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CHAPTER :- 1.

FORMATION OF PARTIAL DIFFERENTIAL EQUATION)

INTRODUCTION :- Partial differential equations are used to mathematically formulate and this aid the solution of physical and other problems involving functions of several variables. Partial differential equations are to great use in Physics, Applied Mathematics and other branches of science. In this chapter we shall learn to form partial differential equation by two methods viz, by eliminating arbitrary constants and by eliminating arbitrary functions.

PARTIAL DIFFERENTIAL EQUATION :-

A differential equation involving partial derivatives of dependent variable with respect to two or more independent variable is called a partial differential equations.

In other words, an equation containing an unknown function of two or more variable and its partial derivatives with respect to these variables is called a partial differential equation. Usually, in case of two variables, we take x and y as independent variables and z as dependent variable.

An equation of the form

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}\right) = 0$$

is a partial differential equation in two variables.

ORDER AND DEGREE OF A PARTIAL DIFFERENTIAL EQUATION :→

The order of a partial differential equation is the order of the highest-order partial derivatives occuring in it.

The degree of a partial differential differential equation is the greatest exponent of the highest order partial derivatives occuring in it, after making it free from radicals and fractions as far as derivatives are concered.

(i) $\left(\frac{\partial z}{\partial x}\right)^3 + 5\left(\frac{\partial z}{\partial y}\right)^4 = z$ is a partial differential equation of first order and fourth degree.

ii) $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2$ is a partial differential equation of first order and first degree.

LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATION :→

A partial Differential equation is said to be a linear if the dependent variable and its partial derivatives occur only in first degree and are not multiplied together. Otherwise the equation is called non-linear partial differential equation.

FORMATION OF EQUATION BY THE ELIMINATION OF ARBITRARY CONSTANT :→

Let $f(x, y, z, a, b) = 0$ — ①

be a relation, where z is regarded as a function of two independent variables x and y, and a and b denote the arbitrary constant.

Differentiating ① partially w.r.t. x

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p = 0 \quad \text{— ②}$$

Diff. ① w.r.t. y, we have

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q = 0 \quad \text{— ③}$$

Eliminating a and b from ①, ②, ③, we obtain a partial differential equation of the form $f(x, y, z, p, q) = 0$ — ④

which is a partial equation of first order.

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Example 71. Find the differential equation by eliminating the arbitrary constant λ and A form the equation
$$Z = Ae^{-\lambda^2 t}\cos\lambda x.$$

Solution → The given equation is
$$Z = Ae^{-\lambda^2 t}\cos\lambda x \quad \text{--- (1)}$$

Differentiating (1) w.r.t. x, we have
$$\frac{\partial Z}{\partial x} = Ae^{-\lambda^2 t}(-\sin\lambda x)\lambda \quad \text{--- (2)}$$

Diff. (1) w.r.t. t, we have
$$\frac{\partial Z}{\partial t} = -Ae^{-\lambda^2 t}\lambda^2\cos\lambda x \quad \text{--- (3)}$$

Diff. (2) w.r.t. x, we have
$$\frac{\partial^2 Z}{\partial x^2} = -Ae^{-\lambda^2 t}\lambda^2\cos\lambda x \quad \text{--- (4)}$$

from (3) and (4), we have
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial Z}{\partial t}$$

which is the required Diff. equation.

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FORMATION OF EQUATION BY THE ELIMINATION OF ARBITRARY FUNCTION :→

suppose u and v are two functions of x, y, z which are connected by the relation
$$f(u, v) = 0 \quad \text{--- (1)}$$

Differentiating (1) w.r.t. x, we get
$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot q\right) = 0 \quad \text{(2)}$$

Now Diff (1) w.r.t. y regarding x as constant
$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot q\right) = 0 \quad \text{(3)}$$

Now we shall eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3)

From (2), $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p\right)=$

$$-\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p\right)=0 \quad \text{--- (4)}$$

From (3), $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q\right)=$

$$-\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right)=0 \quad \text{--- (5)}$$

Dividing (4) and (5) and cross multiplying we get,

$$\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)$$

or $\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q$

$$=\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial x}q\right)$$

$$=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$Pp+Qq=R \quad \text{--- (6)}$$

$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$ written as $\frac{\partial(u,v)}{\partial(y,z)}$

$$Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u,v)}{\partial(z,x)}$$

$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(y,x)}$$

Equation (6) is the required differential equation of first order and first degree.

Example-I Find a partial differential equation by eliminating the function f from the relation:

$$Z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

Sol → The given eq. $Z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ — (1)

Diff. (1) partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log x\right)\left(-\frac{1}{x^2}\right)$$

$$-x^2\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{— (2)}$$

Diff. (1) w.r.t. y, we get

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\cdot\frac{1}{y}$$

$$\left(y\frac{\partial z}{\partial y} - 2y^2\right) = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{— (3)}$$

Divided (2) by (3) we have,

$$\frac{-x^2\frac{\partial z}{\partial x}}{y\left(\frac{\partial z}{\partial y} - 2y^2\right)} = 1$$

$$-x^2 p = yq - 2y^2$$

$$x^2 p + yq = 2y^2 \quad \left[\because p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}\right]$$

which is the required partial differential equation.

CHAPTER :→ 2

First Order Linear Partial differential Equations

Classification of the solution Or Integral of Partial differential Equation

A Function which satisfies a given Partial differential equation is said to be its solution or Integral. There are four types of solutions of a partial differential Equation.

(a) **Complete Integral :→** If

from the Partial differential equation $f(x,y,z,p,q)=0$ a relation $F(x,y,z,p,q)=0$ is obtained, which contain as many arbitrary constant as their are Independent Variables in Partial differential equation, then $F(x,y,z,a,b)=0$ is called Complete Integral.

(b) **Particular solution :→** If

we assign Particular values to arbitrary constants a, b in the Complete Integral, then the solution obtained is called Particular solution.

(c) **General solution :→** The

relation between x, y and z obtained by eliminating arbitrary constant 'a' b/w the equation $F(x,y,z,a,\phi(a))=0$ is called a General solution.

(d) **Singular solution :→** The

relation between x, y and z obtained by eliminating the arbitrary constant 'a' and 'b' between the equation

$F(x,y,z,a,b)=0,\ \frac{\partial F}{\partial a}=0,\ \frac{\partial F}{\partial b}=0$

is called Singular solution.

STANDARD FORM OF LINEAR PARTIAL DIFFERENTIAL EQUATION

A linear partial differential equation of first order of the form $Pp+Qq=R$, where P, Q, R are function of x, y, z is called Lagrange's linear equation.

Working Rule to Solve $Pp+Qq=R$

Step-1 Write the equation as $Pp+Qq=R$ and find P, Q, R by comparing the given equation with $Pp+Qq=R$

Step-2 Write down the Lagrange auxiliary equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Step-3 Solve the auxiliary equation obtain in step 2. Let $u(x,y,z)=a$ and $v(x,y,z)=b$ be two independent solution of it.

Step-4 General solution or Integral of the given equation can be written as

$$f(u,v)=0 \quad \text{or} \quad v=f(u)$$

TYPE - I

Example-1

Solve $p\tan x+q\tan y=\tan z$

Solution: → The given equation is $p\tan x+q\tan y=\tan z$

Comparing it with $Pp+Qq=R$

$P=\tan x,\ Q=\tan y$

$R=\tan z$

Subsidiary Equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad -(1)$$

Taking first two fractions of (1), we have

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\cot x \, dx = \cot y \, dy$$

Integrating

$$\log \sin x = \log \sin y + \log a$$

$$\log \frac{\sin x}{\sin y} = \log a$$

$$a = \frac{\sin x}{\sin y}$$

Taking last two fractions of (1), we have

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\cot y \, dy = \cot z \, dz$$

Integrating

$$\log \sin y = \log \sin z + \log b$$

$$\log \frac{\sin y}{\sin z} = \log b$$

$$b = \frac{\sin y}{\sin z}$$

General Solution is $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

Type - II

Example-2 Solve $z(p-q) = z^2 + (x+y)^2$

Solution:→ The given equation

$$pz - qz = z^2 + (x+y)^2$$

Comparing it with $Pp + Qq = R$

$$P = z, \quad Q = -z, \quad R = z^2 + (x+y)^2$$

Subsidiary Equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2} \quad -(1)$$

Taking First two Fractions

$$dx = -dy$$

Integrating

$$x = -y + a$$

$$x + y = a \quad -(2)$$

Taking the 2nd and 3rd function of (1)

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$$\frac{dy}{-2} = \frac{dz}{z^2+(x+y)^2}$$

$$dy = -2\,\frac{dz}{z^2+a^2}$$

Integrating

$$\int dy = \frac{-1}{2}\int \frac{2z}{z^2+a^2}\,dz + c$$

$$y = \frac{-1}{2}\log(z^2+a^2) + c$$

$$2y = -\log(z^2+a^2) + \log b$$

$$2y = \log\frac{b}{z^2+a^2}$$

$$b = e^{2y}(z^2+a^2) = e^{2x}\left(z^2+(x+y)^2\right)$$

Hence the solution is

$$e^{2y}\left(z^2+(x+y)^2\right) = f(x+y)$$

Type – III

The multipliers P_1, Q_1, R_1 are chosen in such a way so that by Algebra we are in a position to write subsidiary equation as

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$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1}$$

where $PP_1 + QQ_1 + RR_1 = 0$

This gives us $P_1dx + Q_1dy + R_1dz = 0$

This equation can be Integrated to get one solution as $a = u(x,y,z)$

The other Integral can be obtained either by methods discussed in examples already solved or by choosing another set of multipliers.

Type – IV

Example:-3 Solve $(x^2-y^2-z^2)p + 2xyq = 2xz$

Solution:- Comparing the given equation $Pp + Qq = R$

$P = x^2-y^2-z^2$, $Q = 2xy$, $R = 2xz$

Lagrange's subsidiary equations

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{—(1)}$$

Taking 2nd and 3rd fraction of (1)

$$\frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{i.e.} \quad \frac{dy}{y} = \frac{dz}{z}$$

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Integrating

$$\log y = \log z + \log a$$

$$a = \frac{y}{z} \quad \text{---} \quad (2)$$

From (1)

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2x\,y^2 + 2xz^2}$$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)} \quad \text{-} (3)$$

Choosing the last two fraction of (3)

$$\frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\frac{dz}{z} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

Integrating

$$\log b + \log z = \log(x^2+y^2+z^2)$$

$$b = \frac{x^2+y^2+z^2}{z}$$

Hence the solution is $f(a,b)$

$$f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$

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CHAPTER 3

Introduction:— This chapter involves compatable system of first order partial differential equation. We shall discuss Charpit's general method and Jacobi's method to find solutions of partial differential equations of first order.

★ Consider partial differential eqn of first order,

$f(x,y,z,p,q) = 0.$

$g(x,y,z,p,q) = 0.$

This system of equations is said to be compatible if every solution of one is also a solution of the other.

★ **Condition for compatability:**

The given equations are,

$$f(x,y,z,p,q) = 0 \quad \text{---} \quad (1)$$

$$g(x,y,z,p,q) = 0. \quad \text{---} \quad (2)$$

let $J = \dfrac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} \partial f/\partial p & \partial f/\partial q \\ \partial g/\partial p & \partial g/\partial q \end{vmatrix} \neq 0. \quad \text{---} \quad (3)$

Here J = Jacobian of f and g.

It is possible to solve (1) and (2)

$$\boxed{\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0.}$$

which is required condition of compatability.

★ General Methods of SOLUTION:-

• CHARPIT'S METHOD:-

Let the given eq^n,

$$f(x,y,z,p,q) = 0 \quad \text{—— (1).}$$

$$\boxed{\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{df}{0}}$$

Ex:- Q.1 Find the complete integral of :-

$$pxy + pq + qy = yz.$$

Solution:- Here $f(x,y,z,p,q) = pxy + pq + qy - yz = 0$ —— (1)

Charpit's auxiliary equations are,

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$= \frac{df}{0} \quad \text{—— (2).}$$

Now,

$$\frac{\partial f}{\partial x} = py, \quad \frac{\partial f}{\partial y} = px + q - z, \quad \frac{\partial f}{\partial z} = -y,$$

$$\frac{\partial f}{\partial p} = xy + q, \quad \frac{\partial f}{\partial q} = p + y.$$

$\therefore$ from (1), Charpit's auxiliary equations are,

$$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dp}{py - py} = \frac{dq}{px+q-z-qy} \quad \text{—— (3).}$$

eq^n (3) gives, $\frac{dp}{0}$ = each fraction.

eq^n (2) gives, $dp = 0$, $p = c$ (const)

$$cxy + cq + qy - yz = 0,$$

$$(c+y)q = yz - cxy.$$

$$q = \frac{yz - cxy}{c+y}.$$

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$$dz = p\,dx + q\,dy.$$

$$dz - c\,dx = \frac{y(z-cx)}{c+y}\,dy.$$

$$\left(\frac{dz - c\,dx}{z-cx}\right) = \frac{y}{c+y}\,dy.$$

$$\left(\frac{dz - c\,dx}{z-cx}\right) = \left(1 - \frac{c}{c+y}\right)dy.$$

$$\log(z-cx) = y - c\log(c+y) + b.$$

$$\log(z-cx) + c\log(c+y) = y + b.$$

which is complete integral.

★ **JACOBI'S METHOD:-**

This method is used for solving partial differential equations involving three or more independent variables.

$$\boxed{\frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_3}{-\frac{\partial f}{\partial p_3}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}}}$$

This is Jacobi's method — (★)

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Working rule for Jacobi's Method:-

Let the given equation with three independent variables be.

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0. \quad \text{— (1)}$$

Step 1:- consider Jacobi's auxiliary equations

$$\frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dx_3}{-\frac{\partial f}{\partial p_3}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}}$$

Step 2:- Solve these equations to obtain two additional equations,

$$f_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \quad \text{— (2)}$$

and

$$f_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \quad \text{— (3)}$$

where a_1 and a_2 are arbitrary constants.

Step 3:- verify that relations (2) and (3) satisfy the conditions

$$(f_1, f_2) = \sum_{r=1}^{3}\left(\frac{\partial f_1}{\partial x_r}\cdot\frac{\partial f_2}{\partial p_r} - \frac{\partial f_1}{\partial p_r}\cdot\frac{\partial f_2}{\partial x_r}\right) = 0. \quad \text{— (4)}$$

Step 4:- If eqⁿ (4) is satisfied, then solve eqⁿˢ (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3.

Substitute these values of p_1, p_2, p_3 in
$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$.
and integrate to get the complete integral.

Ex-2. ☆ Q2. Find the complete integral of $p_1 x_1 + p_2 x_2 = p_3^2$ using Jacobi's method.

Soln:- Here, $f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 x_1 + p_2 x_2 - p_3^2 = 0$. — (1)

$\therefore \frac{\partial f}{\partial x_1} = p_1, \frac{\partial f}{\partial x_2} = p_2, \frac{\partial f}{\partial x_3} = 0, \frac{\partial f}{\partial p_1} = x_1, \frac{\partial f}{\partial p_2} = x_2$
$, \frac{\partial f}{\partial p_3} = -2p_3.$

Jacobi's auxiliary equations are,

$$\frac{dx_1}{\frac{-\partial f}{\partial p_1}} = \frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_2}{\frac{-\partial f}{\partial p_2}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_3}{\frac{-\partial f}{\partial p_3}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}}.$$

$$\frac{dx_1}{x_1} = \frac{dp_1}{p_1} = \frac{dx_2}{-x_2} = \frac{dp_2}{p_2} = \frac{dx_3}{2p_3} = \frac{dp_3}{0}. \quad \text{— (2)},$$

from first two fractions of (2), we have,

$$\frac{-dx_1}{x_1} = \frac{dp_1}{p_1} \text{ i.e. } \left(\frac{1}{x_1}\right)dx_1 + \left(\frac{1}{p_1}\right)dp_1 = 0.$$

Integrating, $\log x_1 + \log p_1 = \log a_1$
$x_1 p_1 = a_1$

let $f_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 p_1 = a_1$ — (3)

from third & fourth fractions of (2).

$$\frac{dx_2}{-x_2} = \frac{dp_2}{p_2} \text{ i.e. } \left(\frac{1}{x_2}\right)dx_2 + \left(\frac{1}{p_2}\right)dp_2 = 0$$

Integrating, $\log x_2 + \log p_2 = \log a_2$.
$x_2 p_2 = a_2$
let $f_2(x_1, x_2, x_3, p_1, p_2, p_3) = x_2 p_2 = a_2$ — (4)

Now, $(f_1, f_2) = \sum_{r=1}^{3}\left(\frac{\partial f_1}{\partial x_r}\cdot\frac{\partial f_2}{\partial p_r} - \frac{\partial f_1}{\partial p_r}\cdot\frac{\partial f_2}{\partial x_r}\right)$

$$\Rightarrow \left(\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial p_1} - \frac{\partial f_1}{\partial p_1}\cdot\frac{\partial f_2}{\partial x_1}\right) + \left(\frac{\partial f_1}{\partial x_2}\cdot\frac{\partial f_2}{\partial p_2} - \frac{\partial f_1}{\partial p_2}\cdot\frac{\partial f_2}{\partial x_2}\right)$$
$$+ \left(\frac{\partial f_1}{\partial x_3}\cdot\frac{\partial f_2}{\partial p_3} - \frac{\partial f_1}{\partial p_3}\cdot\frac{\partial f_2}{\partial x_3}\right)$$

$\Rightarrow (p_1)\cdot(0) - (x_1)(0) + (0)(x_2) - (0)\cdot(p_2) + (0)(0) + (0)\cdot(0)$

$= 0$.

Thus, $(f_1, f_2) = 0$ is verified for the relations ③ and ④.

③ and ④ may be taken as additional equations,

from ③,

$$\boxed{p_1 = \frac{a_1}{x_1}}$$

from ④,

$$\boxed{p_2 = \frac{a_2}{x_2}}$$

from ①,

substituting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = \left(\frac{a_1}{x_1}\right)dx_1 + \left(\frac{a_2}{x_2}\right)dx_2 + \sqrt{a_1+a_2}\cdot dx_3$$

Integrate

$$z = a_1 \log x_1 + a_2 \log x_2 + x_3\sqrt{a_1+a_2}$$

CHAPTER – 4

Introduction :–

In this chapter, a partial differential equation in which the dependent variable and its partial derivatives occur only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a linear partial differential equations.

General linear partial differential equations of order n is the form

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + A_n \frac{\partial^n z}{\partial y^n}\right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \dots + B_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right) + \dots + \left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}\right) + N_0 z + f(x,y)$$

Linear Homogeneous Partial Differential Equations of order n:

If all the derivatives appearing in the differential equation are of the same order, then the equation is called linear homogeneous partial differential equation.

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-1}\partial y^2} + \dots + A_n \frac{\partial^n z}{\partial y^n} = f(x,y)$$

Method to Find complementary Function :-

$$F(D, D') = f(xy)$$

The given eq^n is

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n) z = f(x,y)$$

complementary function is the solution of

$$[A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n] z = 0$$

Taking L.H.S as

$$[(D - m_1 D')(D - m_2 D')(D - m_3 D') \dots (D - m_n D')] z = 0$$

The solution is

$$(D - m_1 D') z = 0, (D - m_2 D') z = 0, \dots, (D - m_n D') z = 0$$

Method to Find particular Integral (P.I) :-

$$F(D, D') z = f(x,y)$$

$$P.I = \frac{1}{F(D, D')} f(x,y)$$

Consider the equation

$$(D^2 + K_1 DD' + K_2 D'^2) z = f(x,y)$$

$$F(D, D') z = f(x,y)$$

Example - 1. Solve $\frac{\partial^3 z}{\partial x^3} - \frac{4\partial^3 z}{\partial x^2 \partial y} + \frac{4\partial^3 z}{\partial x \partial y^2} = 2\sin(3x+2y)$

Solution:- The given equation in symbolic form is

$$(D^3 - 4D^2 D' + 4DD'^2) z = 2\sin(3x+2y)$$

$$m^3 - 4m^2 + 4m = 0$$

$$m(m^2 - 4m + 4) = 0$$

complementary function is

$$z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$$

$$P.I = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 2\sin(3x+2y)$$

$$= \frac{1}{D(D^2 - 4DD' + 4D'^2)} 2\sin(3x+2y)$$

$$= 2\left(\frac{1}{D^2 - 4DD' + 4D'^2}\right)\frac{1}{D}\sin(3x+2y)$$

$$= 2\frac{1}{D^2 - 4DD' + 4D'^2}\left(-\frac{\cos(3x+2y)}{3}\right)$$

$$= -\frac{2}{3}\cdot\frac{1}{D^2 - 4DD' + 4D'^2}\cos(3x+2y)$$

$$= \left(-\frac{2}{3}\right)\frac{1}{-9 - 4[-(3)(2)] + 4(-4)}\cos(3x+2y)$$

$$= \left(-\frac{2}{3}\right)\frac{1}{-9 + 24 - 16}\cos(3x+2y)$$

$$= \frac{2}{3}\cos(3x+2y)$$

The general solution is

$$z = C.F + P.I$$

$$z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{2}{3}\cos(3x+2y)$$

Working Rule to Find C.F of Linear Homogeneous Partial Differential Equation:-

Let the given equation be

$$F(D, D')z = f(x,y)$$

Step 1. Write m in place of D and 1 in plane of D' in the coefficient of z on L.H.S of (1)

Step 2. (a) If $m_1, m_2, \ldots m_n$ are distinct roots, the the C.F is

$$z = \phi_{r+1}(y + m_{r+1}x) + \phi_2(y + m_{r+2}x) + \ldots + \phi_n(y + m_n x) + \phi_1(y + Mx)$$

$$+ x\phi_2(y+Mx) + \ldots + x^{r-1}\phi_r(y+Mx)$$

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Working rule to find P.I. in case of Linear Homogeneous Equation

$$F(D,D')z = f(x,y)$$

Case I. If $f(x,y) = e^{ax+by}$,

$$P.I = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$$

Case II If $f(x,y) = \sin(mx+ny)$ or $\cos(mx+ny)$

$$P.I = \frac{1}{F(D,D')} \sin(mx+ny) \text{ or } \cos(mx+ny)$$

Case III If $f(x,y) = x^m y^n$

$$P.I = \frac{1}{F(D,D')} x^m y^n$$

Case IV When $f(x,y)$ is any other function of (x,y),

$$P.I = \frac{1}{F(D,D')} f(x,y)$$

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Example-2. Solve the differential equation

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

Solution:- The given eqn in symbolic form is

$$(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$$

Auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$(m-2)(m^2 - m - 2) = 0$$

$$(m-2)(m-2)(m+1) = 0$$

$$m = 2, 2, -1$$

Complementary function is

$$z = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x)$$

$$P.I = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$$= \frac{1}{1-6+32} e^{x+2y}$$

$$= \frac{1}{27} e^{x+2y}$$

The general soln is $z = C.F + P.I$

$$z = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{1}{27}e^{x+2y}$$

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Lagrange's auxiliary equations

$$\frac{dx}{a_r} = \frac{dy}{-b_r} = \frac{dz}{c_r z}$$

Taking first two members

$$\frac{dx}{a_r} = \frac{dy}{-b_r}$$

Integrating, $b_r x + a_r y = c$

Taking first and last term

$$\frac{dx}{a_r} = \frac{dz}{c_r z}$$

$$\frac{dz}{z} = \frac{c_r}{a_r} dx$$

Integrating

$$\log z = \frac{c_r}{a_r} x + \log c_1$$

$$\log \frac{z}{c_1} = \frac{c_r}{a_r} x$$

$$\frac{z}{c_1} = e^{\frac{c_r}{a_r} x}$$

$$z = c_1 e^{\frac{c_r}{a_r} x}$$

$$c_1 = \phi(c)$$

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$$z = e^{\frac{c_r}{a_r} x} \phi(b_r x + a_r y) \text{ if } a_r \neq 0$$

In case $a_r = 0$

$$\frac{dy}{-b_r} = \frac{dz}{c_r z}$$

$$\frac{dz}{z} = -\frac{c_r}{b_r} dy$$

Integrating

$$\log z = -\frac{c_r}{b_r} y + \log c_2$$

$$z = c_2 e^{-\frac{c_r}{b_r} y}$$

$$c_2 = \frac{z}{e^{-\frac{c_r}{b_r} y}}$$

General solution is

$$c_2 = \psi(c)$$

$$\frac{z}{e^{-\frac{c_r}{b_r} y}} = \psi(b_r x + a_r y)$$

$$z = e^{-\frac{c_r}{b_r} y} \psi(b_r x + a_r y)$$

CHAPTER - 5

PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS REDUCIBLE TO EQUATIONS WITH CONSTANT COEFFICIENTS.

Partial differential equation with constant coefficients:

$$A_0 x^n \frac{\partial^n z}{\partial x^n} + A_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + A_n y^n \frac{\partial^n z}{\partial y^n} + \dots = f(x,y) \quad \text{--- (1)}$$

To reduce (1) to the differential equation with constant coefficients we use the substitution.

$$x = e^X \text{ and } y = e^Y$$
$$X = \log x \text{ and } Y = \log y. \quad \text{--- (2)}$$

Now $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x}$.

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \quad \text{--- (3)}$$

Writing in symbolic form.

$$x\frac{\partial}{\partial x} = \frac{\partial}{\partial X} = D.$$

Now differentiating (3) w.r.t x.

$$\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial X}\left(\frac{\partial z}{\partial X}\right) \cdot \frac{\partial X}{\partial x}$$

$$= x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial X^2} \cdot \frac{1}{x}.$$

$$= x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial X^2}.$$

$$= x^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial X^2} - x\frac{\partial z}{\partial x}.$$

$$= x^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial X^2} - \frac{\partial z}{\partial X}$$

$$= D^2 z - Dz.$$

$$= D(D-1)z. \qquad \left(\because \frac{\partial}{\partial X} = D\right)$$

Similarly we can show that

$$x^n\frac{\partial^n z}{\partial x^n} = D(D-1)(D-2)\dots(D-n+1)z.$$

Likewise we can show that →

$$y\frac{\partial z}{\partial y} = D'z \text{ where } \frac{\partial}{\partial Y} = D'$$

$$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z.$$

$$y^n\frac{\partial^n z}{\partial y^n} = D'(D'-1)(D'-2)\ldots(D'-n+1)z.$$

We have seen above that →

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}$$

$$\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial X}\left(\frac{\partial z}{\partial Y}\right)\cdot\frac{\partial X}{\partial x}$$

$$y\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial X \partial Y}\cdot\frac{1}{x}$$

$$xy\frac{\partial^2 z}{\partial x \partial y} = DD'z.$$

IIy we can seen:-

$$x^m y^n \frac{\partial^{m+n} z}{\partial x^m \partial y^n} = D(D-1)\ldots(D-m+1)D'(D'-1)$$

$$\ldots(D'-n+1)z.$$

$$x^m y^n \frac{\partial^{m+n} z}{\partial x^m \partial y^n} = D(D-1)\ldots(D-m+1)$$

$$D'(D'-1)\ldots(D'-n+1)z.$$

Substituting these values in given equation, it reduces to the form $F(D,D')z = V$ which is the equation with constant cofficients and can be solved.

Chapter ⇒ 6

Classification and canonical forms of 2nd order linear Partial diff. equation ⇒

☆ Classification of 2nd order linear partial Diff. eqⁿ ⇒

let the general linear partial diff. eqⁿ of second order of a function of two independent variable x and y be the form

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x}$$

$$s = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}$$

The classification depend on the part $Rr + Ss + Tt$ which is also said to be the Principle part of (1)

The eqⁿ (1) is said to be
1. hyperbolic if $S^2 - 4RT > 0$
2. Parabolic if $S^2 - 4RT = 0$
3. elliptic if $S^2 - 4RT < 0$

☆ Canonical forms ⇒

Working Rule for Reduction to canonical form ⇒

Step-1 Write the given eqⁿ in the standard form

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0$$

Eqⁿ (1) may be hyperbolic or Parabolic or elliptic acc. to as $S^2 - 4RT >$, $=$, or < 0

Step-2 Write the Quadratic equation $R\lambda^2 + S\lambda + T = 0$, find it root

root λ_1 and λ_2 (say) which may be real and distinct or real or equal or complex roots.

Step = 3 Consider the equation

$$\frac{dy}{dx} + \lambda_1 = 0$$

$$\frac{dy}{dx} + \lambda_2 = 0$$

on solving, we have $f_1(x,y) = $ const and $f_2(x,y) = $ const

Step = 4 Take $u = f_1(x,y)$ and $v = f_2(x,y)$

If $\lambda_1 = \lambda_2$ the we get only one function $u = f(x,y)$ (say) and we are to choose v in the such a manner that it must be independent of u for this verify that Jacobian of u and v =

$$= \frac{\partial(u,v)}{\partial(x,y)} \neq 0$$

Step = 5 find p, q, r, s and t in terms of u and v. If λ_1 and λ_2 are complex conjugate, then, $u = \alpha + i\beta$, $v = \alpha + i\beta$ (say) are function of two new independent variable so, in such cases find p, q, r, s and t term of α and β

Step = 6 Substituting the value of p, q, r, s and t in given eq^n (1) we get the required canonical form.

Conclusion :-

Types of eq^n	Corresponding Canonical form
① Hyperbolic if $S^2 - 4RT > 0$	$\frac{\partial^2 z}{\partial u \partial v} = \phi_1\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$
② Parabolic if $S^2 - 4RT = 0$	$\frac{\partial^2 z}{\partial u^2} = \phi_2\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$
③ Elliptic if $S^2 - 4KT < 0$	$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi_3\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$

Monge's Partial Differential Equations of Second Order

⇒ Working Rule for Solving Problem by Monge's Method Based on two Intermediate Integrals:

Step 1.

Write the given equation as

$$Rr + Ss + Tt = V$$

Step 2.

Substitute the value of

$$r = \frac{dp - s\,dy}{dx} \quad \text{and}$$

$$t = \frac{dq - s\,dx}{dy}$$

to get the Monge's subsidiary Equations as

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad \text{-(i)}$$

and

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0 \quad \text{---(2)}$$

Step 3.

Factorize (2) into two distinct factors.

Step 4.

Corresponding to each factor, find two intermediate integrals. Slove these to find p and q.

Step 5.

Use these value of p and q in $dz = p\,dx + q\,dy$ and integrate to get the general solution.

Example 1.

Solve $r - t\cos^2 x + p\tan x = 0$.

Sol. The given differential equation is

$$r - t\cos^2 x + p\tan x = 0 \quad \text{---(i)}$$

As in previous examples, putting $r = \frac{dp - s\,dy}{dx}$ and $t = \frac{dq - s\,dx}{dy}$ in (i)

$$\left[\frac{dp - s\,dy}{dx}\right] - \left[\frac{dq - s\,dx}{dy}\right]\cos^2 x + p\tan x = 0$$

Multiplying by $dx \cdot dy$ we have

$$dp\,dy - s\,dy\,dy - (dq\,dx - s(dx)^2)\cos^2 x + p\tan x\,dx\,dy = 0$$

or

$$(dp\,dy - dq\,dx\cos^2 x + p\tan x\,dy\,dx) - s[(dy)^2 - (dx)^2\cos^2 x] = 0$$

Monge's subsidary equations are

$$dp\,dy - \cos^2 x\, dq\, dx + p\tan x\, dy\, dx = 0 \quad \text{...(2)}$$
$$(dy)^2 - \cos^2 x\,(dx)^2 = 0 \quad \text{...(3)}$$
$$dy + \cos x\, dx = 0 \quad \text{...(4)}$$
$$dy - \cos x\, dx = 0 \quad \text{...(5)}$$

To find one intermediate integral

Integrating (5)

$$y - \sin x = c_1 \quad \text{...(6)}$$

Using $dy = \cos x\, dx$ in (2) we have

$$dp\cos x\, dx - \cos^2 x\, dq\, dx + p\tan x \cos x\, dx\, dx = 0$$

Dividing by $\cos^2 x\, dx$.

$$dp\sec x - dq + p\tan x\sec x\, dx = 0$$
$$dq = dp\sec x + p\tan x\sec x\, dx$$
$$dq = d(p\sec x)$$

Integrating $\quad q = p\sec x + c_2 \quad \text{(7)}$

$$q - p\sec x = c_2$$

Thus one intermediate integral is

$$q - p\sec x = \phi_1(y - \sin x) \quad \text{...(8)}$$

To find other intermediate integral:

from (4) $\quad dy = -\cos x\, dx$

Integrating $\quad y + \sin x = c_3 \quad \text{....(9)}$

Putting $dy = -\cos x\, dx$ in (1)

$$dp(-\cos x)dx - dq\, dx\cos^2 x + p\tan x(-\cos x\, dx)dx = 0$$

Dividing by $-\cos^2 x\, dx$; we have

$$dp\sec x\, dx + dq + p\tan x\sec x = 0$$
$$dq = -d(p\sec x)$$

Integrating,

$$q = -p\sec x + c_4$$
$$q + p\sec x = c_4 \quad \text{..(10)}$$

from (9) & (10), the second intermediate integral is

$$q + p\sec x = \phi_2(y + \sin x).$$

Solving (8) & (11) for p and q, we get

$$q = \frac{1}{2}\left[\phi_2(y+\sin x) + \phi_1(y - \sin x)\right]$$

$$p = \frac{1}{2\sec x}\left[\phi_2(y+\sin x) - \phi_1(y-\sin x)\right]$$

Substituting these value in

$dz = p\,dx + q\,dy$, we have

$$dz = \frac{1}{2}\cos x\left[\phi_2(y+\sin x) - \phi_1(y-\sin x)\right]dx$$

$$+ \frac{1}{2}\left[\phi_2(y+\sin x) + \phi_1(y-\sin x)\right]dy$$

$$dz = \frac{1}{2}\phi_2(y+\sin x).(dy + \cos x\, dx) + \frac{1}{2}\phi_1$$

$$(y - \sin x)(dy - \cos x\, dx)$$

Integrating,

$$z = \frac{1}{2}f_1(y+\sin x) + \frac{1}{2}f_2(y - \sin x)$$

Hence

$$z = F_1(y+\sin x) + F_2(y-\sin x)$$

is the complete solution.

NOTE

1. In the above method when two values of λ are equal, we get only one intermediate integral say $u_1 = f_1(v_1)$. With this integral and with one of $u_1 = a$, and $v_1 = b$, we get p and q to solve $dz = p\,dx + q\,dy$.

2. In case suitable value of p and q are not determined from the two intermediate integrals then one of the integrals say $u_1 = f_1(v)$ together with $u_2 = a$ and $v_2 = b$ is used to solve for p and q. Then $dz = p\,dx + q\,dy$ given the solution.

(i) λ - quadratic is

$$\lambda^2 (UV + RT) + \lambda US + U^2 = 0$$

(ii) Two intermediate integrals are given by

$$U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp = 0$$
$$U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq = 0$$
$$U\,dy + \lambda_2 T\,dx + \lambda_2 U\,dp = 0$$
$$U\,dx + \lambda_1 R\,dy + \lambda_1 U\,dq = 0$$

Example 1. Solve $2pr + 2qt - 4pq(rt - s^2) = 1$.

Solution. Comparing the given equation with

$$Rr + Ss + Tt + U(rt - s^2) = V$$

$R = 2p$, $S = 0$, $T = 2p$, $U = -4pq$ and $V = 1$

λ - quadratic is

$$\lambda^2 (UV + RT) + \lambda US + U^2 = 0$$

i.e. $\lambda^2(-4pq + 4pq) + 0\lambda + 16p^2q^2 = 0$

$$0\lambda^2 + 0\lambda + 16p^2q^2 = 0$$

Since the coefficient of both λ^2 and λ are 0

$\therefore$ Both roots of quadratic λ are infinite.

Thus the only intermediate integrals is given by

$$U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp = 0 \qquad (1)$$
$$U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq = 0 \qquad (2)$$

Dividing (1) and (2) by λ_1 & λ_2 respectively, we

$$\frac{1}{\lambda_1} U\,dy + T\,dx + U\,dp = 0$$

$$\frac{1}{\lambda_2} U\,dx + R\,dy + U\,dq = 0$$

As λ_1 and $\lambda_2 \to \infty$, therefore.

$$2q\,dx - 4pq\,dp = 0 \Rightarrow q\,dx - 2pq\,dp = 0 \qquad -(3)$$
$$2p\,dy - 4pq\,dq = 0 \Rightarrow p\,dy - 2pq\,dq = 0 \qquad -(4)$$

from (3) and (4) we have

$$dx - 2p\,dp = 0 \quad \text{and} \quad dy - 2q\,dq = 0$$

Integrating,

$$x - p^2 = a \qquad -(5)$$

$$y - q^2 = b \quad \text{--(6)}$$

$\therefore$ Only intermediate integral is

$$x - p^2 = f(y - q^2)$$

from (5) and (6)

$$p = \pm (x-a)^{1/2} \text{ and } q = \pm (y-b)^{1/2}$$

Putting these values in

$$dz = p\,dx + q\,dy, \text{ we have}$$

$$dz = \pm (x+a)^{1/2} dx \pm (y-b)^{1/2} dy$$

Integrating

$$z = \pm \frac{2}{3}(x-a)^{3/2} \pm \frac{2}{3}(y-b)^{2/3} + c$$

which is the complete integral.
where c_1, c_2, c_3 are arbitrary constants

Chapter - 8

CHAPTR = 8

Characteristics of second order partial Differential equations and Cauchy's problem

★ **Introduction →**

Characteristice equation and Characteristice curves.

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

Then the corresponding quadratic

$$R\lambda^2 + S\lambda + T = 0$$

$$\lambda = \frac{-S \pm \sqrt{S^2 - 4RT}}{2R}$$

Ordinary differential

$$\frac{dy}{dx} + \left[\frac{-S \pm \sqrt{S^2 - 4RT}}{2R}\right] = 0$$

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

Hence Characteristic equation

Case-I $S^2 - 4RT$ Hyperbolic

$$\frac{dy}{dx} + \lambda_1(x,y) = 0$$

$$\frac{dy}{dx} + \lambda_2(x,y) = 0$$

$$\frac{dy}{dx} + \left(\frac{-S \pm \sqrt{S^2 - 4RT}}{2R}\right) = 0$$

Integrating these ordinary diff equ
Characteristics curv
$\phi(x,y) = c_1$ $\psi(x,y) = c_2$
where c_1 & c_2 are Constant

Case II when $S^2 - 4RT = 0$ Parabolic

$$\frac{dy}{dx} + \lambda(x,y) = 0 \qquad \frac{dy}{dx} = \frac{S}{2R}$$

integrating one family

Example

$$x^2\frac{\partial^2 y}{\partial x^2} + 2xy\frac{\partial^2 y}{\partial x \partial y} + y^2\frac{\partial^2 y}{\partial y^2} = 0$$

Solution $\frac{\partial^2 y}{\partial x^2} = r$, $\frac{\partial^2 y}{\partial x \partial y} = s$, $\frac{\partial^2 y}{\partial y^2} = t$

$$x^2 r + 2xys + y^2 t = 0 \quad \text{— (1)}$$

$R = x^2$, $S = 2xy$, $T = y^2$

$S^2 - 4RT$

$(2xy)^2 - 4 \times x^2 \times y^2$

$4x^2y^2 - 4x^2y^2 = 0$

Hence prove in parabolic
Now quadratic equation
$R\lambda^2 + S\lambda + T = 0$

$$x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

$$(x\lambda + y)^2 = 0$$

$$\lambda = -\frac{y}{x}, -\frac{y}{x}$$

the roots are equal.

Corresponding Characteristics equ

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$$\frac{dy}{dx} + \lambda = 0$$

$$\frac{dy}{dx} - \frac{y}{x} = 0$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x} \quad \text{—} \quad (1)$$

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

Integrating (2) $\frac{y}{x} = c_1$ or $y = c_1 x$

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Method of Separation of Variables;

Wave, Heat and Laplace Equations.

Wave Equations;

(1) One-Dimensional wave Equations;

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

which is said to be one-dimensional wave equation.

where $c^2 = \frac{T}{\rho}$

Two-Dimensional wave equation;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

which is said to be Two-dimensional wave equation,

where $c^2 = \frac{T}{\rho}$

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(2) Heat Equations:-

(i) One-dimensional Heat Equation:-

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

Which is said to be one-dimensional Heat Equation.

(ii) Two-dimensional Heat Equation:-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

Which is said to be two-dimensional Heat Equation.

(3) Laplace Equations:-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or

$$\nabla^2 u = 0$$

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Which is said to be Laplace's equation in two dimensions.

Here ∇ is said to be Laplacian operator.

Method of separation of Variables:-

$$u(x,t) = X(x).T(t)$$

Case-1 when $K>0$

$$K = p^2$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = p^2 \quad \text{and} \quad \frac{1}{c^2T}\frac{d^2T}{dt^2} = p^2$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$T = c_3 e^{pct} + c_4 e^{-pct}$$

$$u(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct})$$