine $A_\rho$ $A_\phi$ and $A_z$

Sol We know that

$$e_\rho = \cos\phi i + \sin\phi j$$

$$e_\phi = -\sin\phi i + \cos\phi j \quad e_z = k$$

Solving the Ist two equations for $i$ and $j$ we have

$$i = \cos\phi e_\rho - \sin\phi e_\phi$$
$$j = \sin\phi e_\rho + \cos\phi e_\phi$$

$$A = zi + 2xj + yk \quad \text{Given}$$

$$z(\cos\phi e_\rho - \sin\phi e_\phi) - 2x(\sin\phi e_\rho + \cos\phi e_\phi)$$

$$= (z\cos\phi - 2x\sin\phi)e_\rho - (z\sin\phi + 2x\cos\phi)e_\phi + ye_z$$

Since $x = \rho\cos\phi$ $y = \rho\sin\phi$
therefore

Q. Evaluate $\int f \cdot dr$ where $f = xyi + yzj + zxk$ the curve $C$ is $r = ti + t^2 j + t^3 k$ and $t$ varies from $-1$ to $1$

Sol Curve $C$ is $r = xi + yj + zk$

$= ti + t^2 j + t^3 k$

$x = t \quad y = t^2 \quad z = t^3$

from (1) $dr/dt = i + 2tj + 3t^2 k$

$f = xyi + yzj + zxk$

$t^3 i + t^5 j + t^4 k$

Now

$$\int_C f \cdot dr = \int_C \left(f \cdot \frac{dr}{dt}\right) dt$$

$$\int_{-1}^{1} (t^3 i + t^5 j + t^4 k) \cdot (i + 2tj + 3t^2 k) dt$$

$$= \int_{-1}^{1} (t^3 + 2t^6 + 3t^6) dt$$

$$= \int_{-1}^{1} (t^3 + 5t^6) dt$$

where $C$ is the curve in $xy$ plane $y = 2x^2$ from

Sol Let $x = t \quad y = 2t^2$

Parametric equations of the curve $C$ are $x = t \quad y = 2t^2$

Since the integration is to be performed in $xy$ plane

$r = xi + yj = ti + 2t^2 j$

$dr/dt = i + 4tj$

At $(0,0)$ $t = 0$ and at $(1,2)$ $t = 1$

$$\int_C f \cdot dr = \int_C \left(f \cdot \frac{dr}{dt}\right) dt =$$

$$\int_C (3xyi + y^2 j)(i + 4tj) dt$$

$$= \int_0^1 (3xy - 4y^2 t) dt$$

$$\int_0^1 (6t^3 - 16t^5) dt$$

$$\left[\frac{6}{4}t^4 - \frac{16}{4}t^6\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{-14}{12} = \frac{-7}{6}$$

Q. Find the circulation of f around the curve C when $f = yi + zj + xk$ and C is the circle $x^2 + y^2 = 1$, $x = 0$

Circulation of f along the curve C is $\oint_C f \cdot dr$

Equation of circle is $x^2 + y^2 = 1$, $z = 0$

Its parametric equations are

$x = \cos\theta \quad y = \sin\theta \quad z = 0$

Now $r = xi + yj + zk = \cos\theta i + \sin\theta j + 0k$

$dr = (-\sin\theta i + \cos\theta j + 0k)d\theta$

$f = yi + zj + xk = \sin\theta i + 0j + \cos\theta k$

$(-\sin\theta i + \cos\theta j + 0k)d\theta$

$\Rightarrow \oint_C (\sin\theta i + 0j + \cos\theta k) \cdot (-\sin\theta i + \cos\theta j + 0k)d\theta$

$\Rightarrow \oint_0^{2\pi} -\sin^2\theta \, d\theta$

$\Rightarrow -\int_0^{2\pi} \left[\frac{1 - \cos 2\theta}{2}\right] d\theta$

$\Rightarrow -\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$-\left[\frac{2\pi}{2} - 0 (0 - 0)\right] = -\pi$

Q. If $f = (2x+y)i + (3y-x)j$ evaluate $\int_C f \cdot dr$ where C is the curve in xy plane consisting of the straight lines from (0,0) to (2,0) and then to (3,2)

Sol $\int_C f \cdot dr = \int_C [(2x+y)i + (3y-x)j] \cdot (dxi + dyj)$

$\int_C f \cdot dr = \int (2x+y)dx + (3y-x)dy)$

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On OA $y=0$ and $dy=0$ and $x$ varies from 0 to 2

Now equation of line AB is

$$y-0 = \frac{2-0}{3-2}(x-2)$$

$$y = 2(x-2) = 2x-4$$

$$y = 2x-4$$

$dy = 2dx$ and $x$ varies from

from

$$\int_C f\cdot dr = \int_0^2 2x\,dx + \int_2^3 (2x+2x-4)dx$$

$$+ (6x-12-x)2dx$$

$$[x^2]_0^2 + \int_2^3 (4x-4)dx + 2\int_2^3 (5x-12)dx$$

$$4 + \left[\frac{4x^2}{2} - 4x\right]_2^3 + 2\left[\frac{5x^2}{2} - 12x\right]_2^3$$

$$4 + [(18-12) - (8-8)] + 2\left[\left(\frac{45}{2} - 36\right) - (10-$$

$$4+6+2\left[-\frac{27}{2}+14\right] = 4+6+1$$

$$= 11$$