Maa Omwati Degree Collage

Hassanpur

NOTES

Subject:- Properties of Matter Kinetic Theory & Relativity

B.SC Sem:-II

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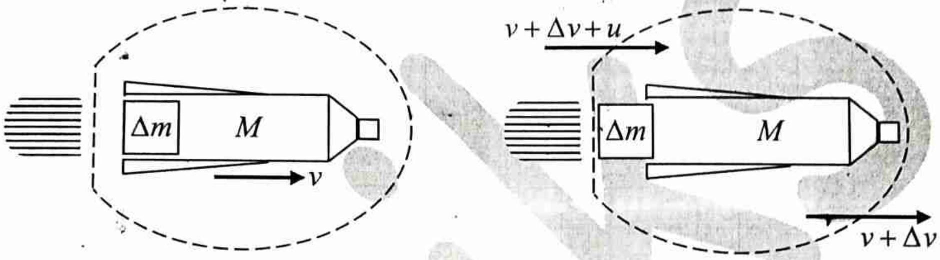
To analyze the motion of the rocket in detail, we must equate the external force on the system, F, with the rate of change of momentum, dP/dt. Consider the rocket at time t. Between t and $t+\Delta t$ a mass of fuel Δm is burned and expelled as gas with velocity u relative to the rocket. The exhaust velocity u is determined by the nature of the propellants, the throttling of the engine, etc., but it is independent of the velocity of the rocket.

The sketches below show the system at time t and at time $t+\Delta t$. The system consists of Δm plus the remaining mass of the rocket M. Hence the total mass is $M+\Delta m$.



Time t Time $t+\Delta t$

The velocity of the rocket at time t is $v(t)$, and at $t+\Delta t$ it is $v+\Delta v$.

The initial momentum is $P(t)=(M+\Delta m)$

the final momentum is $P(t+\Delta t)=M(v+\Delta v)+\Delta m(v+\Delta v+u)$

The change in momentum is $\Delta P = P(t+\Delta t)-P(t)=M\Delta v+(\Delta m)u$ assume $\Delta m.\Delta v=0$

Therefore, $\frac{dP}{dt}=\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t}=M\frac{dv}{dt}+u\frac{dm}{dt}$.

Note that we have defined u to be positive in the direction of v. In most rocket applications, u is negative, opposite to v. It is inconvenient to have both m and M in the equation. dm/dt is the rate of increase of the exhaust mass. Since this mass comes from the rocket,

$$\frac{dm}{dt}=-\frac{dM}{dt}.$$

equating the external force to dP/dt, we obtain the fundamental rocket equation

$$F=M\frac{dv}{dt}-u\frac{dM}{dt}.$$

It may be useful to point out two minor subtleties in our development. The first is that the velocities have been expressed with respect to an inertial frame, not a frame attached to the rocket. The second is that we took the final velocity of the element exhaust gas to be $v+\Delta v+u$ rather than $v+u$.

Example: (a) A rocket is moving vertically in absence of gravity .if initial exhaust speed and mass of rocket is u and M_0 respectively. What will be speed if mass became αM_0 after time $t=t_f$. (assume $t=0$ velocity of rocket is v_0 :)

(b) How situation will different if a gravitational field is present.

Solution: (a) If there is no external force on a rocket, $F=0$ and its motion is given by

$$M\frac{dv}{dt}=u\frac{dM}{dt} \qquad \text{or} \qquad \frac{dv}{dt}=\frac{u}{M}\frac{dM}{dt}$$

Generally the exhaust velocity i is constant, in which case it is easy to integrate the equation of motion.

$$\int_{t_0}^{t_f}\frac{dv}{dt}dt=u\int_{t_0}^{t_f}\frac{1}{M}\frac{dM}{dt}dt=u\int_{M_0}^{M_f}\frac{dM}{M}$$

or $$v_f-v_0=u\ln\frac{M_f}{M_0}=u\ln\alpha$$

(b) If a rocket takes off in a constant gravitational field,

$M_g=M\frac{dv}{dt}-u\frac{dM}{dt}$, where u and g are directed down and are assumed to be constant.

$$\frac{dv}{dt}=\frac{u}{M}\frac{dM}{dt}+g$$

Integrating with respect to time, we obtain

$$v_f-v_0=u\ln\left(\frac{M_f}{M_0}\right)+g\left(t_f-t_0\right)\Rightarrow v_f=v_0+u\ln(\alpha)+gt_f.$$

MCQ (Multiple Choice Questions)

Q1. Match the following List I with List II.

[redacted]

A. Elastic collision	1. Momentum constant but K.E. is not constant
B. Ineslastic collision	2. Moment and K.E. is constant.
C. Perfectly inelastic collision	3. $0 < e < 1$
D. Partly elastic collision	4. $e = o$

	A	B	C	D
(a)	2	4	3	1
(b)	2	1	4	3
(c)	1	2	3	4
(d)	4	3	1	2

[redacted] $\left(-10\hat{i}+35\hat{j}+3\hat{k}\right)\, cm/s$ respectively. The velocity of the centre of mass is given by

(a) $2k\, m/s$ (b) $\left(-8\hat{i}+28\hat{j}\right) cm/s$

(c) $\left(\frac{10}{2}\hat{i}-\frac{35}{6}\hat{j}+3\hat{k}\right) cm/s$ (d) $2\hat{k}\ cm/s$

Q3. A $2\,kg$ mass is acted upon by a force which gives a time dependent displacement of $x=\frac{t^2}{2}$. The work done in the time interval 0 to t_0 is

(a) $0.5t_0$ (b) t_0 (c) t_0^2 (d) t_0^3

[redacted] mass M. If the body of mass M moves with velocity v, then the velocity of the body of mass m after the impact is

(a) $\frac{m-M}{m+M}u$ (b) $\frac{m-M}{m+M}v$ (c) $\frac{mu+Mv}{m+M}$ (d) $\frac{mu-Mv}{m+M}$

Q5. The mass of a rocket is M and the total mass of rocket and the fuel is M_0. The average exhaust velocity of gases ejected from rocket motors is u are the final velocity attained by the rocket after using all fuel is v. The final velocity v is proportional to

(a) $\log\left(\frac{M_0}{M}\right)$ (b) $\left(\frac{Mu}{M_0}\right)$

(c) $(M_0 - M)$ (d) $(M_0 - M)^{-1}$

Q6. A block of mass $2\ kg$ is attached with a spring of spring constant $4900\ Nm^{-1}$ and the system is kept on smooth, horizontal plane the other end of spring is attached with a wall initially, spring is stretched by $5\ cm$ from its natural length and the block is at rest. Now, suddenly an impulse of $4\ kg\,ms^{-1}$ is given to the block towards the wall. The velocity of block, when spring acquires natural length is

(a) $5\ ms^{-1}$ (b) $3\ ms^{-1}$ (c) $6\ ms^{-1}$ (d) None

Q7. Pick out the **WRONG** alternative.

(a) A force is said to be conservative if the work done by the force, when a particle moves in a closed loop, is zero.

(b) If the work done by a force is path independent then the force is said to be conservative.

(c) Work done by frictional force is path independent.

(d) Frictional force is a non-conservative.

Q8. Pick out the correct alternative.

(a) The concept of potential energy can be associated with any type of force.

(b) The mechanical energy of a system is the sum of kinetic potential and internal energy.

(c) The total mechanical energy of a system remains constant if the internal forces are conservative and external forces do no work.

(d) Suppose the internal forces within a system are conservative and external forces do negative work then the mechanical energy of the system increases.

Q9. Pick out the **WRONG** alternative:

(a) When an object of mass m moves from height h_1 above the earth surface to height h_2 the change in its gravitational potential energy is $mg(h_2 - h_1)$.

(b) A block of mass m slides down a frictionless incline, then the speed of block of the bottom is $\sqrt{2gh}$ where h is the height of incline.

(c) When the free end of a spring is displaced from x_1 to x_2, the change in its potential energy is $\frac{1}{2}K(x_2^2 - x_1^2)$, where k is the spring constant.

(d) The change in potential energy of a system is equal to the work done by the conservative force on the system.

Q10. A particle is rotated in a vertical circle by connecting it to a spring of length l and keeping to other end of the string fixed. The minimum speed of the particle when the string is horizontal for which the particle will complete the circle is

(a) $\sqrt{gl}$ (b) $\sqrt{2gl}$ (c) $\sqrt{3gl}$ (d) $\sqrt{5gl}$

Q11. Two springs A and $B(k_A = 2k_B)$ are stretched by applying forces of equal magnitude of the four ends. If the energy stored in A is E, that in B is

(a) $\frac{E}{2}$ (b) $2E$ (c) E (d) $\frac{E}{4}$

Q12. Consider the following two statements

(A) Linear momentum of the system remains constant.

(B) Centre of mass of the system remains at rest.

(a) A implies B and B implies A

(b) A does not imply B and B does not imply A

(c) A implies B and B does not imply A

(d) B implies A but A does not imply B

Q13. A body falling vertically downwards under gravity breaks in two parts of unequal masses. The centre of mass of the two parts taken together shifts horizontally towards

(a) heavier piece

(b) lighter piece

(c) does not shift horizontally

(d) depends on the vertical velocity of the time of breaking

Q14. A body of rest breaks into two pieces of equal masses. The parts will move

(a) in the same direction

(b) along different lines

(c) in opposite directions with equal speeds

(d) in opposite directions with unequal speeds

Q15. Pick out the correct alternative

(a) If the net external force acting on a system is zero, its acceleration is non-zero.

(b) If the velocity of centre of mass is constant, then linear momentum is necessarily constant.

(c) During collision of a system of particles linear momentum is not constant.

(d) Conservation of momentum for a system implies the conservation of energy for that system.

Q16. Two balls are thrown simultaneously in air. The acceleration of the centre of mass of the two balls while in air

(a) depends on the direction of motion of the balls

(b) depends on the masses of two balls

(c) depends on the speed of the two balls

(d) is equal to g

Q17. The potential energy function associated with the force $\vec{F} = 4xy\hat{i} + 2x^2\hat{j}$ is

(a) $u = -2x^2y$

(b) $u = -2x^2y + \text{constant}$

(c) $u = 2x^2y + \text{constant}$

(d) Not defined

Q18. When a spring is stretched by $2cm$, it stores $100\ J$ of energy. If it is stretched further by $2cm$, the stored energy will be increased by

(a) $100\ J$ (b) $200\ J$ (c) $300\ J$ (d) $400\ J$

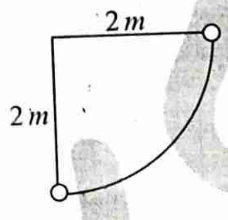
Q19. The bob of a pendulum of length $2m$ lies at P. When it reaches Q, it loses 10% of its total energy due to air resistance, The velocity at Q is (Take $g = 10 m/s^2$).

(a) $6 m/s$

(b) $1\ m/s$

(c) $2\ m/s$

(d) $8\ m/s$



Q20. A block moving horizontally on a smooth surface with a speed of $20\ m/s$ breaks into two equal parts continuing in the same direction. If one of the parts moves of $30\ m/s$, with what speed does the second part move

(a) $5\ m/s$ (b) $10\ m/s$ (c) $15\ m/s$ (d) $20\ m/s$

Q21. A car of mass M is moving with a uniform velocity v on a horizontal road when a man drops himself on it from above. Taking the mass of the man to be m, the velocity of the car after the event is

(a) $\frac{3mv}{M+m}$ (b) $\frac{mv}{M+m}$ (c) $\frac{Mv}{M+m}$ (d) $\frac{(M+m)v}{M}$

Q22. You lift a suitcase from the floor and keep it on a table. The work done by you on the suitcase does not depend on

(a) the path taken by the suitcase (b) the time taken by you in doing so

(c) the weight of the suitcase (d) your weight

NAT (Numerical Answer Type)

Q23. 2 kg mass is tied to one end of a string 2 m in length which is hanging vertically. What minimum horizontal speed should be imparted to the mass such that it just reaches the top of the vertical circle with equal to the. length of the string? (Take $g = 10\ m/s^2$) ________ m/s

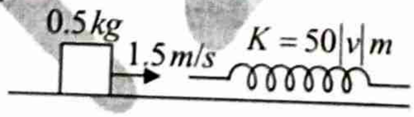
Q24. A $10^4\ kg$ rocket standing on its launch pad can expel exhaust gas at the speed of $2\times10^3\ m/s$ Just to lift the rocket from the ground, the exhaust gas must be ejected at ____ $kg\,s^{-1}$

Q25. A 25 kg mass, starting from rest at the top, slides down a plane that makes an angle of 30^0 with the horizontal. When it reaches the bottom of the 10 m long slide, its velocity is 8 m/s. The work done by the force of friction is closest to a value of ________ J

Q26. An object of mass m moving with a velocity v approaching a second object of the same mass but rest. The total kinetic energy of the two objects viewed from the centre of mass is............ αmv^2 the value of α

Q27. Work done when a force $\vec{F} = (\hat{x} + 2\hat{y} + 3\hat{z})$ acting on a particle takes it from the point $\vec{r}_1 = (\hat{x} + \hat{y} + \hat{z})m$ to the point $\vec{r}_2 = (\hat{x} - \hat{y} + 2\hat{z})m$ is ________ J

Q28. A mass of 0.5 kg moving with a speed of 1.5 m/s on a horizontal smooth surface, collides with a weightless spring of force constant $k = 50\ N/m$. The maximum compression of the spring would be ____________ m.



Q29. A moving body of mass m and velocity 3 m/s collides with a motionless body of mass $2m$ and sticks to it. Now the velocity of the combined mass would be ________ m/s.

Q30. A block of mass M is attached to the lower end of a vertical spring. The spring is hung from the ceiling and has force constant value k. The mass is released from rest with the spring initially unstretched. The maximum extension produced in the length of the spring will be ________ Mg/k.

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Q31. A particle is projected at an angle of 60^0 with the horizontal with a kinetic energy E. The kinetic energy of the highest point is αE, then value of α is _______.

Q32. A uniform chain of length $2\ m$ is kept on a table such that a length of $60\,cm$ hangs freely from the edge of the table. The total mass of the chain is $4\ kg$. The work done in pulling the entire chain on the table is __________ J.

Q33. A particle of mass $4m$ explodes into three fragments. Two of the fragments each of mass m are found to move with a speed is each in mutually perpendicular directions. The total energy released in the process of explosion is __________ mv^2.

Q34. Two particles of mass 1 kg and 3 kg move towards each other under their mutual force of attraction. No other force acts on them. When the relative velocity of approach of two particles is $2\,m/s$, their centre of mass has a velocity of $0.5\ m/s$. When the relative velocity of approach becomes $3\,m/s$, the velocity of centre of mass is __________

Q35. A particle of mass $10\ kg$ is moving with velocity of $\left(2\hat{i}+3\hat{j}\right)m/s$. From time $t=0$, a force $10\,\hat{j}$ starts acting on the particle. The x- component of momentum of the particle at any other time t is ____________.

Q36. Two blocks of masses $10\ kg$ and $4\ kg$ are connected by a spring of negligible mass and placed on a frictionless horizontal surface. An impulse gives a velocity of $14\ m/s$ to the heavier block in the direction of the lighter block. The velocity of centre of mass is _____.

Q37. A $4.0\ kg$ block sliding on a frictionless surface explodes into two fragments of $2.0\ kg$ parts. One moves with $3.0\ m/s$ due north and other at $6.0\ m/s$, 30^0 north east. The original speed of the block is ______________.

MSQ (Multiple Select Questions)

Q38. If for a system of N particles of different masses $m_1, m_2 ... m_N$ with position vectors $\vec{r}_1, \vec{r}_2, ... \vec{r}_N$ and corresponding velocities $\vec{v}_1, \vec{v}_2, ... \vec{v}_N$ respectively such that $\sum_i \vec{v}_1 = 0$, then

(a) linear momentum must be constant

(b) total angular momentum must be independent of the choice of the origin

(c) the total force on the system must be zero

(d) total torque on the system must be zero

Q39. A ball bits the floor and rebounds, after elastic collision in this case

(a) the magnitude of the momentum of the ball just after collision is same as that just before collision

(b) the mechanical energy of the ball remains the same in the collision.

(c) total momentum of ball and earth is conserved

(d) total kinetic energy of ball and earth is conserved

Q40. A net force $F_x(t) = A + Bt^2$ in the $+x$ direction is applied to a boy of mass m. The force starts at $t = 0$ and continues until time t

(a) Impulse of the force is $2Bt$

(b) His speed at time t is $\frac{A}{m}t + \frac{B}{3m}t^3$

(c) Impulse of the force is $At + \frac{B}{3}t^3$

(d) Speed at time t is $\frac{2B}{m}t$

Q41. A block of mass m moving on a smooth horizontal plane with a velocity v_0 collides with a stationary block of mass M at the back of which a spring of spring constant k is attached as shown in figure. Select the correct option(s).

(a) Velocity of centre of mass $\frac{m}{m+M}v_0$

(b) Initial kinetic energy of the system in centre of mass frame is $\frac{1}{4}\left(\frac{mM}{m+M}\right)v_0^2$

(c) Maximum compression in the spring is $v_0\sqrt{\frac{mM}{(M+m)}\frac{1}{k}}$

(d) When the spring is in state of maximum compression the kinetic energy in the centre of mass frame is zero.

Q42. A body moving towards a finite body at rest collides with it. It is possible that

(a) both bodies come to rest

(b) both bodies move after collision

(c) the moving body came to rest and stationary body starts moving

(d) the stationary body remain stationary the moving body change its velocity

Q43. Pick out the **CORRECT** alternative(s):

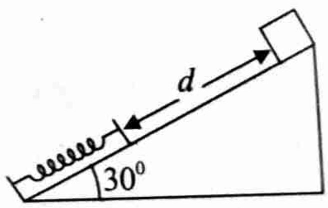
(a) If center of mass of three particles is at rest and it is known that two of them are moving along different lines, then the third must also be moving.

(b) If centre of mass remains at rest, then net work done by the forces acting on the system must be zero.

(c) If centre of mass remains at rest, then the net external force must be zero.

(d) For two particles collision, the linear momentum is conserved but for three particles it is not conserved.

Q44. A block of $4\ kg$ mass starts at rest and slides a distance d down a friction less incline (angle 30^0) when it runs into a spring of negligible mass. The block slides an additional difference of $25\ cm$ before it is brought to rest momentarily by compressing the spring. The force constant of the spring is $400\ N/m$.



(i) The value of d is

(ii) When the block strikes the spring its speed is $\sqrt{3.75}\ m/s$

(iii) At maximum compression the potential energy of the spring is $12.5\ J$

(iv) The change in gravitational P.E. of the block from the top and till the position of momentary rest is $15\ J$

Q45. A body moving towards a body of finite mass at rest collides with it. It is possible that

(a) both bodies come to rest.

(b) both bodies move after collision.

(c) the moving body stops and the body at rest starts moving.

(d) the stationary body remains stationary.

Q46. In an elastic collision between two particles.

(a) the total kinetic energy of the system is always conserved.

(b) the kinetic energy of the system before collision is equal to the kinetic energy of the system after collision.

(c) the linear momentum of the system is always conserved.

(d) the linear momentum before collision is equal to the linear momentum after collision.

Q47. A particle is taken from point A to point B under the influence of a force field. Now it is taken from B to A and it is observed that the work done in taking the particle from B to A is not equal to the work done in taking it from A to B. If W_{nc} and W_c are the work done by non-conservative and conservative forces and ΔU and ΔK are change in potential and change in kinetic energy then

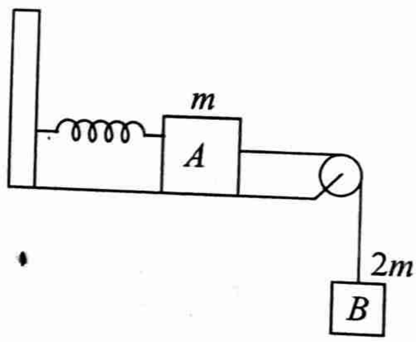
(a) $W_{nc} - \Delta U = \Delta K$

(b) $W_{nc} = -\Delta U$

(c) $W_{nc} + W_c = \Delta K$

(d) $W_{nc} - \Delta U = -\Delta K$

Q48. In the adjoining figure block A is of mass m and block B is of mass $2\,m$. The spring has force constant K. All the surfaces are smooth and the system is released from rest with spring unstretched.



(a) The maximum extension of the spring is $\frac{4mg}{K}$

(b) The speed of block A when extension in spring is $\frac{2mg}{K}$, is $2g\sqrt{\frac{2m}{3K}}$

(c) The net acceleration of block B when the extension in the spring is maximum is $\frac{2}{3}g$

(d) Tension in the thread for extension of $\frac{2mg}{K}$ in spring is mg.

Q49. At two positions kinetic energy and potential energy of a particle are $K_1 = 10\,J,\ U_1 = -20\,J, K_2 = 20\,J, U_2 = -10\,J$. In moving from 1 to 2.

(a) Work done by conservative forces is positive

(b) Work done by conservative forces is negative

(c) Work done by all the forces is positive

(d) Work done by all the forces is negative

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Q50. The potential energy function of a particle in the x - y plane is given by $U = k(x+y)$ where k is a constant

(a) The work done by the conservative force in moving a particle from $(1,1)$ to $(2,3)$ is $-3k$

(b) The work done by the conservative force in moving the particle from $(0,0)$ to $(1,1)$ is $-2k$

(c) The work done by the conservative force in moving the particle from $(1,1)$ to $(2,2)$ is $+2k$

(d) The work done by the conservative force in moving the particle from $(0,0)$ to $(3,3)$ is $-6k$

Q51. A block is placed on top of an inclined plane inclined at 37^0 with the horizontal. The length of the plane is $5m$. The block slides down the plane and reaches the bottom. Take $g = 10 m/s^2$, $\sin 37^0 = \frac{3}{5}$ and $\cos 37^0 = \frac{4}{5}$.

(a) If the plane is smooth the block reaches the bottom with a speed of $2\sqrt{15}\ m/s$

(b) If the plane is smooth the block reaches the bottom with a speed of $3\sqrt{15}\ m/s$

(c) If the plane has the coefficient of friction 0.25, the block reaches the bottom with a speed of $2\sqrt{10}\ m/s$

(d) If the plane has coefficient of friction 0.25, the block reaches the bottom with a speed of $5\sqrt{10}\ m/s$

Q52. You lift a suitcase from the floor and keep it on a table. The work done by you on the suitcase does not depend on

(a) the path taken by the suitcase
(b) the time taken by you in doing so
(c) the weight of the suitcase
(d) your weight

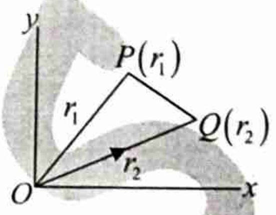
Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Ans. 2: (d)

Solution: If r_1, r_2 are position vectors of two masses m_1 and m_2 then the position vector of their centre of mass R_{CM} is given as $(m_1+m_2)R_{CM} = m_1r_1 + m_2r_2$



Differentiating w.r.t. time, we get $(m_1+m_2)v_{cm} = m_1v_1 + m_2v_2$

$$(100+20)v_{CM} = 100\left(2\hat{i}-7\hat{j}+3\hat{k}\right)+20\left(-10\hat{i}+35\hat{j}-3\hat{k}\right)$$

$$120v_{CM} = 200\hat{i}-700\hat{j}+300\hat{k}+\left(-200\hat{i}\right)+700\hat{j}-60\hat{k} = 240\hat{k}$$

$$v_{CM} = \frac{240}{120}\hat{k} = 2\hat{k}\ cm/s$$

Ans. 3: (c)

Solution: The displacement is given as $x=\frac{t^2}{2} \Rightarrow dx = \frac{2}{2}t\,dt$ Work done $\Rightarrow dW = F.dx = madx$

(ii) Differentiating equation (i) w.r.t. time, we get

Acceleration $\frac{d^2x}{dt^2} = \frac{2}{2} = 1$ So force $F = ma = 2\times 1 \Rightarrow F = 2N$

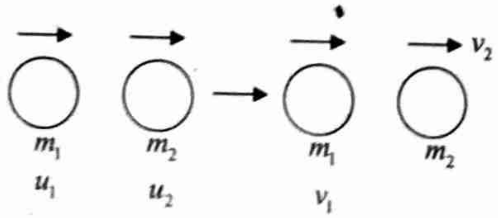
Work done $dW = 2dx = 2t\,dt \Rightarrow W = \int_0^{t_0} t\,dt = 2.\frac{t_0^2}{2} = t_0^2$

Ans. 4: (a)

Solution: When the collide elastically then their velocities after collision are given as

$$v_1 = \frac{(m_1-m_2)u_1 + 2m_2u_2}{m_1+m_2} \quad \text{(i)}$$

$$v_2 = \frac{2m_1u_1 + (m_2-m_1)u_2}{m_1+m_2}$$



Before collision After collision

Here, $m_1 = m, m_2 = M, u_1 = u, u_2 = 0$

And $v_2 = v$

Putting all these values in equation (i), we get

$$v_1 = \frac{(m-M)u+0}{m+M} \Rightarrow v_1 = \left(\frac{m-M}{m+M}\right)u$$

Ans. 5: (a)

Solution: The velocity of rocket

$M_0 =$ mass at $t=0$ $v_0 =$ velocity at $t=0$ $v = v_0 + v_e \log\frac{M_0}{M}, v \propto \log\frac{M_0}{M}$

Ans. 6: (b)

Solution: Applying conservation of energy

$$= \frac{1}{2}kx^2 + \frac{m^2v^2}{2m} = \frac{1}{2}mv'^2 + 0 \Rightarrow mv'^2 = kx^2 + \frac{m^2v^2}{m}$$

$$\Rightarrow \quad 2\times v'^2 = 4000\times\left(5\times10^{-2}\right)^2 + \frac{4^2}{2} \Rightarrow 2v'^2 = 0.4\times25+8 \Rightarrow v' = 9 \Rightarrow v' = 3$$

Ans.7: (c)

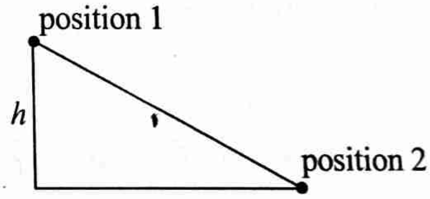
Solution: A force is said to be conservative if the work done by it in a closed loop is zero. Alternatively if the work done by a force is path independent, the force is said to be conservative. Frictional force does not satisfy any of these properties, hence it is a non-conservative force.

Ans.8: (c)

Solution: The concept of potential energy can be associated with only a conservative force. The mechanical energy of a system is the sum of kinetic and potential energy. Internal conservative forces can not change the mechanical energy. The mechanical energy of the system changes by the amount of work done by the external force.

Ans. 9: (d)

Solution: Gravitational potential energy at a height h is mgh, hence $\Delta U = mg(h_2 - h_1)$. Mechanical energy of position 1 = Mechanical energy of position 2, hence $0 + mgh = \frac{1}{2}mv^2 + 0$



$\therefore v = \sqrt{2gh}$. The change in spring potential energy is $\Delta U = \frac{1}{2}K\left(x_2^2 - x_1^2\right)$

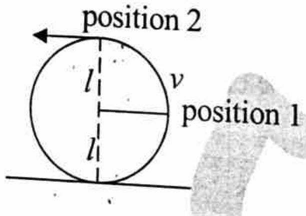
The change in potential energy of a system is equal to the negative of the work done by the conservative force on the system.

Ans. 10: (c)

Solution: The minimum speed at the top for the particle to complete the circle $=\sqrt{gl}$. Now applying the principle of conservation of mechanical energy.

Mechanical energy of position 1 = Mechanical energy of position 2

$$\frac{1}{2}mv^2+mgl=\frac{1}{2}m\left(\sqrt{gl}\right)^2+2mgl$$



or by solving we get $v=\sqrt{3gl}$

Ans. 11: (b)

Solution: $F=k_Ax_A=k_Bx_B$ or $\frac{x_B}{x_A}=\frac{k_A}{k_B}=2$

Hence $\frac{E_B}{E_A}=\frac{\frac{1}{2}k_Bx_B^2}{\frac{1}{2}k_Ax_A^2}=\left(\frac{1}{2}\right)\cdot 2^2=2 \quad \therefore E_B=2E_A=2E$

Ans. 12: (d)

Solution: Linear momentum of a system is $\vec{P}=M\vec{V}_{cm}$

If $\vec{P}$ is constant we can only conclude that $\vec{V}_{cm}$ is constant, if $\vec{V}_{cm}$ is constant then $\vec{P}$ is constant.

Ans. 13: (d)

Solution: Since the motion of centre of mass is decided by the net external force acting on the system and the net external force acting on the system and the net external force is the same (both before and after breaking), hence centre of mass does not shift at all.

Ans. 14: (c)

Solution: From conservation of momentum $M\times 0=\frac{M}{2}\vec{v}_1+\frac{M}{2}\vec{v}_2 \Rightarrow \vec{v}_1=-\vec{v}_2$

Ans. 15: (b)

Solution: From the relation $\vec{F}_{ext}=M\vec{a}_{cm}$ if $\vec{F}_{ext}=0$, then $\vec{a}_{cm}=0$

From the relation $\vec{P}=M\vec{V}_{cm}$, if $\vec{V}_{cm}$ is constant, $\vec{P}$ is constant. During collision the linear momentum is necessarily constant (in the absence of external forces).

Conservation of linear momentum and conservation of energy are entirely different concepts.

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Ans. 16: (d)

Solution: From the relation $\vec{F}_{ext} = m\vec{a}_{cm}$ we have $m\vec{g} = m\vec{a}_{cm}$ or $\vec{a}_{cm} = \vec{g}$

Ans. 17: (b)

Solution: From the relation $\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j}$

We see that the most general expression is $U = -2x^2 y + \text{constant}$.

Ans. 18: (c)

Solution: From the question,

$$\frac{1}{2}k(2\,cm)^2 = 100 \quad \text{(i)}$$

$$\frac{1}{2}k(4\,cm)^2 = E \quad \text{(ii)}$$

Hence $\frac{E}{100} = 4 \Rightarrow E = 400\,J$, change in energy $= 400 - 100 = 300\,J$.

Ans. 19: (a)

Solution: Work done by air resistance = decrease in mechanical energy

$$\Rightarrow 0.1\,mgl = \frac{1}{2}mv^2 - mgl \qquad \therefore v = \sqrt{1.8\,gl} = \sqrt{36} = 6\ m/s$$

Ans. 20: (b)

Solution: $M \cdot 20 = \frac{M}{2} \cdot 30 + \frac{M}{2}v$ or $40 = 30 + v \Rightarrow v = 10\ m/s$

Ans. 21: (b)

Solution: Even after the vertical fall, the horizontal component of momentum is not affected, hence

$$Mv = (M+m)v' \quad \text{or} \quad v' = \frac{mv}{M+m}$$

Ans. 22: (c)

Solution: since the suitcase is at rest in both situation, hence

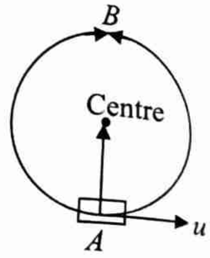
$W_g + W_{you} = 0$ (work KE theory)

$W_{you} = -W_g = -(-mgh) = mgh$. Hence only option (c) is correct

NAT (Numerical Answer Type)

Ans. 23: $10\ m/s$

Solution: If u is velocity at the lowest point A then in order en reach the mass at highest point B the velocity u should be given as $u^2 \geq 5gr$.



$$u = \sqrt{5gr} = \sqrt{5\times 10\times 2} = \sqrt{100} = 10\ m/s$$

Ans. 24: $49\ kg\,s^{-1}$

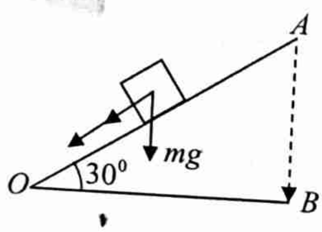
Solution: If $\frac{dm}{dt}$ is the rate exhaust gas then it is related with speed of rocket as

$$v\frac{dm}{dt} = Mg \Rightarrow 2\times 10^3\frac{dm}{dt} = 10^4\times 9.8 \Rightarrow \frac{dm}{dt} = \frac{9.8\times 10^4}{2\times 10^3} = 49\ kg\,s^{-1}$$

Ans. 25: 450

Solution: By conservation of energy law.

Total energy at A = Total energy at O



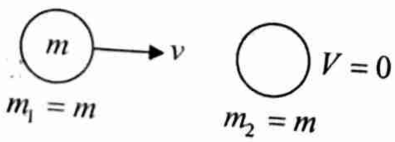
$$\Rightarrow \quad (KE)_A + (PE)_A = (KE)_o + (PE)_O + (\text{Frictional energy})$$

$$\Rightarrow \quad 0 + MgAB = \frac{1}{2}mv^2 + 0 + W_t \Rightarrow Mg10\sin 30 = \frac{1}{2}mv^2 + W_t$$

$$\Rightarrow \quad 10\times 25\times 10\times\frac{1}{2} = \frac{1}{2}\times 25\times (8)^2 + W_t \Rightarrow 1250 = 800 + W_t \Rightarrow W_t = 450\ J$$

Ans. 26: 0.25

Solution: Lets v_1 is velocity of first mass after collision and v_2 is velocity of second mats after collision. By the conservation law momentum



$$m_1 v = m_1 v_1 + m_2 v_2$$

But $\quad m_1 = m_2 = m \Rightarrow mv = mv_1 + mv_2 \Rightarrow v = v_1 + v_2$

By conservation law of energy

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \Rightarrow v^2 = v_1^2 + v_2^2$$

Solving equation (ii) and (iii), we get

Velocity of mass becomes zero whereas velocity of sector mass attain the velocity of first.

Now, the velocity of centre of mass is given

$$(m+m)v_{CM} = mv \Rightarrow v_{CM} = \frac{1}{2}v$$

Hence, the velocity of first w.r.t. centre of mass

$$0 - v_{CM} = -\frac{1}{2}v$$

Total kinetic energy as seen centre of mass

$$\frac{m(v_1 - v_{cm})^2}{2} + \frac{m(v_2 - v_{cm})^2}{2} = \frac{m\left(v_1 - \frac{v}{2}\right)^2}{2} + \frac{m\left(v_1 - \frac{v}{2}\right)^2}{2}$$

using $v^2 = v_1^2 + v_2^2$ and $v = v_1 + v_2$ $= E = \frac{mv^2}{4}$

Ans. 27: -1

Solution: Work done $= \vec{F}.d\vec{r}$

Here, $\vec{F} = \hat{x} + 2\hat{y} + 3\hat{z}\, N$

$\vec{r}_1 = (\hat{x} + \hat{y} + \hat{z}); \vec{r}_2 = (\hat{x} - \hat{y} + 2\hat{z}), dr = (2\hat{y} + \hat{z})$

So, $F = (\hat{x} + 2\hat{y} + 3\hat{z}).(-2\hat{y} + \hat{z}) = 2(-2) + 3(1) = -4 + 3 = -1J$

Ans. 28: 0.15

Solution: Taking block & spring as the system and applying the conservation of mechanical energy

$$\frac{1}{2}\times 0.5 \times (1.5)^2 + 0 = 0 + \frac{1}{2}\times 50 \times x_{max}^2 \Rightarrow x_{max} = 0.15\ m$$

Ans. 29: 1

Solution: From conservation of linear momentum $m \times 3 = (m + 2m)\times v$ or $v = \frac{3m}{3m} = 1\, m/s$

Ans. 30: 2

Solution: Taking the zero of spring potential energy of the location of unstretched spring and the zero of gravitational potential energy at the lowest point.

From conservation of mechanical energy $0 + Mgd = \frac{1}{2}kd^2 + 0$ or $d = \frac{2Mg}{k}$

Ans. 31: 0.25

Solution: The height attained by the projectile

$$H=\frac{v_0^2\sin^2\theta_0}{2g}$$, where v_0 is the speed of projection and θ_0 is the initial angle with the horizontal.

$$=\frac{v_0^2\cdot\frac{3}{4}}{2g} \quad \text{or} \quad H=\frac{3v_0^2}{8g}$$

From conservation of mechanical energy $E+0=\frac{mg\cdot 3v_0^2}{8g}$

But $\frac{1}{2}mv^2=E \quad \Rightarrow E=K+\frac{3}{4}E\Rightarrow K=\frac{1}{4}E$

Ans. 32: 3.6

Solution: Work done in putting the chain = change in gravitational potential energy,

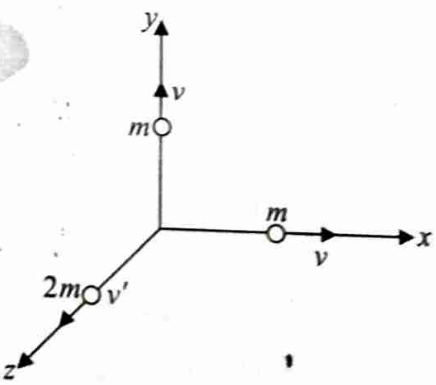
$$=0-\left[-\left(\frac{4}{2}\right)(0.6)\times 10\times(0.3)\right]=3.6\ J$$

Ans. 33: 1.5

Solution: From conservation of momentum

$$0=mv\hat{i}+mv\hat{j}+2m\vec{v}' \quad \text{or} \quad \vec{v}'=-\frac{v}{2}\hat{i}-\frac{v}{2}\hat{j}$$

$$\frac{1}{2}.2mv'^2=\frac{1}{2}.2m.\left(\frac{v^2}{4}+\frac{v^2}{4}\right)=\frac{1}{2}mv^2$$



hence total energy released

$$\frac{1}{2}mv^2+\frac{1}{2}mv^2+\frac{1}{2}mv^2=\frac{3}{2}mv^2=1.5mv^2$$

Ans. 34: $0.5\,m/s$

Solution: When no net external force acts on a system the total linear momentum and hence the velocity of centre of mass remains constant. Hence in both cases velocity of centre of mass $0.5\,m/s$.

Ans. 35: $20\ kg\ ms^{-1}$

Solution: At $t=0$, the x-component of momentum is $10\times 2 = 20\ kg\ ms^{-1}$

Since no of net external force acts on the particle in the x-direction, hence its x-component of momentum remains constant.

Ans. 36: $10\ m/s$

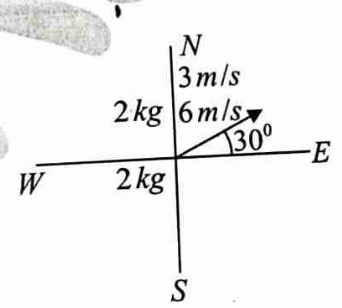
Solution: The velocity of centre of mass is given by

$$V_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

$$\therefore V_{cm} = \frac{10\times 14 + 4.0}{10+4} = 10\ m/s$$

Ans. 37: 3.43

Solution: From conservation of momentum



$$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2$$

or $$4\vec{V} = 2\left(6\cos 30\hat{i} + 6\sin 30\hat{j}\right) + 2\times 3\hat{j}$$

$$4\vec{V} = 2\left(\frac{6.\sqrt{5}}{2}\hat{i} + \frac{6}{2}\hat{j}\right) + 6\hat{j} = 6\sqrt{3}\hat{i} + 3\hat{j} + 6\hat{j} = 6\sqrt{3}\hat{i} + 9\hat{j}$$

$$\vec{V} = \frac{6\sqrt{3}}{4}\hat{i} + \frac{9}{4}\hat{j} \Rightarrow V = \frac{1}{4}\sqrt{108+81} = \frac{\sqrt{189}}{4}\ m/s = 3.43\ m/s$$

MSQ (Multiple Select Questions)

Ans. 38: (a) and (c)

Solution: $$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ..m_N\vec{v}_N}{m_1+m_2+...+m_N} \Rightarrow \vec{v}_{cm} = \frac{\sum_{i=1}^{N} m_i\vec{v}_i}{\sum_{i=1}^{N} m_i} \qquad \because \sum\vec{v}_i = 0$$

$$\Rightarrow \quad \sum\frac{d\vec{v}_i}{dt} = 0 \Rightarrow \frac{1}{m_i}\sum m_i\frac{d\vec{v}_i}{dt} = 0 \Rightarrow \frac{1}{m_i}\sum dF_i = 0 \Rightarrow \sum dF_i = 0$$

Hence, total force on the system is zero so linear momentum is conserve.

Ans. 39: (c) and (d)

Ans. 40: (b) and (c)

Solution: $F = A + Bt^2$

Impulse, $I = \int_0^t F dt = \int_0^t A + Bt^2 = At + \frac{Bt^3}{3}$

Acceleration, $a = \frac{F}{m} = \frac{A + Bt^2}{m} \Rightarrow \frac{dv}{dt} = \frac{A + Bt^2}{m} \Rightarrow \int_0^v dv = \int_0^t \frac{A + Bt^2}{m} dt \Rightarrow v = \frac{A}{m}t + \frac{B}{3m}t^3$

Ans. 41: (a), (c) and (d)

Solution: Velocity of centre of mass $v_{CM} = \frac{m \times v_0 + M \times 0}{m + M} \Rightarrow v_{CM} = \frac{m\omega_0}{m + M}$

For maximum compression, velocity of each mass become same let it is V

$$\Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}(m + M)v^2 \Rightarrow v = \sqrt{\frac{m}{(m + M)}}v_0$$

$$\Rightarrow \quad \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 - \frac{1}{2}\frac{m^2}{(m + M)}v_0^2 - \frac{1}{2}\frac{mM}{(m + M)}v_0^2$$

on solving, $x = u_0\sqrt{\frac{mM}{M + m}\frac{1}{k}}$

Initial $KE \quad = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}Mv_{cm}^2 = \frac{1}{2} \times \frac{m^2}{(m + M)^2}v_0^2[m + M] = \frac{1}{2}\frac{m^2}{(M + m)}v_0^2$

Ans. 42: (b), (c) and d)

Ans. 43: (a) and (c)

Solution: By definition $\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3}{m_1 + m_2 + m_3}$

$\vec{V}_{cm} = 0$ and $\vec{v}_1 \neq 0$ and $\vec{v}_2 \neq 0$ and the lines of $\vec{v}_1$ and $\vec{v}_2$ are different implies $m_1\vec{v}_1 + m_2\vec{v}_2 \neq 0$

hence $\vec{v}_3 = \frac{-(m_1\vec{v}_1 + m_2\vec{v}_2)}{m_3} \neq 0$

If centre of mass is at rest then $\vec{a}_{cm} = 0$ hence $\vec{F}_{\exp} = 0$, but this does not imply net work done by the forces acting on the system to be zero. Linear momentum (in the absence of external force) is conserved for collision of any number of particles.

Ans. 44: (a), (b) and (c)

Solution: Loss of gravitational potential energy = gain in spring potential energy

$$mg\sin\theta(d+0.25) = \frac{1}{2}kd^2 \Rightarrow 4\times 10\times\frac{1}{2}(d+0.25) = \frac{1}{2}\times 400\times(0.25)^2$$

$$d+0.25 = 0.375, d = 0.375\ m \qquad \text{or} \qquad d = 37.5\ cm$$

The speed of the block of the time of striking the spring is

$$mgd\sin 30^0 = \frac{1}{2}mv^2 \quad \Rightarrow 10\times 0.375\times\frac{1}{2} = \frac{1}{2}v^2 \Rightarrow v = \sqrt{3.75}\ m/s$$

At maximum compression potential energy of spring is

$$U = \frac{1}{2}kx^2 \qquad \text{or} \qquad U = \frac{1}{2}\times 400\times(0.25)^2 = 12.5\ J$$

Ans. 45: (b), (c) and (d)

Solution: Since one body is moving the net linear momentum of the system is non-zero. If both bodies come to rest the net linear momentum will be zero which is not possible.

Yes both bodies may move so as to conserve momentum.

If this is an elastic collision of two equal masses, then the moving body will stop and the body at rest will move.

If the stationary body is of infinite mass, it will remain at rest and the moving body will rebound.

Ans. 46: (b), (c) and (d)

Solution: In elastic collision, the kinetic energy is not conserved during the process of collision, but it is conserved just before and just after the collision.

In any type of collision (in the absence of external forces) the momentum is conserved.

Ans. 47: (a) and (c)

Solution: By definition $\quad W_c = -\Delta U \quad$ also $\quad W_c + W_{nc} = \Delta k$

Hence, $\quad -\Delta U + W_{nc} = \Delta k$

Ans. 48: (a)

Solution: Initially the system is at rest and finally the system is at rest. Applying the principle of conservation of mechanical energy for maximum compression.

Loss in gravitational potential energy = given in spring potential energy

$$\Rightarrow \quad 2mgx_{\max} = \frac{1}{2}kx x_{\max}^2. \quad \text{or} \quad x_{\max} = \frac{4mg}{k}$$

when the extension is $\frac{2mg}{k}$, applying it conservation of mechanical energy gives

$$2m.\frac{2mg}{k}g = \frac{1}{2}mv^2 + \frac{1}{2}.2mv^2 \Rightarrow \frac{4m^2g^2}{k} = \frac{3mv^2}{2} \quad \text{or } v = \sqrt{\frac{8mg}{3k}} = 2\sqrt{\frac{2mg}{3k}}$$

Ans. 49: (b) and (c)

Solution: $W_C = -\Delta U$, hence $W_C = -(-10+20) = -10\ J$

From work-kinetic energy theorem

Work done by all forces = charge in kinetic energy

$$\Rightarrow \quad W_{all} = 20 - 10 = 10\ J$$

Ans. 50: (a), (b) and (d)

Solution: Work done by conservation force $= -\Delta U$

$$W_{(1,1)\to(2,3)} = -3k,\ W_{(0,0)\to(1,1)} = -2k,\ W_{(1,1)\to(2,2)} = -2k,\ W_{(0,0)\to(3,3)} = -6k$$

Ans. 51: (a) and (c)

Solution: If the plank is smooth, applying the principle of conservation of mechanical energy

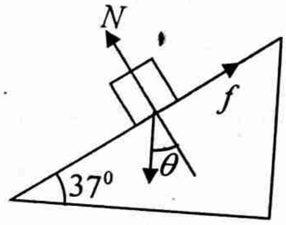
$$5mg\sin 37^0 = \frac{1}{2}mv^2 \quad \text{or} \quad 5\times 10\times\frac{3}{5} = \frac{1}{2}\times v^2 \Rightarrow v^2 = \sqrt{60} = 2\sqrt{15}\ m/s$$

If the plank has friction, work done by friction force = change in mechanical energy

$$\Rightarrow \quad (-\mu mg\cos 37^0)(5) = \left(\frac{1}{2}mv^2 + 0\right) - \left(0 + 5mg\sin 37^0\right)$$

$$\Rightarrow \quad -5\times(0.25)\times 10\times\frac{4}{5} = \frac{v^2}{2} - 5\times 10\times\frac{3}{5} \Rightarrow -10 = \frac{v^2}{2} - 30$$

$$\frac{v^2}{2} = 20.0 \quad \therefore \quad v^2 = 40 \Rightarrow v = 2\sqrt{10}\ m/s$$



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Ans. 52: (a), (b) and (d)

Solution: since the suitcase is at rest in both situation, hence

$$Wg + W_{you} = 0 \text{ (work } KE \text{ theory)}$$

$$W_{you} = -Wg = -(-mgh) = mgh$$

3. Central Force and Kepler's System

3.1 Central Force

In classical mechanics, the **central-force problem** is to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

In central force potential V is only function of r only a central force is always a conservative force; the magnitude F of a central force can always be expressed as the derivative of a time-independent potential energy

$$\vec{\nabla}\times\vec{F}=\frac{1}{r\sin\theta}\left(\frac{\partial F}{\partial\phi}\right)\hat{\theta}-\frac{1}{r}\left(\frac{\partial F}{\partial\theta}\right)\hat{\phi}=0$$

And the force F is defined as $F=-\frac{\partial V}{\partial r}\hat{r}$ (force is only in radial direction)

3.1.1 Angular Momentum and Areal Velocity

The equation of motion in polar coordinate is given by $m(\ddot{r}-r\dot{\theta}^2)=F_r$ and $m(r\ddot{\theta}+2\dot{r}\dot{\theta})=F_\theta$ but for central force

$$\tau=\vec{r}\times\vec{F}_r \qquad \Rightarrow \tau=r\hat{r}\times-\frac{\partial V}{\partial r}\hat{r},$$

External torque $\tau=0$ so angular momentum is conserve

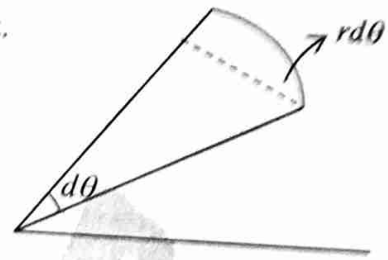
$m(r\ddot{\theta}+2\dot{r}\dot{\theta})=F_\theta$ but for central force $F_\theta=0$ so $m(r\ddot{\theta}+2\dot{r}\dot{\theta})=0\Rightarrow\frac{d(mr^2\dot{\theta})}{dt}=0$ means

Angular momentum of $mr^2\dot{\theta}=J=|\vec{r}\times\vec{p}|$ $\quad\Rightarrow\dot{\theta}=\frac{J}{mr^2}$

$\vec{r}\cdot\vec{J}=\vec{r}\cdot(\vec{r}\times\vec{p})=0$ $\Rightarrow$ $\vec{r}\perp\vec{J}$ hence position vector $\vec{r}$ is perpendicular to angular momentum vector $\vec{J}$ hence $\vec{J}$ is conserve , its magnitude and direction both are fixed so direction of $\vec{r}$ is also fixed.

So motion due to central force is confine into a plane and angular momentum $\vec{J}$ is perpendicular to that plane

Central force problem. Prove that Areal velocity is constant.



For the central force problem. Now $A = \frac{1}{2} r \cdot rd\theta$

Areal velocity $= \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$

$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$ It is given that $\dot{\theta} = \frac{J}{mr^2}$ so $\frac{dA}{dt} = \frac{J}{2m}$

Which means equal area will swept in equal time

3.1.2 Total Energy of the System

Hence total energy is not explicitly function of time t so $\frac{\partial E}{\partial t} = 0$ one can conclude that total energy in central potential is constant.

$$E = \frac{1}{2} m\dot{v}^2 + V(r) \text{ and Velocity } \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

So total energy $E = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$

$$= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2\dot{\theta}^2 + V(r) \text{ it is known } \dot{\theta} = \frac{J}{mr^2}$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) \; r > 0$$

$\Rightarrow E = \frac{1}{2} m\dot{r}^2 + V_{eff}$ Where $\frac{J^2}{2mr^2} + V(r)$ is identified as effective potential $V_{effective}$

3.1.3 Condition for Circular Orbit

From equation of motion in radial part $m(\ddot{r}-r\dot{\theta}^2)=f(r)$

For circular orbit or radius r_0, $r=r_0$ and $\ddot{r}=0$ $-r\dot{\theta}^2=f(r)\Rightarrow -\frac{J^2}{mr^3}=f(r)$ at $r=r_0$

Which can be also derived by $\left.\frac{\partial V_{effective}}{\partial r}\right|_{r=r_0}=0$ and $\dot{\theta}=\omega_0$ is identified as angular frequency in circular orbit.

radius $r=r_0$ of circular orbit is also identified as stable equilibrium point so $\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0}\geq 0$ .if some how particle of mass m is change its orbit without changing is angular momentum and new orbit is bounded then new orbit is identified as elliptical orbit . the angular frequency in new elliptical orbit is $\omega=\sqrt{\frac{\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0}}{m}}$

3.1.4 Equation of Motion and Differential Equation of Orbit

From equation of motion in radial part $m(\ddot{r}-r\dot{\theta}^2)=f(r)\Rightarrow \frac{md^2r}{dt^2}-\frac{J^2}{mr^3}=f(r)$ ....(1)

Where $J=mr^2\dot{\theta}\Rightarrow d\theta=\frac{J}{mr^2}dt\Rightarrow \frac{d}{dt}=\frac{J}{mr^2}\frac{d}{d\theta}$

$$\frac{d^2}{dt^2}=\left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right)=\left(\frac{J}{mr^2}\right)\frac{d}{d\theta}\left(\frac{J}{mr^2}\right)\frac{d}{d\theta}$$

.Substitute in (1)

$$\frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)-\frac{J^2}{mr^3}=f(r)\Rightarrow \frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{d(-1/r)}{d\theta}\right)-\frac{J^2}{mr^3}=f(r)$$

$$-\left(\frac{J^2}{m}\frac{1}{r^2}\frac{d^2(1/r)}{d\theta^2}+\frac{J^2}{mr^3}\right)=f(r)\Rightarrow -\frac{J^2}{mr^2}\left(\frac{d^2(1/r)}{d\theta^2}+\frac{1}{r}\right)=f(r)$$

Let $\frac{1}{r}=u\Rightarrow -\frac{J^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)=f\left(\frac{1}{u}\right)$ (differential equation of an orbit)

Example: Consider that the motion of a particle of mass m in the potential field $V(r)=\frac{kr^2}{2}$ If l is angular momentum.

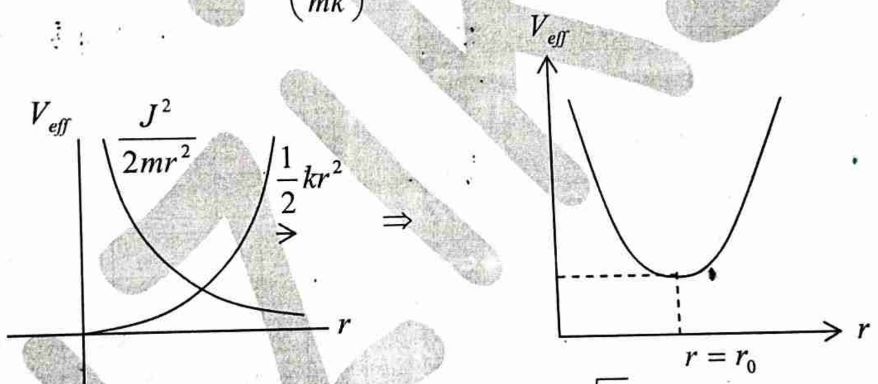
(a) What is effective potential (V_{eff}) of the system. plot V_{eff} vs r

(b) Find value of energy such that motion is circular in nature.

(c) If particle is slightly disturbed from circular orbit such that its angular remain constant. What will nature of new orbit? Find the angular frequency of new orbit in term of m, l, k.

Solution: (a) $V_{eff}=\frac{J^2}{2mr^2}+\frac{1}{2}kr^2$

(b) $\frac{dV_{eff}}{dr}=-\frac{J^2}{mr^3}+kr=0$ at $r=r_0$ so $r_0=\left(\frac{J^2}{mk}\right)^{1/4}$ and $J=m\omega_0 r_0^2$



for circular motion $m\omega_0^2 r_0=kr_0$ where r_0 is radius of circle $\omega_0=\sqrt{\frac{k}{m}}$

total energy $E=\frac{J^2}{2mr^2}+\frac{1}{2}kr^2=\frac{mkr_0^4}{2mr_0^2}+\frac{1}{2}kr_0^2$ $E=kr_0^2$ put $r_0=\left(\frac{J^2}{mk}\right)^{1/4}$ $E=J\sqrt{\frac{k}{m}}$

(c) orbit is elliptical in nature

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}=\frac{3J^2}{mr^4}+k=\frac{3J^2}{m\left(\frac{J^2}{mk}\right)}+k=4k$$

$$\omega=\sqrt{\frac{\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}}{m}}\Rightarrow\omega=\sqrt{\frac{4k}{m}}\Rightarrow\omega=2\sqrt{\frac{k}{m}}\Rightarrow 2\omega_0$$

Example: A particle of mass m moves under the influence of an attractive central force $f(r)$.

(a) What is condition that orbit is circular in nature if J is the angular momentum of particle

(b) If force is in form of $f(r)=\frac{-k}{r^n}$ determine the maximum value of n for which the circular orbit can be stable.

Solution: (a) if $V_{eff}=\frac{J^2}{2mr^2}+V(r)$ for circular stable orbit $\frac{\partial V_{eff}}{\partial r}=0,\ \frac{\partial^2 V_{eff}}{\partial r^2}>0$

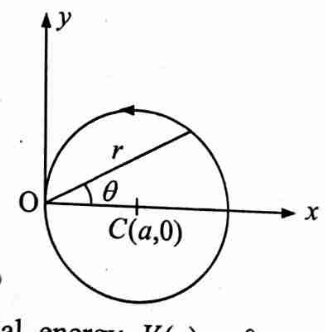
(b) $f(r)=\frac{-k}{r^n}$ for circular motion $\frac{\partial V_{eff}}{\partial r}=0\Rightarrow -\frac{J^2}{mr^3}+\frac{\partial V}{\partial r}=0$

It is given $\frac{\partial V}{\partial r}=-f(r)$ if $f(r)=\frac{-k}{r^n}\Rightarrow \frac{\partial V}{\partial r}=\frac{k}{r^n}$

$$-\frac{J^2}{mr^3}+\frac{k}{r^n}=0\Rightarrow \frac{k}{r^n}=\frac{J^2}{mr^3}$$

$$\frac{\partial^2 V_{eff}}{\partial r^2}>0\Rightarrow \frac{3J^2}{mr^4}-\frac{nk}{r^{n+1}}>0\Rightarrow \frac{3J^2}{mr^4}-\frac{n}{r}\cdot\frac{J^2}{mr^3}>0 \text{ so } n<3$$

Example: A particle of mass m and angular momentum l is moving under the action of a central force $f(r)$ along a circular path of radius a as shown in the figure. The force centre O lies on the orbit.



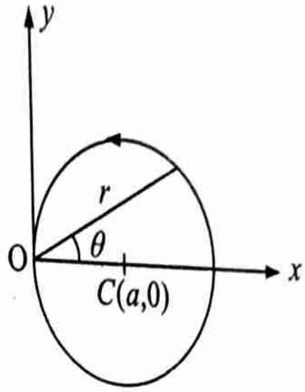
(a) Given the orbit equation in a central field motion.

$$\frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2u^2}f\text{, where } u=\frac{1}{r}.$$

Determine the form of force in terms of l, m, a and r.

(b) Calculate the total energy of the particle assuming that the potential energy $V(r)\to 0$ as $r\to\infty$.

Solution: (a) from the fig $r = 2a\cos\theta$ $\qquad \dfrac{1}{r} = \dfrac{\sec\theta}{2a}$



$$-\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{2am}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta\right)+\frac{\sec\theta}{2a}\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{2am}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta+\sec\theta\right)\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^3\theta}{4a^2m}\left[\tan^2\theta+\sec^2\theta+1\right] = f\left(\frac{1}{u}\right) \Rightarrow -\frac{2J^2\sec^5\theta}{4a^2m} = f\left(\frac{1}{u}\right) \Rightarrow f(r) \propto \frac{1}{r^5}$$

(b) $E = \dfrac{m\dot{r}^2}{2} + \dfrac{J^2}{2mr^2} + V(r)$ $\;,r\to\infty,\; V(r)\to 0\quad \dfrac{J^2}{2mr^2}\to 0\quad r\to\infty$

$E = \dfrac{m\dot{r}^2}{2}$ and $r = 2a\cos\theta$ and $\dot{r} = -2a\sin\theta\dot{\theta}$, $\;\dot{\theta} = \dfrac{J^2}{mr^2}$ as $r\to\infty$

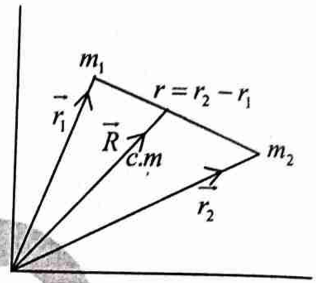
hence $\;\dot{\theta} = \dfrac{J^2}{mr^2}\to 0$ so $\;\dot{r}\to 0$ so $\;E = 0$

3.2 Two Body Problem

Reduction of two body central force problem to the equivalent one body problem

A system of two particles of mass m_1 and m_2 whose instantaneous position vectors of inertial frame with origin O are r_1 and r_2 respectively.

Vector m_2 relative to m_1 is $\vec{r} = \vec{r}_2 - \vec{r}_1$



The potential energy is V is only function for distance between the particles

So $V = V(|\vec{r}_2 - \vec{r}_1|)$

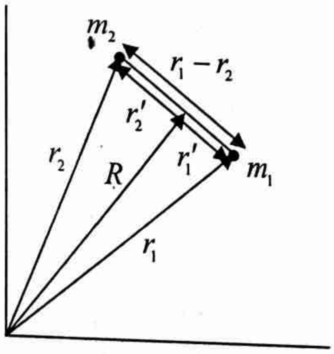
Total energy of the system is given by in lab frame

$$E = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 + V(r_2 - r_1)$$

Let the position vectors of m_1 and m_2 be r_1 and r_2. The position vector of the center of mass, measured from the same origin, is

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

The center of mass lies on the line joining m_1 and m_2. To show this, suppose first that the tip of R does not lie on the line, and consider the vectors r_1', r_2' from the tip of R to m_1 and m_2. From the sketch we see that



$$\vec{r}_1' = \vec{r}_1 - \vec{R}$$

$$\vec{r}_2' = \vec{r}_2 - \vec{R}$$

$$\vec{r}_1' = \vec{r}_1 - \frac{m_1\vec{r}_1}{m_1+m_2} - \frac{m_2\vec{r}_2}{m_1+m_2} = \frac{m_2}{m_1+m_2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2' = \vec{r}_2 - \frac{m_1\vec{r}_1}{m_1+m_2} - \frac{m_2\vec{r}_2}{m_1+m_2} = -\left(\frac{m_1}{m_1+m_2}\right)(\vec{r}_1 - \vec{r}_2)$$

r_1' and r_2' are proportional to $r_1 - r_2$, the vector from m_1 to m_2. Hence r_1' and r_2' lie along the line joining m_1 and m_2 as shown. Furthermore,

$$r_1' = \frac{m_2}{m_1+m_2}|r_1 - r_2| = \frac{m_2}{m_1+m_2}r \text{ and } r_2' = \frac{m_1}{m_1+m_2}|r_1 - r_2| = \frac{m_2}{m_1+m_2}r$$

$E=\frac{1}{2}m_1\left|\vec{\dot{r}}_1\right|^2+\frac{1}{2}m_2\left|\vec{\dot{r}}_2\right|^2+V\left(r_2-r_1\right)$. The total energy is transformed

$$E=\frac{1}{2}\left(m_1+m_2\right)\left|\dot{R}\right|^2+\frac{1}{2}\frac{m_1m_2}{m_1m_2}\left|\vec{\dot{r}}\right|^2-V\left(r\right)$$

Centre of mass moving with constant momentum and equation of motion for three generalized co-ordinates or will not terms in R and $\dot{R}$. Hence discuss the motion of system one can ignore $\frac{1}{2}\left(m_1+m_2\right)\dot{R}^2$.

So Energy in centre of mass reference frame reduce to $E=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}\left|\vec{\dot{r}}\right|^2+V\left(r\right)$

Where $\mu=\frac{m_1m_2}{m_1+m_2}$ is reduce to one body system in centre of mass reference frame.

3.2.1 Kepler's Problem

Kepler discuss the orbital motion of the sun and Earth system under the potential $V(r)=-\frac{k}{r}$ where $k=Gm_sm_e$ it is given m_s and m_e is mass of Sun and Earth .Although kepler discuss sun and earth system but method can be used for any system which is interacting with potential $V(r)=-\frac{k}{r}$

The reduce mass for sun and earth system is $\mu=\frac{m_em_s}{m_e+m_s}\Rightarrow\frac{m_e}{1+\frac{m_e}{m_s}}=m_e\ ,m_s>>m_e$

Let us assume mass of earth $m_e=m$

3.2.2 Kepler's First Law: Every planet (earth) moves in an elliptical orbit around the sun, the sun is being at one of the foci. Where sun and earth interact each other with potential $V(r)=-\frac{k}{r}$ we solve equation of motion in center of mass reference frame with reduce mass $\mu=m_e=m$

3.2.3 Equation of Motion:

$$m\ddot{r}-mr\dot{\theta}^2=-\frac{k}{r^2} \quad \text{put } \dot{\theta}=\frac{J}{mr^2}$$

$$m\ddot{r}-\frac{l^2}{mr^3}=-\frac{k}{r^2}$$

Equation or orbit is given by $\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right]=-f\left(\frac{1}{u}\right)$

$$f(r)=-\frac{k}{r^2} \quad f\left(\frac{1}{u}\right)=-ku^2 \Rightarrow \frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right]=+ku^2 \Rightarrow \frac{d^2u}{d\theta^2}+u=\frac{ku^2m}{J^2u^2}$$

$$\frac{d^2u}{d\theta^2}+\left(u-\frac{km}{J^2}\right)=0 \text{ put } u-\frac{km}{J^2}=y \text{ so } \frac{d^2u}{d\theta^2}=\frac{d^2y}{d\theta^2}$$

The equation reduce to $\frac{d^2y}{d\theta^2}+y=0$

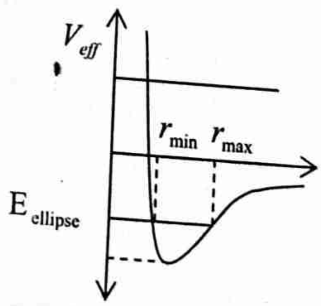
The solution of equation reduce to $y=A\cos\theta$ $u-\frac{km}{J^2}=A\cos\theta \Rightarrow u=\frac{km}{J^2}+A\cos\theta$

$$\frac{1}{r}=\frac{km}{J^2}+A\cos\theta \Rightarrow \frac{\frac{J^2}{km}}{r}=1+\left(\frac{AJ^2}{km}\right)\cos\theta$$

Put $\frac{J^2}{km}=l$ and $e=\frac{AJ^2}{km}$ the equation reduce to $\frac{l}{r}=1+e\cos\theta$ which is equation of conics where l is latus rectum and e is eccentricity.

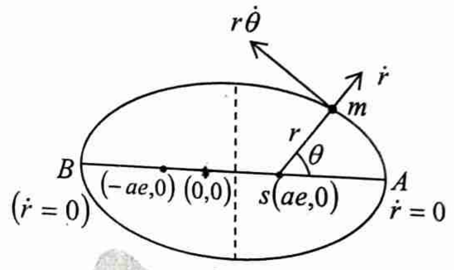
In a central force potential which is interacting with potential $V(r)=-\frac{k}{r}$ can be any conics section depending on eccentricity e.

Now we are discuss the case specially of elliptical orbit as Kepler discuss for planetary motion .



Total energy $E=\frac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}-\frac{k}{r}$ where $V_{effective}=\frac{J^2}{2mr^2}-\frac{k}{r}$ with constant angular momentum J.

If one will plot $V_{effective}$ Vs r it is clear that for negative energy the orbit is elliptical which is shown in figure. Earth is orbiting in elliptical path with sun as focus as shown in figure.



Let equation of this ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b=a\sqrt{1-e^2}$

minimum value of r is $(a-ae)$ value and maximum value of r is $(a+ae)$ $r_{max}+r_{min}=2a$ from plot of effective potential it is identified r_{max} and r_{min} is the turning point so so at these points radial velocity is zero

$E=\frac{J^2}{2mr^2}-\frac{k}{r}\Rightarrow 2mEr^2+2mkr-J^2=0$ given equation is quadratic in term of r for their root at r_{max} and r_{min}. Using theory of quadratic equation sum of root $r_{max}+r_{min}=-\frac{2mk}{2mE}$

$\Rightarrow E=-\frac{k}{2a}$ which is negative .

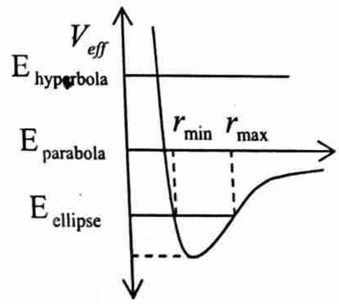
3.2.4 Relationship between Energy and Eccentricity.

For central potential $V(r)=-\frac{k}{r}$ the solution of orbit is $\frac{l}{r}=1+e\cos\theta$ with $l=\frac{J^2}{km}$

The energy is given by $E=\frac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}-\frac{k}{r}$

the solution of orbit is $\frac{l}{r}=1+e\cos\theta$ with $l=\frac{J^2}{km}$

so $\frac{-l}{r^2}\dot{r}=-e\sin\theta\dot{\theta}$ where $\dot{\theta}=\frac{J}{mr^2}$ so $\dot{r}=\frac{eJ\sin\theta}{ml}$,



after putting the value of $\frac{l}{r}=1+e\cos\theta$ and $\dot{r}=\frac{eJ\sin\theta}{ml}$ with $l=\frac{J^2}{km}$ in equation of energy, one will get $e = \sqrt{1 + \frac{2EJ^2}{mk^2}}$

the condition on energy for possible nature of orbit for potential

$E > 0$ $\quad e > 1$ Hyperbola

$E = 0$ $\quad e = 1$ Parabola

$E < 0$ $\quad e < 1$ Ellipse

$E = -\frac{mk^2}{2J^2}$ $\quad e = 0$ circle

3.2.5 Kepler's Second Law:

Equal Area will swept in equal time or Areal velocity is constant.

$\frac{dA}{dt} = \frac{J}{2m}$ (which is derived earlier)

3.2.6 Kepler's Third Law:

The square of time period(T) of revolution in elliptical orbit is proportional to cube of semi major axis a ie $T^2 \propto a^3$

$$\frac{dA}{dt} = \frac{J}{2m} \Rightarrow \int dA = \frac{J}{2m}\int dt \Rightarrow \pi ab = \frac{J}{2m}T \ (\pi ab \text{ is the area of ellipse})$$

$$\pi a \cdot a\sqrt{1-e^2} = \frac{J}{2m}T \text{ it is given } e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \quad \text{and} \quad E = -\frac{k}{2a}$$

$$e^2 = 1 - \frac{2kJ^2}{2amk^2} \Rightarrow 1 - e^2 = \frac{2kJ^2}{2amk^2}$$

$$T^2 = \frac{4m^2}{J^2}\pi^2 a^2 \cdot a^2 (1-e^2) \text{ put value of } 1 - e^2 = \frac{2kJ^2}{2amk^2} = \frac{J^2}{amk}$$

$$T^2 = \frac{4m^2}{J^2}\pi^2 a^4 (1-e^2) \Rightarrow T^2 = \frac{4m^2}{J^2}\pi^2 a^4 . \frac{J^2}{mak} = \frac{4\pi^2 ma^3}{k}$$

$$\Rightarrow T^2 = \frac{4\pi^2 ma^3}{k} \text{ if } k = Gm_s m \text{ then } \Rightarrow T^2 = \frac{4\pi^2 a^3}{Gm_s} \text{ where } m_s \text{ is mass of the sun.}$$

Example: Given a classical model of tritium atom with atom with nucleus of charge +1 and a single.

Electron in a circular orbit of radius r_0 suddenly the nucleus emits a negatron and changes to charge +2 (the emitted negatron escapes rapidly and we can forget about it.) the orbit suddenly has a new situation.

(a) Find the ratio of the electron's energy after to before the emission of the negatron

(b) Describe qualitatively the new orbit

(c) Find the distance of closest and the farthest approach for the new orbits in units of r_0

Solution: (a) As the negatron leaves the system rapidly, we can assume that its leaving has no effect on the position and kinetic energy of the orbiting electron.

From the force relation for the electron,

$$\frac{mv_0^2}{r_0} = \frac{e^2}{4\pi\varepsilon_0 r_0^2}, \text{ and we find its kinetic energy } \frac{mv_0^2}{2} = \frac{e^2}{8\pi\varepsilon_0 r_0}$$

and its total mechanical energy

$$E_1 = \frac{mv_0^2}{2} - \frac{e^2}{4\pi\varepsilon_0 r_0} = -\frac{e^2}{8\pi\varepsilon_0 r_0}$$

before the emission of the negatron. After the emission the kinetic energy of the electron is still $\frac{e^2}{8\pi\varepsilon_0 r_0}$, while its potential energy suddenly changes to $\frac{-2e^2}{4\pi\varepsilon_0 r_0} = \frac{-e^2}{2\pi\varepsilon_0 r_0}$

Thus after the emission the total mechanical energy of the orbiting electron is

$$E_2 = \frac{mv_0^2}{2} - \frac{2e^2}{4\pi\varepsilon_0 r_0} = \frac{-3e^2}{8\pi\varepsilon_0 r_0}, \text{ giving } \frac{E_2}{E_1} = 3.$$

In other words, the total energy of the orbiting electron after the emission is three times as large as that before the emission.

(b) As $E_2 = \frac{-3e^2}{8\pi\varepsilon_0 r_0}$, the condition equation (i) for circular motion is no longer satisfied and the new orbit is an ellipse.

(c) Conservation of energy gives

$$\frac{-3e^2}{8\pi\varepsilon_0 r_0}=\frac{-e^2}{2\pi\varepsilon_0 r_0}+\frac{m\left(\dot{r}^2+r^2\dot{\theta}^2\right)}{2}$$

At positions where the orbiting electron is at the distance of closest or farthest approach to the atom, we have $\dot{r}=0$, for which

$$\frac{-3e^2}{8\pi\varepsilon_0 r_0}=\frac{mr^2\dot{\theta}^2}{2}-\frac{e^2}{2\pi\varepsilon_0 r}=\frac{J^2}{2mr^2}-\frac{e^2}{2\pi\varepsilon_0 r}$$

Then with

$$J^2=m^2v_0^2r_0^2=\frac{me^2r_0}{4\pi\varepsilon_0}$$

the above becomes

$$3r^2-4r_0r+r_0^2=0$$

with solutions $r=\frac{r_0}{3}$, $\quad r=r_0$

Hence the distances of closest and farthest approach in the new orbit are respectively

$$r_{\min}=\frac{1}{3},\qquad r_{\max}=1$$

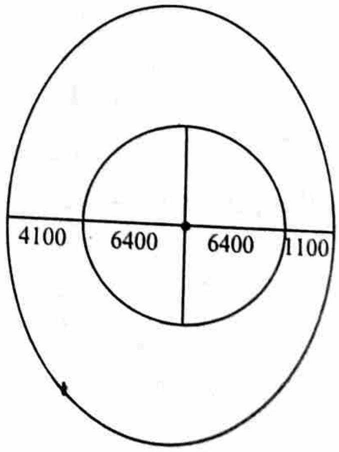
Example: A satellite of mass $m=2000$ kg is in elliptical orbit about earth. At perigee it has an altitude of 1,100 km and at apogee it has altitude 4,100 km. assume radius of the earth is $R_e=6,400$ km. it is given $GmM_e=8\times10^{17}\,J.m$

(a) What is major axis of the orbit?

(b) What is eccentricity of the orbit?

(c) What is angular momentum of the satellite?

(d) How much energy is needed to fix satellite in to orbit from surface of earth?



Solution: $r_{max} = 4100 + 6400 = 10500\,km$

$r_{min} = 1100 + 6400 = 7500\,km$

(a) $r_{max} + r_{min} = 2a \Rightarrow 18000 = 2a \Rightarrow a = 9000\ km$

$$e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$

(b) $\Rightarrow e = \dfrac{10500 - 7500}{10500 + 7500} = \dfrac{3000}{18000} = \dfrac{1}{6} \Rightarrow \quad e = \dfrac{1}{6}$

It is given $k = 8 \times 10^{17}\ J \cdot m$

(c) $E = -\dfrac{k}{2a} = -\dfrac{8 \times 10^{17}}{18000 \times 10^3}$ $, E_f = -\dfrac{8}{18} \times 10^{11} J = -4.5 \times 10^{10} J$

$$e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \Rightarrow \left(\frac{1}{6}\right)^2 = 1 + \frac{2EJ^2}{mk^2}$$

$$\frac{1}{36} - 1 = \frac{2EJ^2}{mk^2} \Rightarrow 140 \times 10^{26} = J^2 \Rightarrow J = \sqrt{140} \times 10^{13} = 1.2 \times 10^{14}\ kgm/\sec^2$$

(d) When satellite is at surface of earth,

$R = 6400\ Km$

$$E_i = -\frac{GMm}{R} = \frac{-8 \times 10^{17}}{6400 \times 10^3} = \frac{-10^{17}}{800 \times 10^3} = \frac{-10^{12}}{8} = -12.5 \times 10^{10}$$

$$E_f = -\frac{GMm}{2a} = -4.5 \times 10^{10} J \Rightarrow \Delta E = E_f - E_i = 8 \times 10^{10} J$$

Example: For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that perihelion distance of the parabola is one-half the radius of the circle.

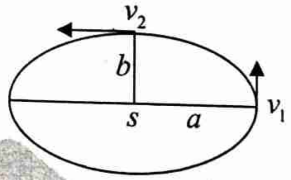
Solution: For Kepler's problem $\dfrac{l}{r} = 1 + \cos\theta$, for circular orbit $e = 0 \Rightarrow \dfrac{l}{r_c} = 1$ and for parabola

$e = 1\ \dfrac{l}{r} = 1 + \cos\theta \qquad r_p$ is minimum when $\cos\theta$ is maximum.

$$\frac{l}{r_p} = 2 \text{ and } \frac{l}{l_c} = 1 \Rightarrow \frac{r_p}{r_c} = \frac{1}{2}$$

Example: A planet of mass m moves in the inverse square central force field of the Sun of mass M. If the semi-major and semi-minor axes of the orbit are a and b, respectively, the find total energy of the planet by assuming sun is at center of ellipse .

Solution: Assume Sun is at the centre of elliptical orbit.



Conservation of energy $\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$

Conservation of momentum $L = mv_1a = mv_2b$

$$v_2 = v_1\left(\frac{a}{b}\right)$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2 - v_1^2\frac{a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2-a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}$$

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{a} = GMm\frac{b}{a}\frac{1}{(b+a)} - \frac{GMm}{a}$$

$$= \frac{GMm}{a}\left(\frac{b}{(b+a)} - 1\right) = \frac{GMm}{a}\left(\frac{b-b-a}{(b+a)}\right) = -\frac{GMm}{(b+a)}$$

MCQ (Multiple Choice Questions)

Q1. A planet moves round the sun. At a point P, it is closest to the sun at distance r_1 and has speed v_1 At another point Q, when it is farthest from the sun at a distance r_2, what is its speed ?

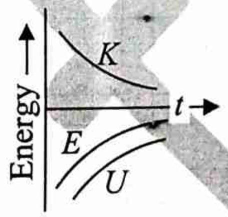
(a) $\frac{r_1^2 v_1}{r_2^2}$ (b) $\frac{r_1 v_1}{r_2}$ (c) $\frac{r_2^2 v_2}{r_1^1}$ (d) $\frac{r_2 v_1}{r_1}$

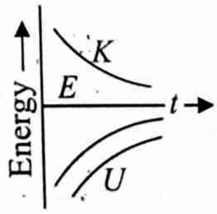
Q2. Two masses m and M are initially at rest at infinite distance apart They approach each other due to gravitational interaction What is their speed of approach at the instant when they are at a distance d apart?

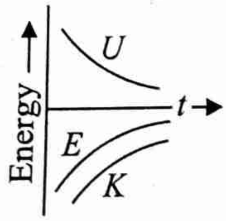
(a) $\left(\frac{2G\left(M^2+m^2\right)^{\frac{1}{2}}}{d}\right)^{\frac{1}{2}}$ (b) $\left(\frac{2GMm}{d(M+m)}\right)^{\frac{1}{2}}$

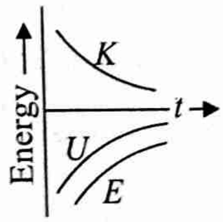
(c) $\left(\frac{2G(M+m)}{d}\right)^{\frac{1}{2}}$ (d) $\left(\frac{GMm}{d(M+m)}\right)^{\frac{1}{2}}$

Q3. Which one of t1 following diagrams correct depicts the variation of kinetic energy (K) potential energy (U) and total energy (E) of a body in circular planetary motion? (r is the radius of the circle).

(a) 

(b) 

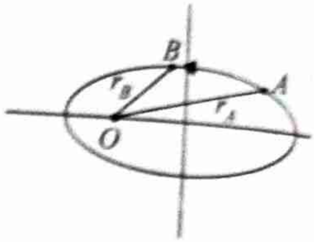
(c) 

(d) 

Q4. A planet of mass m moves in a circular orbit of radius r_0 in the gravitational potential $V(r)=-\frac{k}{r}$, where k is a positive constant. The orbit angular momentum of the planet is

(a) $2r_0km$ (b) $\sqrt{2r_0km}$ (c) r_0km (d) $\sqrt{r_0km}$

Q5. The figure given shows a satellite of mass m orbiting elliptically about the centre O of the Earth. The mass of the Earth is M. The work done on the satellite as it moves from A to B in its orbit is

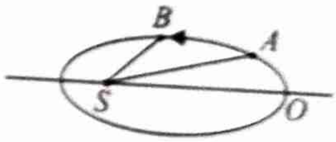


(a) $\frac{GMm}{r_B}\left(1-\frac{r_B}{r_A}\right)$ (b) $\frac{GMm}{r_A}\left(1-\frac{r_A}{r_B}\right)$ (c) $GMM\left(1-\frac{r_B}{r_A}\right)$ (d) $\frac{GMm}{r_B^2-r_A^2}$

Q6. If the gravitational force is assumed to vary inversely as the nth power of distance r, then how does the time period of a, planet the sun depend upon r?

(a) r^n (b) r^{-n} (c) $r^{(n+1)/2}$ (d) $r^{(n-1)/2}$

Q7. The figure below shows the motion of a planet around the sun in an elliptical orbit with sun as the focus. Areas SOA and SAB shown in the figure can be assumed to be equal. If t_1 and t_2 represent the times for the planet to move from O to A and A to B respectively; then



(a) $t_1 < t_2$ (b) $t_1 > t_2$ (c) $t_1 = t_2$ (d) None of these

Q8. If R is the radius of the earth ρ is mean density and G the gravitational constant, then the earth's surface potential will be nearly equal to

(a) $\frac{\pi\rho G}{R^2}$ (b) $\frac{4}{3}\pi R^3\rho G$ (c) $\frac{4}{3}\pi\rho R^2 G$ (d) $\frac{4}{3}\pi\frac{R}{\rho}G$

Q9. Three objects S_1, S_2 and S_3 having same mass m are in different elliptic orbits of same semi-major axis, about an object of mass M. If the eccentricities of the orbits are e_1, e_2 and e_3 respectively such that $e_1 < e_2 < e_3$ and if E_1, E_2 and E_3 are their respective mechanical energies, then which one of the following is correct?

(a) $E_1 < E_2 < E_3$ (b) $E_1 > E_2 > E_3$

(c) $E_1 = E_2 = E_3$ (d) E_1, E_2 and E_3 cannot be compared

Q10. A particle is moving in an inverse square field If the total energy of the particle is positive, then trajectory of particle is

(a) circular (b) elliptical (c) parabolic (d) hyperbolic

Q11. In a central force field, the trajectory of a particle of mass m and angular momentum J in plane polar coordinates is given by,

$$\frac{1}{r} = \frac{m}{J^2}(1 + \varepsilon \cos\theta)$$

where, ε is the eccentricity of the particle's motion. Which one of the following choice for ε gives rise to a parabolic trajectory?

(a) $\varepsilon = 0$ (b) $\varepsilon = 1$ (c) $0 < \varepsilon < 1$ (d) $\varepsilon > 1$

Q12. Match the following List I with List II.

List I	List II
A. Eccentricity $e > 0$, energy $(E) > 0$	1. Circular orbit
B. Eccentricity $e = 1$ $E = 0$	2. Elliptical orbit
C. $e < 1, E < 0$	3. Parabolic orbit
D. $e = 0$, $E = -\frac{mk^2}{2J^2}$	4. Hyperbolic orbit

	A	B	C	D
(a)	1	2	3	4
(b)	2	1	3	4
(c)	3	4	1	2
(d)	4	3	2	1

Q13. A satellite in a circular orbit about the earth has a kinetic energy E_K. What is the minimum amount of energy to be added, so that it escape from the earth?

(a) $\frac{E_K}{4}$ (b) $\frac{E_K}{2}$ (c) E_K (d) $2E_K$

Q14. Two particles of identical mass move in circular orbits under a central potential $V(r)=\frac{1}{2}kr^2$. Let l_1 and l_2 be the angular momenta and r_1, r_2 be the radii of the orbits respectively. If $\frac{J_1}{J_2}$, the value of $\frac{r_1}{r_2}$ is:

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 2 (d) $1/2$

Q15. A planet of mass m moves in the gravitational field of the Sun (mass M). If the semi-major and semi-minor axes of the orbit are a and b respectively, the angular momentum of the planet is

(a) $\sqrt{2GMm^2(a+b)}$ (b) $\sqrt{2GMm^2(a-b)}$ (c) $\sqrt{\frac{2GMm^2ab}{a-b}}$ (d) $\sqrt{\frac{2GMm^2ab}{a+b}}$

NAT (Numerical Answer Type)

Q16. If a planet revolves round the sun in a circular orbit of radius a with a period of revolution T, then (k being a positive constant) if $T\propto a^{\alpha}$ then will value of α is given by ....

Q17. Consider a satellite going round the earth in a circular orbit at a height of $2R$ from the surface of the earth, where R is the radius of the earth. The speed of the satellite is .... $\left(\frac{gR}{\beta}\right)^{\alpha}$ then value of α......... and value of β is ....

Q18. A body of mass m moves under the action of a central force with potential $V(r)=Ar^3\,(A>0)$ where r about the origin. Then kinetic energy will the orbit be a circle of radius R is αAR^3 what is value of α ________?

Q19. A comet moves in an elliptical orbit with an eccentricity of $e=0.20$ around a star. The distance between the perihelion and the aphelion is $1.0\times10\,km$. If the speed of the comet at perihelion $60\,km/s$ then the speed of the comet at the aphelion is.............km/s

Q20. The acceleration due to gravity (g) on the surface of Earth is approximately 2.6 times that on the surface of Mars. Given that the radius of Mars is about one half the radius of Earth, the ratio of the escape velocity on Earth to that on Mars is approximately ....

MSQ (Multiple Select Questions)

Q21. A satellite is in elliptical orbit the earth (radius $=6400\ km$). At perigee it has an altitude of $1100\ km$ and the apogee its apogee its altitude is $4100\ km$.

(a) Then The major axis of the orbit is $18,000\ km$

(b) Then The major axis of the orbit is $9,000\ km$

(c) The eccentricity is 0.16

(d) Eccentricity is 0.33

Q22. A planet (mass M_P) is moving around the sun (mass M_S) in an elliptical orbit of semimajor axis a. Total energy of the planet is

(a) $-\frac{GM_PM_S}{4a}$ (b) $-\frac{2GM_PM_S}{a}$ (c) $\frac{GM_PM_S}{2a}$ (d) $-\frac{GM_PM_S}{2a}$

Q23. Consider the following statements for a particle moving in an elliptic orbit under the influence of a central force.

(a) The radius vector covers equal area in equal time.

(b) The motion takes place in a plane.

(c) The angular momentum is a constant of motion.

(d) The energy remains constant and negative

Q24. Consider the following expressions n respect of earth's motion around the sun in a circular orbit of radius r with a linear speed v and angular speed ω. Then which of the following expressions are correct?

(a) v is proportional to $r^{\frac{-1}{2}}$

(b) v is proportional to $r^{\frac{-3}{2}}$

(c) $\omega\, v$ is proportional to $r^{\frac{-5}{2}}$

(d) ω is proportional to $r^{\frac{-3}{2}}$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Solution: closest from the sun $= r_1$, speed $= v_1$

Farthest from the sun $= r_2$

According to conservation of momentum $mr_1v_1 = mr_2v_2$

$$v_2 = \frac{r_1 v_1}{r_2}$$

Ans. 2: (c)

Solution: if reduce mass $\mu = \frac{mM}{m+M}$ then $\frac{1\mu v^2}{2} = 0 - \left(-\frac{GMm}{d}\right) \Rightarrow v^2 = \frac{2G(M+m)}{d}$ (acceleration due to gravity upward from the earth)

$$v = \sqrt{\frac{2G(M+m)}{d}}$$

Ans. 3: (a)

Solution: For circular motion

$$\frac{mv^2}{r} = \frac{k}{r^2} \Rightarrow K = \frac{mv^2}{2} = \frac{k}{r^2} \quad E = K + U \quad U = -\frac{k}{r} \quad E = -\frac{k}{2r}$$

Ans. 4: (d)

Solution: $V_{effctive} = \frac{J^2}{2mr^2} - \frac{k}{r} \Rightarrow \frac{dV_{effect}}{dr} = -\frac{J^2}{mr^3} + \frac{k}{r^2} = 0$ at $r = r_0$

so $J = \sqrt{r_0 km}$

Ans.5: (a)

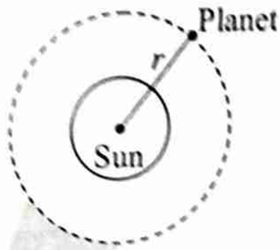
Solution: By conservation law of energy

$$\Rightarrow \quad dW = (PE)_A - (PE)_B \quad \text{[By equation (i)]}$$

$$= \left(-\frac{GMm}{r_A}\right) - \left(-\frac{GMm}{r_B}\right) = \frac{GMm}{T_B} - \frac{GMm}{T_A} = \frac{GMm}{r_B}\left[1 - \frac{r_B}{r_A}\right]$$

Ans. 6: (c)

Solution: If a planet revolve with velocity v radius r then the centripetal force it gives as



$$F = \frac{mv^2}{r} \qquad \text{(i)}$$

The attractive M between sun and planet is given as

$$F = -\frac{k}{r^n} \qquad \text{(ii)}$$

k being a constant

By equations (i) and (ii) $\frac{k}{r^n} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{k}{mr^{n-1}} \Rightarrow v = \left(\frac{k}{mr^{n-1}}\right)^{\frac{1}{2}}$

Hence, time period $= \frac{2\pi r}{v} = \frac{2\pi r}{\left(\frac{k}{mr^{n-1}}\right)^{\frac{1}{2}}} = 2\pi r\left(\frac{mr^{n-1}}{k}\right)^{\frac{1}{2}} = \frac{2\pi r . r^{\frac{n-1}{2}}\sqrt{m}}{\sqrt{k}} = 2\pi\sqrt{\frac{m}{k}}\, r^{\frac{n+1}{2}}$

Hence, time period is proportional to $r^{\frac{n+1}{2}}$

Ans. 7: (c)

Solution: Kepler's second law states that the areal velocity swept out by a radius drawn from sun to a planet is constant, i.e.,

$$\frac{dA}{dt} = \text{constant}$$

Ans. 8: (c)

Solution: Potential at earth surface $V = -\frac{GM_e}{R_e}$ (i)

Mass of earth $M_e = \text{volume} \times \text{density} = \frac{4}{3}\pi R_e^3 \rho$ (ii)

By equation (i) and (ii) becomes

$$V = -\frac{G}{R_e}\left(\frac{4}{3}\pi R_e^3 \rho\right) = -\frac{4}{3}\pi R^2 \rho G$$