MAA OMWATI DEGREE COLLEG
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NOTES

SUBJECT : Advanced Calculus

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Continuous Functions

Chapter-Ist

A function f is said to be continuous at a if is defined at $x = a$ and $\lim_{x \to a} F(x) = F(a)$

Another definition :-

Let f be a function defined in a neighbourhood of a. Then the function f is said to be continuous at a if for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|F(x) - F(a)| < \varepsilon$$

whenever $|x - a| < \delta$

Equivalently, a function f is said to be continuous at $x = a$,

$$\lim_{x \to a^-} F(x) = \lim_{x \to a^+} f(x) = F(a)$$

Discontinuous Functions :-

A function f is said to be discontinuous at a point $x = a$ if f is not continuous at $x = a$.

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* Discontinuity of first kind:-

If $\lim_{x \to a^-} F(x)$ and $\lim_{x \to a^+} F(x)$ both exist but are unequal then F is said to have a discontinuity of first kind.

* Discontinuity of second kind:-

If either $\lim_{x \to a^-} F(x)$ or $\lim_{x \to a^+} F(x)$ or both of them do not exist, then F is said to have a discontinuity of second kind.

* Mixed discontinuity:- If a function F has a discontinuity of second kind on one side a and on the other side it may be continuous or may have discontinuity of first kind, then the function is said to have mixed discontinuity.

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(v) Removable discontinuity:

If $\lim_{x \to a} F(x)$ exists, but not equal to $F(a)$, then the function F has removable discontinuity at $x = a$.

Infinite discontinuity: A function F has an discontinuity at $x = a$ if either $\lim_{x \to a^-} F(x)$ or $\lim_{x \to a^+} F(x)$ or both have infinite values.

Thm: If a function f is continuous at $x = a$, then $|F|$ is also continuous at $x = a$. Is the converse true? If not show by an exam.

Proof: Suppose that F is continuous at $x = a$ and $\langle x_n \rangle$ be any sequence converging.

$\lim_{n \to \infty} x_n = a$ and $\lim_{n \to \infty} F(x_n) = F(a)$

or $\lim_{n\to\infty} |F| x_n = \lim_{n\to\infty} |F(x_n)|$

$= |\lim_{n\to\infty} F(x_n)| = |F(a)|$

$= |F|(a)$

$\therefore$ $|F|$ is continuous at a.

The converse of the theorem is not alway true.

For example,

Consider $F(x) = \begin{cases} 1, & x \geq 0 \\ -1 & x < 0 \end{cases}$

$|F|(x) = 1$ for all $x \in R$

At $x = 0$, $\lim_{x\to 0^+} F(x) = 1 = 1$

$\lim_{x\to 0^-} F(x) = -1$

$\therefore$ $\lim F(x)$ does not exist.

But $\lim_{x\to 0} |F|(x) = 1$

It shows that $|F|$ is continuous at $x = 0$ but F is not continuous at $x = 0$

Exp. By definition, prove that

$F(x) = \begin{cases} x^2 \cos\frac{1}{x} & x \neq 0 \\ 0 & , x = 0 \end{cases}$ is

continuous at $x = 0$.

Sol. $F(x) = x^2 \cos\frac{1}{x}$

$F(0) = 0$

Let $\varepsilon > 0$ be any given real number.

$|F(x) - F(0)| = |x^2 \cos\frac{1}{x} - 0|$

$= |x^2 \cos\frac{1}{x}|$

$= |x^2| \, |\cos\frac{1}{x}|$

$\leq |x^2|$ $\left[\because |\cos\frac{1}{x}| \leq 1\right]$

Hence

$|F(x) - F(0)| < \varepsilon$ if $|x|^2 < \varepsilon$

$|F(x) - F(0)| < \varepsilon$ if $|x| < \sqrt{\varepsilon}$

Taking $\sqrt{\varepsilon} = \delta$

$|F(x) - F(0)| < \varepsilon$

whenever $|x - 0| < \delta$

Hence the function f is continuous at $x=0$.

Uniform Continuity:-

A function f is said to be uniformly continuous on the interval $[a,b]$ if for given $\epsilon > 0$, there exists $\delta > 0$

$$|f(x_1) - f(x_2)| < \epsilon$$
whenever $|x_1 - x_2| < \delta$

Prove that function $f(x) = 2x^2 + 3x - 4$ is uniformly continuous on $[-2,2]$

Sol: Let $\epsilon > 0$ be a given number.

Let x_1, x_2 be any two points $[-2,2]$

$\therefore |x_1| \leq 2$ and $|x_2| \leq 2$

$\therefore |f(x_1) - f(x_2)|$

$= |2x_1^2 + 3x_1 - 4 - 2x_2^2 + 3x_2 - 4|$

$= |2(x_1^2 - x_2^2) + 3(x_1 - x_2)|$

$= |(x_1 - x_2)[2(x_1 + x_2) + 3]|$

$= |x_1 - x_2| \, |2(x_1 + x_2) + 3|$

$\leq |x_1 - x_2| \, |2(|x_1| + |x_2|) + 3|$

$\leq |x_1 - x_2| \, |2(2+2) + 3|$

$= 11|x_1 - x_2|$

$\therefore |f(x_1) - f(x_2)| < \epsilon$ if $11|x_1 - x_2| < \epsilon$

$|f(x_1) - f(x_2)| < \epsilon$ if $|x_1 - x_2| < \frac{\epsilon}{11}$

$|f(x_1) - f(x_2)| < \epsilon$
whenever $|x_1 - x_2| < \delta$
when $\delta = \frac{\epsilon}{11}$

Hence f is uniformly cont. $[-2,2]$.

Chapter - 2nd

The derivative and mean value theorems.

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Derivability of a function:-

A function f defined on $[a,b]$ is said to be derivable or diff at $c \in (a,b)$ if $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. This limit is called derivative of f at c and denoted by $f'(c)$. The process of obtaining $f'(c)$ is called differentiation.

Exp If $y = v^3 + 2v^2 + 5$, $v = 3u+1$

$u = 9x+1$ find $\frac{dy}{dx}$.

$\frac{dy}{dv} = 3v^2 + 4v$

$\frac{dv}{du} = 3$

$\frac{du}{dx} = 9$

$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$

$(3v^2+4v) \cdot 3 \cdot 9$

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$= 27(3(3u+1)^2 + 4(3u+1)]$

$= 27[3(9u^2+1+6u) + 12u+4]$

$= 27[27u^2 + 30u + 7]$

$= 27[27(9x+1)^2 + 30(9x+1)+7]$

$= 27[27(81x^2+1+18x) + 270x+30+7]$

$= 27[2187x^2 + 756x + 64]$ Ans

Darboux's thm:-

If a function F defined on $[a,b]$ is such that

(i) F is derivable on $[a,b]$

(ii) $F'(a)$ and $F'(b)$ are of opposite signs.

Then there exists some point $c \in (a,b)$ such that $F'(c) = 0$

Proof: Let us suppose that $F'(a) > 0$ and $F'(b) < 0$.

As $F'(a) > 0$ therefore there exists some $\delta_1 > 0$ such that

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$F(x) > F(a)$ for all $x \in (a, a+\delta_1)$ — (1)

As $F'(b) < 0$, therefore there exists some $\delta_2 > 0$ such that

$F(x) > F(b)$ for all $x \in (b-\delta_2, b)$ — (2)

As the function F is derivable on $[a,b]$ and every derivative function is continuous also.

$\therefore$ F is continuous on $[a,b]$

$\Rightarrow$ F is bounded on $[a,b]$ and attain its bounds.

Thus there exists $c, d \in [a,b]$ such that

$F(c) = \sup. F$ on $[a,b] = u$

$F(d) = \inf. F$ on $[a,b] = l$

Now u is $\sup. F$ on $[a,b]$.

$F(x) \leq u$ for all $x \in [a,b]$ — (5)

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$\therefore$ $F(x) \leq u$ for all $x \in [a,b]$

Claim that $c \neq a$ or $c \neq b$

If $c = a$ then

$F(c) = F(a) = u$

From (1)

$F(x) > u$ for all $x \in (a, a+\delta_1)$

This is a contradiction to (5)

Thus $c \neq a$.

If $c = b$ then $F(c) = F(b) = u$

From (2)

$F(x) > u$ for all $x \in (b-\delta_2, b)$

This is a contradiction to (5)

Thus $c \neq b$

Hence our claim that $c \neq a$ or $c \neq b$ is true.

Finally we shall show that

$F'(c) = 0$

Case I. $F'(c) > 0$:

then exists some $\delta_3 > 0$ such that

$$F(x) > F(c) \text{ for all } x \in (c, c+\delta_3)$$

$$F(x) > u \text{ for all } x \in (c, c+\delta_3)$$

which is a contradiction to (5) and thus $F'(c) > 0$ is not possible

Case II If $F'(c) < 0$:

then exists some $\delta_4 > 0$ such that

$$F(x) > F(c) \text{ for all } x \in (c-\delta_4, c)$$

$$F(x) > u \text{ for all } x \in (c-\delta_4, c)$$

which is contradiction to (5) and thus $F'(c) < 0$ is not possible.

thus there exists some $c \in (a,b)$ such that

$$F'(c) = 0$$

Similarly we can prove $F'(c) = 0$ for some $c \in (a,b)$ such that $F'(a) < 0$, $F'(b) \ngtr 0$, then there exists some $c \in (a,b)$ such that $F'(c) = 0$.

Rolle's thm :-

Let F be a function defined on $[a,b]$ such that

(i) F is continuous on $[a,b]$

(ii) F is derivable on (a,b)

(iii) $F(a) = F(b)$

then there exists atleast one real number c lying b/w a and b such that $F'(c) = 0$

Proof:

Since the function f is continuous on $[a,b]$, therefore

F is bounded on $[a,b]$ and attain its bounds. there $c, d \in [a,b]$ such that $F(c) = M$ and $F(d) = m$ where M and m are Sub. f and inf. f.

Two diff Case :-

Case 1. $M = m$ Sub. f = inf. f

Here f is a constant function over $[a,b]$ and consequently $f'(x) = 0$ for all $x \in [a,b]$.

$F'(c) = 0$ when $c \in (a,b)$

Case II $M \neq m$.

A $F(a) = F(b)$ therefore either M or m is diff from

$F(a) = F(b)$, Suppose that $M \neq F(a)$ and $M \neq F(b)$

As $F(c) = M$ therefore c is diff from both a and b and c lies in the open interval (a,b)

Since $F(c)$ is the supremum of f on $[a,b]$ therefore

$F(x) \leq F(c)$ for all $x \in [a,b]$

$F(c-h) \leq F(c)$	$F(c+h) \leq F(c)$
$\therefore F(c-h) - F(c) \leq 0$	$\therefore F(c+h) - F(c) \leq 0$
Dividing by $-h < 0$	Dividing by $h > 0$
$\frac{F(c-h) - F(c)}{-h} \geq 0$	$\frac{F(c+h) - F(c)}{h} \leq 0$

Taking limit as $h \to 0$

$\lim_{h \to 0} \frac{F(c-h) - F(c)}{-h} \geq 0$ $\lim_{h \to 0} \frac{F(c+h) - F(c)}{h} \leq 0$

$L\,F'(c) \geq 0$ — (2) $R\,F'(c) \leq 0$ (3)

Since $F(x)$ is exists at $x = c$

$L\,F(c) = R\,F(c) = F(c)$

this is possible only if $F(c) = 0$

Similarly we can prove that

$F'(d) = 0$ where $d \in (a,b)$

Hence there exists atleast one $c \in (a,b)$ such that $f'(c) = 0$

Exp. Prove that Rolle's thm $F(x) = x^2 - 5x + 6$ in the interval $[2,3]$

Sol. $F(x) = x^2 - 5x + 6$

$F(x)$ is a polynomial function of x and is therefore continuous and derivable for all real x.

(a) $F(x)$ is continuous in $[2,3]$

(b) $F(x)$ is derivable in $(2,3)$

(c) $F(2) = (2)^2 - 5(2) + 6$
$= 4 - 10 + 6 = 0$

$F(3) = (3)^2 - 5 \cdot 3 + 6$
$9 - 15 + 6 = 0$

$F(2) = F(3)$

Thus all the condition of Rolle's thm are satisfied. Hence there must exist atleast one point $x = c$ such that $f'(c) = 0$

Diff (i) w.r.t x.

$F'(x) = 2x - 5 \Rightarrow F'(c) = 2c - 5$

$F'(c) = 0 \Rightarrow 2c - 5 = 0$

$c = \frac{5}{2} \in (2,3)$

Hence Rolle's thm holds for $F(x) = x^2 - 5x + 6$ in $[2,3]$

Lagrange's Mean value thm.

Statement :- If a function $F(x)$ is defined in closed interval $[a,b]$ such that

(i) $F(x)$ is continuous in closed interval $[a,b]$

(ii) $F(x)$ is derivable in open interval (a,b) then then exists atleast one point $c \in (a,b)$ such

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Proof: Let us consider a function

$$F(x) = f(x) + Ax$$

where A is a constant to be determined such that

$$F(a) = F(b)$$

$F(a) = f(a) + Aa$ and
$F(b) = f(b) + Ab$

$$f(a) + Aa = f(b) + Ab$$

$$f(b) - f(a) = -A(b-a)$$

$$-A = \frac{f(b)-f(a)}{b-a} \quad \text{— (3)}$$

Now $f(x)$ is given to be continuous in $[a,b]$ and Ax being a polynomial in x, is always cont. in $[a,b]$

thus $F(x)$ the sum of two cont. function $f(x)$ and $A(x)$ is also continuous in $[a,b]$ and Also $F(x)$, the sum of two derivable function $f(x)$ and Ax in (a,b) is also derivable in (a,b)

thus we conclude that
$F(x) = f(x) + A(x)$ is

(i) continuous in the closed interval $[a,b]$ $(a \le x \le b)$

(ii) derivable in the open interval (a,b) $a < x < b$

(iii) $F(a) = F(b)$

$\therefore$ $F(x)$ satisfies all the three condition of Rolle's thm and hence there must exist atleast one value 'c' of x in open interval (a,b) such that

$$F'(c) = 0$$

$$F(x) = f(x) + A(x)$$

$F'(x) = f'(x) + A$

$F'(c) = f'(c) + A$

$F'(c) = 0 \Rightarrow f'(c) + A = 0$

$f'(c) = -A$ — (4)

From (3) and (4) we have

$$\frac{f(b) - f(a)}{b-a} = f'(c) \text{ for some } c \in (a,b)$$

Exp.

$f(x) = 2x - x^2$

$f(x)$ being a polynomial function of x is continuous for all reals

$\therefore f(x)$ is continuous $[0,1]$

$f'(x) = 2-2x$ which exists for all $x \in (0,1)$. Thus $f(x)$ is derivable in $(0,1)$

Thus both the condition of Lagrange's mean value thm are satisfied. Hence there must exist atleast one poin $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b-a} = f'(c) \text{ when } a=0, b=1$$ — (1)

$f'(x) = 2-2x$

$f'(c) = 2-2c$

$f(a) = f(0) = 0-0 = 0$

$f(b) = f(1) = 2-1 = 1$

$\frac{1-0}{1-0} = 2-2c$

$2-2c = 1$

$2c = 1 \Rightarrow c = \frac{1}{2} \in (0,1)$

Hence the Lagrange's mean value thm is ~~so~~ verified.

Exp. Verify the Cauchy's mean value thm for $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$

Here $f(x) = \;$ $f(x) = x^2$

$g(x) = x^3$

Exp. (i) f and g being polynomial function are cont. in [1,2]

(ii) f and g being polynomial function are derivable in (1,2)

(iii) $g'(x) = 3x^2 \neq 0$ for all $x \in (1,2)$

Thus the condition of Cauchy's mean value thm are satisfied. So there exists some point $c \in (1,2)$, such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{4-1}{8-1} = \frac{2c}{3c^2}$$

$$\frac{3}{7} = \frac{2}{3c}$$

$c = \frac{14}{9}$ which lies in (1,2)

Hence Cauchy's mean value thm is verified.

Chapter-3rd

Indeterminate Form

Definition:- If a function involves the independent variable in such a manner that for a certain assigned value of that variable, its value is undefined, then such a function is said to be of indeterminate form.

Exp. By L'Hospital rule, evaluate

$$\lim_{x \to 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)}$$

Sol.

$$\lim_{x \to 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)}$$

$$\lim_{x \to 0} \frac{2\tan^{-1} x \cdot \frac{1}{1+x^2}}{\frac{2x}{1+x^2}} = \lim_{x \to 0} \frac{\tan^{-1} x}{x}$$

$$\lim_{x \to 0} \frac{\frac{1}{1+x^2}}{1} \quad \lim_{x \to 0} \frac{1}{1+x^2} = 1$$

L' Hospital's Rule to Evaluate the indeterminate form $\frac{\infty}{\infty}$

Evaluate $\lim_{x \to \infty} \frac{x^3 - 8x^2 + 2x + 1}{x^4 - x^2 + 2x - 3}$

put $x = \frac{1}{y}$ For $x \to \infty$, $y \to 0$

$$\lim_{x \to \infty} \frac{x^3 - 8x^2 + 2x + 1}{x^4 - x^2 + 2x - 3}$$

$$= \lim_{y \to 0} \frac{\frac{1}{y^3} - 8 \cdot \frac{1}{y^2} + 2 \cdot \frac{1}{y} + 1}{\frac{1}{y^4} - \frac{1}{y^2} + 2 \cdot \frac{1}{y} - 3}$$

$$= \lim_{y \to 0} \frac{y - 8y^2 + 2y^3 + y^4}{1 - y^2 + 2y^3 - 3y^4} = \frac{0}{1}$$

$= 0$

Evaluation of indeterminate forms 0^0, 1^∞, and ∞^0

Exp. Evaluate $\lim_{x \to 0} (\cos x)^{\cot^2 x}$

Sol. Here the limit takes the form 1^∞

Let $y = \lim_{x \to 0} (\cos x)^{\cot^2 x}$

$$\log y = \lim_{x \to 0} \cot^2 x \log x$$

$$= \lim_{x \to 0} \frac{\log \cos x}{\tan^2 x}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2 \tan x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{-1}{2} \cos^2 x$$

$$= \frac{-1}{2}$$

$$\log y = \frac{-1}{2}$$

$$y = e^{-1/2}$$

$$\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$$

$$= e^{-1/2}$$

Chapter 4th

Limit and continuity of Function of two Variable.

Def. A number l is said to be the limit of function $f(x,y)$ as $(x,y) \to (a,b)$ if for any real number $\varepsilon > 0$, there exist a real number $\delta > 0$ such that for all points (x,y) other that (a,b)

$$|f(x,y) - l| < \varepsilon \text{ for } 0 < |x-a| < \delta \text{ and } 0 < |y-b| < \delta$$

$$|f(x,y) - l| < \varepsilon \text{ for } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = l$$

Exp Let $f : R^2 \to R$ be defined by $f(x,y) = x \sin\frac{1}{y} + y \sin\frac{1}{x}$

$x \neq 0,\ y \neq 0$

Prove that $\lim_{(x,y) \to (0,0)} (x,y) = 0$

Sol. Let $\varepsilon > 0$ be any given real number then

$$|f(x,y) - 0| = \left|x \sin\frac{1}{y} + y \sin\frac{1}{x}\right|$$

$$\leq \left|x \sin\frac{1}{y}\right| + \left|y \sin\frac{1}{x}\right|$$

$$= |x| \left|\sin\frac{1}{y}\right| + |y| \left|\sin\frac{1}{x}\right|$$

$$\leq |x| + |y|$$

$$= |x-0| + |y-0| \quad -(1)$$

Let $|x-0| < \varepsilon/2$ and $|y-0| < \varepsilon/2$

$$|f(x,y) - 0| \leq |x-0| + |y-0|$$

$$< \varepsilon/2 + \varepsilon/2$$

$\therefore |f(x,y) - 0| \leq \varepsilon$ ~~$|x-0| + |y-0|$~~

whenever

$|x-0| < \varepsilon/2$

$|y-0| < \varepsilon/2$

Exp. Let $F: R^2 \to R$ be defined as

$$F(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that
$\lim_{(x,y) \to (0,0)} F(x,y)$ does not exist

Sol. If $(x,y) \to (0,0)$ along the line $y = mx$ but $(x,y) \neq 0,0$

$$F(x,y) = \frac{xy}{x^2+y^2} = \frac{m(mx)}{x^2+m^2x^2}$$

$$= \frac{m}{1+m^2}$$

$$\lim_{x-y \to 0,0} F(x,y) = \frac{m}{1+m^2}$$
along $y = mx$

This limit is diff for diff value of m. Since the limit of the function depends on the manner of approach to (0,0) therefore $\lim F(x,y)$ does not

Partial Differentiation
Chapter - 5th

def:-
Let F be a function of two independent variable x and y, the partial derivative of f with respect to x is the function F_x defined as

$$F_x(x,y) = \lim_{\delta x \to 0} \frac{F(x+\delta x, y) - F(x,y)}{\delta x}$$

Similarly the partial derivative of F with respect to y is the function F_y

$$F_y(x,y) = \frac{F(x, y+\delta y) - F(x,y)}{\delta y}$$

Exp. If $x = r\cos\theta$ and $y = r\sin\theta$
Prove that
$$\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$$

Sol. $x = r\cos\theta$
$y = r\sin\theta$
To eliminate θ from the give

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eqn by squaring and adding these,

$r^2 = (x^2+y^2)$

$r = (x^2+y^2)^{1/2}$

$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x$

$= \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y$

$= \frac{y}{r}$

Again $\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \frac{\partial r}{\partial x}}{r^2}$

$= \frac{r - x \cdot \frac{x}{r}}{r^2}$

$= \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}$ — (1)

$[\because r^2 = x^2+y^2]$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$

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LHS

$\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3} \cdot \frac{x^2}{r^3}$

$= \frac{x^2 y^2}{r^6}$

RHS $\left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$

$= \left[\frac{\partial}{\partial x} \cdot \frac{\partial r}{\partial y}\right]^2 = \left[\frac{\partial}{\partial x}\left(\frac{y}{r}\right)\right]^2$

$= \left[-\frac{y}{r^2} \cdot \frac{\partial r}{\partial x}\right]^2$

$= \left(-\frac{y}{r^2} \cdot \frac{x}{r}\right)^2 = \left(-\frac{xy}{r^3}\right)^2$

$\frac{x^2y^2}{6}$

LHS = RHS.

Hence prove that.

Homogeneous function:-

A function $F(x,y)$ is said to be a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n.

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Euler's thm:

Let u be a homogeneous function of order n in x and y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof.

Since u is a homogeneous function of degree n in x and y therefore we can express it

$$u = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot \left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) \quad -(1)$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$x^{n-1} f'\left(\frac{y}{x}\right)$$

$$y\frac{\partial u}{\partial y} = yx^{n-1} f'\left(\frac{y}{x}\right) \quad -(2)$$

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Adding (1) and (2) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot x^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot u$$

Exp. If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$ prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Sol.

$$u = \tan^{-1} \frac{x^3+y^3}{x-y}$$

$$\tan u = \frac{x^3+y^3}{x-y}$$

$$= \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\left(\frac{y}{x}\right)\right]}$$

$$= x^2 f\left(\frac{y}{x}\right)$$

$\therefore$ $\tan u$ is a homogeneous function of degree 2 in x and y

$\therefore$ By Euler's thm:

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial x}(\tan u) = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\cdot\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \quad \text{— (A)}$$

Exp. Find the total derivative of u with respect to t, when $u = e^x \sin y$, when $x = \log t$, $y = t^2$. Also verify by direct calculation.

Sol

$$u = e^x \sin y \quad \text{— (1)}$$

$$\left.\begin{aligned} x &= \log t \\ y &= t^2 \end{aligned}\right\} \quad \text{— (2)}$$

Now u is function of x, y and x, y are themselves function t. therefore u is a composite function of t.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$e^x \sin y\cdot\frac{1}{t} + e^x \cos y\cdot 2t$$

$$= t\cdot\sin t^2\cdot\frac{1}{t} + t\cdot\cos t^2\, 2t$$

$$\left[\because x = \log t \quad e^x = t\right.$$

$$\sin t^2 + 2t^2\cos t^2 \quad \text{— (3)}$$

By direct method :-

$$u = e^x \sin y$$

$$= e^{\log t}\sin t^2 = t\sin t^2$$

Diff w.r.t.:

$$\frac{du}{dt} = \sin t^2 + 2t^2\cos t^2$$

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Taylor's theorem for function of two variables:

Statement: If $f(x,y)$ is a function of two independent variable x and y possessing continuous partial derivatives of the n^{th} order in some neighbourhood of a point (a,b) and $(a+h, b+k)$ is any point of this neighbourhood

$$f(a+h, b+k)$$

$$= f(a,b) + df(a,b) + \frac{1}{2!}d^2 f(a,b)$$

$$+ \cdots \frac{1}{(n-1)!} d^{n-1} f(a,b) + \frac{1}{n!} d^n f(a+\theta h, b+\theta k),$$

$$d = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) \quad d^n = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$$

Proof:

Let $g = f(x,y)$ such that

$x = a + ht$, $y = b + kt$,

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$$\frac{dx}{dt} = h \text{ and } \frac{dy}{dt} = k$$

Since g is a function of x, y and x, y are themselves function of t, so g is a composition function.

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt}$$

$$h\frac{\partial g}{\partial x} + k\cdot\frac{\partial g}{\partial y}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)g$$

$$\therefore \frac{d}{dt} = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$$

$$\frac{d^2}{dt^2}g = \frac{d}{dt}\left(\frac{dg}{dt}\right)$$

$$= \frac{d}{dt}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)g$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)g$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 g$$

Proceeding like this

$$\frac{d^n g}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n g$$

Now g is a function of single variable t and $g^n(t)$ is continuous in $[0,1]$, therefore, by Maclaurin's then there exists atleast one number θ, $0 < \theta < 1$ such

$$g(t) = g(0) + t\,g'(0) + \frac{t^2}{2!}g''(0)$$

$$+ \cdots \frac{t^{n-1}}{(n-1)!}g^{n-1}(0) + \frac{t^n}{n!}g^n(\theta t)$$

For $t = 1$

$$g(1) = g(0) + g'(0) + \frac{1}{2!}g''(0) + \cdots$$

$$\frac{1}{(n-1)!}g^{n-1}(0) + \frac{1}{n!}g^n(\theta t) \quad \text{—} \; (1)$$

$$g(t) = F(x,y) = F(a+ht,\, b+kt)$$

$$g(1) = F(a+h,\, b+k)$$

$$g(0) = F(a,b)$$

Since $g^n(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n g(t)$

$$g^n(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n g(0)$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n F(a+\theta h,\, b+\theta k)$$

$$= d^n F(a+\theta h,\, b+\theta k)$$

putting these values in (1)

$F(a+h,\, b+k)$

$$= F(a,b) + dF(a,b) + \frac{1}{2!}d^2F(a,b)$$

$$+ \frac{1}{(n-1)!}d^{n-1}F(a,b) + \frac{1}{n!}d^nF(a+\theta h,\, b+\theta k)$$

Exp. Expand $e^x \sin y$ in powers of x and y as far as terms of third degree.

Sol $F(x,y) = e^x \sin y$

$\therefore F(0,0) = 0$

$f_x(x,y) = e^x \sin y$

$f_{xx}(x,y) = e^x \sin y$

$f_y(x,y) = e^x \cos y$

$f_{yy}(x,y) = -e^x \sin y$

$f_{xy}(x,y) = e^x \cos y$

$f_{xxx}(x,y) = e^x \sin y$

$f_{xxy}(x,y) = e^x \cos y$

$f_{xyy}(x,y) = -e^x \sin y$

$f_{yyy}(x,y) = -e^x \cos y$

$f_x(0,0) = 0$

$f_{xx}(0,0) = 0$

$f_y(0,0) = 1$

$f_{yy}(0,0) = 0$

$f_{xy}(0,0) = 1$

$f_{xxx}(0,0) = 0$

$f_{xxy}(0,0) = 1$

$f_{xyy}(0,0) = 0$

$f_{yyy}(0,0) = -1$

By Taylor's thm:

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$$

$$e^x \sin y = 0 + [0 + y] + \frac{1}{2!}[0 + 2xy + 0] + \frac{1}{3!}[3x^2y - y^3] + \text{——}$$

$$y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3$$

Chapter - 6th
Differentiability of Functions of Two Variable

Diff Let (a,b) be a point in the domain of the function $F: R^2 \to R$ and $(a+h, b+k)$ be a point in the neighbourhood of point (a,b). The function F is said to be differentiable at point (a,b) if

$$F(a+h, b+k) - F(a,b)$$
$$= Ah + Bk + h\phi(h,k) + k\psi(h,k)$$

where A, B are constants independent of h, k and $\phi(h,k) \to 0$, $\psi(h,k) \to 0$ as $h,k \to (0,0)$

Exp. Show that the function

$$F(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2}, & xy \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous and possesses first order partial derivative but not differential at the origin.

Sol.

$$|F(x,y) - F(0,0)| = \left|\frac{x^3-y^3}{x^2+y^2} - 0\right|$$

putting $x = r\cos\theta$, $y = r\sin\theta$

$$|F(x,y) - F(0,0)| = \left|\frac{r^3 (\cos^3\theta - \sin^3\theta)}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$

$$= |r\cos^3\theta - r\sin^3\theta|$$

$$\leq r|\cos^3\theta| + |r\sin^3\theta|$$

$$\leq |r| + |r|$$

$$= 2|r| = 2\sqrt{x^2+y^2} < \varepsilon$$

$|F(x,y) - F(0,0)| < \varepsilon$ if $2\sqrt{x^2+y^2} < \varepsilon$

~~$|F(x,y) - F(0,0)| < \varepsilon$ if~~

$$\sqrt{(x-0)^2 + (y-0)^2} < \frac{\varepsilon}{2}$$

$|F(x,y) - F(0,0)| < \varepsilon$ if $\sqrt{(x-0)^2+(y-0)^2} < \delta$

$$\delta = \frac{\varepsilon}{2}$$

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$\therefore$ the function f is continuous at the point $(0,0)$

(ii). $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{h-0}{h} = 1$

$f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k}$

$= \dfrac{-k-0}{k} = -1$

Hence the first order partial derivatives exist at origin

(iii) Suppose that f is diff. at $(0,0)$

$\therefore$ $f(h,k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + h\phi(h,k) + k\psi(h,k)$

where $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$

as $h,k \to (0,0)$

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$\therefore$ $\dfrac{h^3 - k^3}{h^2 + k^2} - 0$

$h(1) + k(-1) + h\phi(h,k) + k\psi(h,k)$ — (1)

put $h = r\cos\theta$ and $k = r\sin\theta$

$\therefore$ $\phi\ h,k \to 0$ and $\psi\ h,k \to 0$

$\therefore$ Eqn (1)

$\dfrac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2} = r\cos\theta - r\sin\theta$

$\dfrac{r^3(\cos^3\theta - \sin^3\theta)}{r^2} = r(\cos\theta - \sin\theta)$

$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$

$(\cos\theta - \sin\theta)(\cos^2\theta + \sin^2\theta + \cos\theta\sin\theta)$
$= \cos\theta - \sin\theta$

$(\cos\theta - \sin\theta)(1 + \cos\theta\sin\theta) - (\cos\theta - \sin\theta)$
$= 0$

$(\cos\theta - \sin\theta)[1 + \cos\theta\sin\theta - 1] = 0$

$(\cos\theta - \sin\theta)(\cos\theta\sin\theta) = 0$

which is not true for all value of θ. Hence our supposition is wrong.

$\therefore$ F is not differentiable at $(0,0)$

Maximum and minimum of a function of two variables
Chapter - 7th

Defin:

$\therefore$ Maximum Value:- A function $F(x,y)$ is said to have a maximum value at a point (a,b) if there exists a neighbourhood N of (a,b) such that

$$F(a,b) > F(a+h, b+k) \text{ for all } (a+h, b+k) \in N$$

A function $F(x,y)$ is said to have a maximum value at a point (a,b) if $F(a,b) > F(a+h, b+k)$ for small values of h and k positive or negative.

Minimum Value:-

A function $F(x,y)$ is said to have a minimum value at a point (a,b) if there exists a neighbourhood N of (a,b) such that

$$F(a,b) < F(a+h, k+k) \text{ for all } (a+h, b+k) \in N$$

Extreme value: A function $f(x,y)$ is said to have extreme value at (a,b) if it has either a maximum or minimum at (a,b).

Absolute maximum:- A function $f(x,y)$ is said to have an absolute maximum at (a,b) if $f(a,b) > f(x,y)$ for all (x,y) in the domain of f.

Absolute minimum:- A function $f(x,y)$ is said to have an absolute minimum at (a,b) if

$f(a,b) < f(x,y)$ for all x,y in the domain of f.

Exp. Examine for extreme value $f(x,y) = 3x^2 - y^2 + x^3$

$$f(x,y) = 3x^2 - y^2 + x^3$$

$$\frac{\partial f}{\partial x} = 6x + 3x^2$$
$$= 3x(2+x)$$

$$\frac{\partial^2 f}{\partial x^2} = 6 + 6x$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial^2 f}{\partial y^2} = -2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$\frac{\partial f}{\partial x} = 0$, and $\frac{\partial f}{\partial y} = 0$

$3x(2+x) = 0$ $\quad x = 0, -2$

and $-2y = 0$ $\quad y = 0$

Thus extreme point are $(0,0)$ and $(-2,0)$

At $(0,0)$

$$A = \frac{\partial^2 f}{\partial x^2} = 6$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$C = \frac{\partial^2 f}{\partial y^2} = -2$$

$$AC - B^2 = -12 < 0$$

$\therefore$ $(0,0)$ is a saddle point

At $(-2, 0)$

$A = -6, B = 0 \quad C = -2$

$\therefore AC - B^2 = 12 > 0$ and $A = -6 < 0$

$(-2, 0)$ is a point of maximum

and maximum value

$$f(-2, 0) = -12 - 0 + (-8)$$
$$= 4$$

Ans

Exp. Find the volume of the largest rectangular Parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol For the rectangular parallelopiped inscribed in the ellipsoid to be maximum by symmetry, the centre of the parallelopiped is the same as the centre of the ellipsoid which is the origin.

$\therefore$ $2x, 2y, 2z$ are the dimensions of the parallelopiped

$$\text{Volume} = 8xyz$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$(x > 0, y > 0, z > 0)$

Let $f(x, y, z) = 8xyz$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Let $\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

For stationary points, by Lagrange's method,

$$\frac{\partial f}{\partial x} + k \frac{\partial \phi}{\partial x} = 0$$

$$8yz + k \frac{2x}{a^2} = 0 \quad - (1)$$

$$\frac{\partial F}{\partial y} + k\frac{\partial \phi}{\partial y} = 0$$

$$8zx + k\frac{2y}{b^2} = 0 \quad \text{— (3)}$$

$$\frac{\partial F}{\partial z} + k\frac{\partial \phi}{\partial z} = 0$$

$$8xy + k\frac{2z}{c^2} = 0 \quad \text{— (4)}$$

Multiplying (2), (3), (4) by (x, y, z) respectively and adding we have

$$24xyz + 2k\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$12xyz + k = 0$$

$$k = -12xyz \quad \text{— (5)}$$

$\therefore$ from (2),

$$8yz + \frac{2x}{a^2}(-12xyz) = 0$$

$$8yza^2 - 24x^2yz = 0$$

$$8yz(a^2 - 3x^2) = 0$$

$$8yz \neq 0 \qquad a^2 - 3x^2 = 0$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly

$$y = \frac{b}{\sqrt{3}}$$

$$z = \frac{c}{\sqrt{3}}$$

$\therefore$ Maximum value of

$$F(x, y, z) = 8xyz$$

$$= 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}}$$

$$= \frac{8abc}{3\sqrt{3}} \text{ Ans}$$

Curves in space.

Chapter - 8th

Definition:-

Analytic Function:-

A function f is said to be analytic over I if it is single valued and possesses continuous derivatives of all order at every point of I.

Regular curve:-

The vector curve r, defined on I is said to be regular, if $\frac{dr}{dt}$, derivative w.r.t parameter t never vanishes.

$\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are never zero.

Unit vector Along the tangent to a Given curve:-

$$P = \frac{dr}{ds}$$

$$\frac{dr}{ds} = r'$$

Formula for length of Arc of a curve.

$$S = \int_{t_0}^{t} ds = \int_{t_0}^{t} |\dot{r}|\, dt$$

$$= \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

Qup. Find the normal form of the curve.

$2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 6t k$, $-\infty < t < \infty$

The eqn of the given curve is

$$r(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 6t k$$

Diff w.r.t. t, we have.

$$\frac{dr}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 6k$$

$$\therefore \left|\frac{dr}{dt}\right| = \sqrt{4\sin^2 t + 4\cos^2 t + 36}$$

$$\sqrt{4(\sin^2 t + \cos^2 t) + 36}$$

$$= \sqrt{4+36} = \sqrt{40}$$

$$2\sqrt{10}.$$

Let S be the length of arc from $t=0$ to any point t on the curve then

$$S = \int_0^t \left|\frac{dr}{dt}\right| dt$$

$$\int_0^t 2\sqrt{10}\,dt = 2\sqrt{10}\,t$$

$$S = 2\sqrt{10}\,t \Rightarrow t = \frac{S}{2\sqrt{10}}$$

putting the value of t in (1) we get

$$r = 2\cos\frac{S}{2\sqrt{10}}\,\hat{i} + 2\sin\frac{S}{2\sqrt{10}}\,\hat{j} + \frac{3S}{2\sqrt{10}}\,\hat{k}$$

which is the normal form of the given curve

★ Eqn of a Tangent line at a point on a space curve

$$R = r + u\,\dot{r} \quad \text{when} \quad \frac{r}{\frac{dS}{dt}} = u$$

★ Cartesian form of eqn of tangent line

$$\frac{X-x}{\dot{x}} = \frac{Y-y}{\dot{y}} = \frac{Z-z}{\dot{z}}$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$

$$\dot{z} = \frac{dz}{dt}$$

★ To find the eqn of the tangent line to the curve of intersection of $F_1(x,y,z)=0$ and $F_2(x,y,z)=0$

$$\frac{X-x}{\frac{\partial F_1}{\partial y}\cdot\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\cdot\frac{\partial F_2}{\partial y}} = \frac{Y-y}{\frac{\partial F_1}{\partial z}\cdot\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\cdot\frac{\partial F_2}{\partial z}}$$

$$= \frac{Z-z}{\frac{\partial F_1}{\partial x}\cdot\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\cdot\frac{\partial F_2}{\partial x}}$$

Exp. Find the eqn of tangent line at the point $t=1$ to the curve

$$x = 1+t,\ y = -t^2,\ z = 1+t^2$$

$$\frac{dx}{dt} = 1,\quad \frac{dy}{dt} = -2t,\quad \frac{dz}{dt} = 2t$$

At $t=1$

$x = 2,\ y = -1,\ z = 2$

$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -2, \quad \frac{dz}{dt} = 2$

$\therefore$ Eqn of the tangent line is

$$\frac{X-x}{\frac{dx}{dt}} = \frac{Y-y}{\frac{dy}{dt}} = \frac{Z-3}{\frac{dz}{dt}}$$

$$\frac{X-2}{1} = \frac{Y+1}{-2} = \frac{Z-2}{2} \quad \underline{\text{Ans}}$$

Osculating plane :-

Definition :- The osculating plane at a point P of a curve of class ≥ 2, is the limiting position of the plane which contains the tangent line at P and a neighbouring point Q on the curve as $Q \to P$.

☆ Eqn of the osculating plane at a point P of the curve.

$$[R - r(s) \quad r'(s) \quad r''(s)] = 0$$

☆ Eqn of the osculating plane from above definition

$$[R - r(t) \quad \dot{r}(t) \quad \ddot{r}(t)] = 0$$

☆ Cartesian form of osculating plane.

$$R - r(s) \quad r'(s) \quad r''(s) = 0$$

or

$$[R - r(t) \quad \dot{r}(t) \quad \ddot{r}(t)] = 0$$

☆ Order of contact b/w curve and surface.

Case I :- $F'(t_0) = 0$

In this case t_0 is said to be simple zero of $F(t)$

The curve and the surface have contact of zero order.

Case II:

$$F'(t_0) = 0 \text{ and } F''(t_0) \neq 0$$

In this case t_0 is said to be double zero of $F(t_0)$. The curve and the surface have double point contact and this contact is of first order.

Case III:

$$F'(t_0) = F''(t_0) = 0 \text{ and } F'''(t_0) \neq 0$$

In this case t_0 is said to be triple zero of $F(t)$. The curve and the surface have three point contact and this contact is of second order.

Q.) Find the eqn of the plane that has three contact at the origin with the curve

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1$$

Sol. Any plane through the origin is

$$ax + by + cz = 0$$

Eqn of given curve are

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1$$

At origin, we have

$$t = 1$$

points which are common to (1) and (2) are obtained by solving (1) and (2). Substituting the value of x, y, z from 2 in (1), we have

$$F(t) = a(t^4 - 1) + b(t^3 - 1) + c(t^2 - 1) = 0 \quad (3)$$

Since the plane has three point contact with the curve at the origin

$$F'(1) = 0 \quad F''(1) = 0 \quad (4)$$

$$F'(t) = 4at^3 + 3bt^2 + 2ct,$$
$$F''(t) = 12at^2 + 6bt + 2c \quad (5)$$

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From (4) and (5) we have

$$f'(1) = 4a + 3b + 2c = 0$$
$$f''(1) = 12a + 6b + 2c = 0$$

Solving these eqn

$$\frac{a}{6-12} = \frac{b}{24-8} = \frac{c}{24-36}$$

$$\frac{a}{3} = \frac{b}{-8} = \frac{c}{6}$$

Hence the eqn of the required plane is

$$3x - 8y + 6z = 0 \text{ Ans}$$

★ Eqn of Tangent Plane at any point of the surface

$$F(x, y, z) = 0$$

the $(R - r) \cdot \nabla F = 0$

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★ Show that normal plane is perpendicular to the osculating plane.

$$\dot{r} \cdot (\dot{r} \times \ddot{r}) = 0$$

★ Eqn of osculating plane at a point on the curve of intersection of two surface.

$$F(r) = 0 \quad , \quad \psi(r) = 0$$

$$\frac{(R-r) \cdot \nabla F}{\ddot{r} \cdot (\nabla F)''} = \frac{(R-r) \cdot \nabla \psi}{\ddot{r} \cdot (\nabla \psi)''}$$

Ex Find the osculating plane at (x_1, y_1, z_1) on the curve of intersection of cylinders

$$x^2 + y^2 = a^2, \quad y^2 + z^2 = b^2$$

Sol. The given eqn of surface are

$$F(x, y, z) = x^2 + z^2 - a^2 = 0$$
$$\psi(x, y, z) = y^2 + z^2 - b^2 = 0$$

Diff both eqn w.r.t (t) we have

$xx' + zz' = 0$ and $yy' + zz' = 0$

which gives

$$\frac{x'}{\frac{1}{x}} = \frac{y'}{\frac{1}{y}} = \frac{z'}{\frac{1}{z}}$$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \frac{\partial^2 f}{\partial z^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial x \partial z} = 0 \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$$

$\underline{\underline{\theta_1}}$

$$\frac{\partial \psi}{\partial x} = 0 \qquad \frac{\partial \psi}{\partial y} = 2y \qquad \frac{\partial \psi}{\partial z} = 2z$$

$$\frac{\partial^2 \psi}{\partial x^2} = 0 \qquad \frac{\partial^2 \psi}{\partial y^2} = 2 \qquad \frac{\partial^2 \psi}{\partial z^2} = 2$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = 0 \qquad \frac{\partial^2 \psi}{\partial y \partial z} = 0 \qquad \frac{\partial^2 \psi}{\partial z \partial x} = 0$$

We can find their value at (x_1, y_1, z_1), using the formula obtained.

$$\frac{(X - x_1)2x_1 + (Y - y_1)0 + (Z - z_1)2z_1}{\frac{2}{x_1^2} + \frac{2}{z_1^2}} = \frac{2y_1(Y - y_1) + 2z(Z - z_1)}{\frac{2}{y_1^2} + \frac{2}{z_1^2}}$$

$$= \frac{(Xx_1 + Zz_1 - x_1^2 + z_1 z)x_1^2 z_1^2}{x_1^2 + z_1^2} = \frac{(Yy_1 + Zz_1 - y_1^2 - z_1^2)y_1^2 z_1^2}{y_1^2 + z_1^2}$$

$$\frac{(Xx_1 + Zz_1 - a^2)x_1^2}{a^2} = \frac{(Yy_1 - Zz_1 - b^2)y_1^2}{b^2}$$

$\left[\because x_1^2 + z_1^2 = a^2 \right.$

$y_1^2 + z_1^2 = b^2$

$$\frac{Xx_1^2+(2z_1-a^2)(a^2-z_1^2)}{a^2} = \frac{Yy_1^3+(2z_1-b^2)(b^2-z_1^2)}{b^2}$$

$$\frac{Xx_1^3-2z_1^3-a^4+a^2(z_1^2+2z_1)}{a^2}$$

$$= \frac{Y y_1^3 - 2z_1^3 - b^4 + b^2(z_1^2+2 \cdot z_1)}{b^2}$$

$$= \frac{Xx_1^3 - 2z_1^3 - a^4}{a^2} = \frac{Yy_1^3 - 2z_1^3 - b^4}{b^2}$$

Circle of Curvature and Spherical Curvature

Chapter - 9th

Exp Show that the radius of spherical curvature of a circular helix $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\cot\alpha$ is equal to the radius of circular curvature.

Sol. Let r be the position vector of any point on the curve

$$r = (a\cos\theta,\ a\sin\theta,\ a\theta\cot\alpha)$$

$$r' = (-a\sin\theta,\ a\cos\theta,\ a\cot\alpha)\frac{d\theta}{ds}$$

$$t = (-a\sin\theta,\ a\cos\theta,\ a\cot\alpha)\frac{d\theta}{ds}$$

Squaring both side of as we get

$$|t^2| = (a^2\sin^2\theta + a^2\cos^2\theta + a^2\cot^2\alpha)\left(\frac{d\theta}{ds}\right)^2$$

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$$1 = a^2(1+\cot^2\alpha)\left(\frac{d\theta}{ds}\right)^2 \quad [\because t^2 = 1]$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2\csc^2\alpha}$$

$$\frac{d\theta}{ds} = \frac{1}{a\csc\alpha} = \frac{\sin\alpha}{a}$$

$$t = (-a\sin\theta, a\cos\theta, a\cot\alpha)\frac{d\theta}{ds}$$

$$t = (-a\sin\theta, a\cos\theta, a\cot\alpha)\frac{\sin\alpha}{a}$$

$$t = \sin\alpha(-\sin\theta, \cos\theta, \cot\alpha) \quad \text{---}(2)$$

Diff (2) w.r.t. s we have

$$t' = \sin\alpha(-\cos\theta, -\sin\theta, 0)\frac{d\theta}{ds}$$

$$Kn = \sin\alpha(-\cos\theta, -\sin\theta, 0)\frac{\sin\alpha}{a}$$

$$Kn = \frac{\sin^2\alpha}{a}(-\cos\theta, -\sin\theta, 0)$$

$[\because t' = Kn]$

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$$|Kn| = \left[\frac{\sin^4\alpha}{a^2}(\cos^2\theta + \sin^2\theta)\right]^{1/2}$$

$$|Kn| = K$$

$K = \frac{\sin^2\alpha}{a}$ which is constant quantity

$\rho = \text{constant} \Rightarrow \rho' = 0$

$\therefore$ If R is the radius of spherical curvature by

$$R^2 = \rho^2 + \sigma^2\rho'^2$$

$$R = \rho$$

Involutes and Evolutes.

Chapter - 10th

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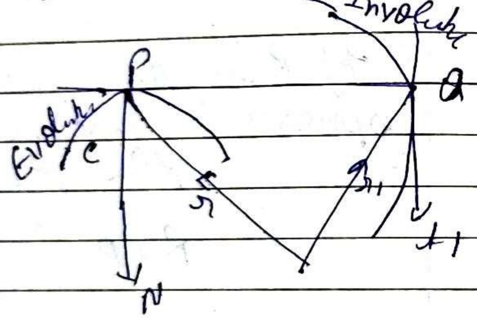
Involutes:-

Let C be a given curve whose eqn is $r = r(s)$, where S denotes the arc length parameter. A curve C_1 is said to be an involute of C_1 if tangent to C is normal to C_1.



From (1) $r_1(s_1) = r(s) + (c-s)t$, which gives the eqn of involute C_1 of C.

* Curvature of the involute.

$$\kappa_1 = \frac{\sqrt{\tau^2 + k^2}}{k(c-s)}$$

* Evolutes:-

$$r(s_1) = r(s) + \lambda n + \mu b$$

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Exp. Find the involutes and evolutes of circular helix.

$x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$.

Sol. (1) the eqn of circular helix

$$r = a\cos u\, i + a\sin u\, j + au\tan\alpha\, k \quad -(1)$$

$$\dot{r} = -a\sin u\, i + a\cos u\, j + a\tan\alpha\, k \quad -(2)$$

and $|\dot{r}| = \sqrt{a^2 + a^2\tan^2\alpha} = a\sec\alpha$ -(3)

$$t = \frac{dr}{du} \cdot \frac{du}{ds} = \frac{\dot{r}}{\frac{ds}{du}}$$

$$t = \frac{-a\sin u\, i + a\cos u\, j + a\tan\alpha\, k}{\frac{ds}{du}} \quad -(4)$$

$$\frac{ds}{du} = |\dot{r}| = a\sec\alpha \quad -(5)$$

$$S = \int_0^u a\sec\alpha\, du = a\sec\alpha\, u$$

$$t = \frac{-a\sin u\,\hat{i} + a\cos u\,\hat{j} + a\tan\alpha\,\hat{k}}{a\sec\alpha}$$

$$t = \frac{-\sin u\,\hat{i} + \cos u\,\hat{j} + \tan\alpha\,\hat{k}}{\sec\alpha} \quad \text{--- (6)}$$

the eqn of involute is given by

$$r_1 = r + (c-s)\,t$$

$$r_1 = a\cos u\,\hat{i} + a\sin u\,\hat{j} + au\tan\alpha\,\hat{k} + (c-s)\left(\frac{-\sin u\,\hat{i} + \cos u\,\hat{j} + \tan\alpha\,\hat{k}}{\sec\alpha}\right)$$

using (1) and (6)

$$r_1 = \left(a\cos u - \frac{(c-s)\sin u}{\sec\alpha}\right)\hat{i} + \left(a\sin u + \frac{(c-s)\cos u}{\sec\alpha}\right)\hat{j} + \left(au\tan\alpha + \frac{(c-s)\tan\alpha}{\sec\alpha}\right)\hat{k} \quad \text{--- (7)}$$

where $S = a\sec\alpha\, u$.

If $r = (x, y, z)$ then from (7), the eqn of involutes are

$$x = a\cos u - (c-s)\sin u\cos\alpha$$
$$y = a\sin u + (c-s)\cos u\cos\alpha$$
$$z = au\tan\alpha + (c-s)\tan\alpha\cos\alpha$$

where $S = a\sec\alpha\, u$.

Ex. Show that the locus of the centre of curvature of a curve is an evolute only when the curve is a plane curve.

Sol. the evolutes of the curve

$$r = r(s)$$
$$r_1 = r + \rho n - \rho\tan(\psi(s) + c)\,b \quad \text{--- (1)}$$

where c is an arbitrary constant.

The locus of centre of curvature of the an curve

$r = r(s)$ is given by

$r_1 = r + \rho n$ — (2)

Eqn (1) and (2) represent the same curve.

$r + \rho n - \rho \tan(\psi(s) + c)\, b$
$= r + \rho n$

$\rho \tan(\psi(s) + c)\, b = 0$

Assume $\rho \neq 0$ then

$\tan(\psi(s) + c)\, b = 0$

$\tan(\psi(s) + c) = 0$ $[\because |b| = 1]$

$\psi + c = m\pi$ when m is an integer

$\frac{d\psi}{ds} = 0$ $\tau = 0$

Hence the curve is a plane curve

Chapter – 11th
Concept of a Surface and Envelopes

Definition of Surface:- A surface is defined as the locus of a point whose cartesian coordinates (x, y, z) are function of two independent variable u, v called parameters.

$x = f(u, v)$
$y = g(u, v)$
$z = h(u, v)$

Implicit Representation of a Surface:-

$F(x, y, z) = 0$

Types of Singularity:- These singularities are due to the nature of the surface. These are independent of the choice of parametric representation. The vertex of the cone is an

Essential singularity :- these singularities are due to the nature of the surface. these are independent of the choice of parametric representation. the vertex is an essential singularity.

(2) Artifical Singularity :-

these singularities are due to the choice of particular parametric representation. the origin or pole in the plane referred to polar coordinates is an artificial singularity.

Cartesian Eqn of the Normal :

$$\frac{X-x}{\frac{\partial f}{\partial x}} = \frac{Y-y}{\frac{\partial f}{\partial y}} = \frac{Z-z}{\frac{\partial f}{\partial z}}$$

Exp. Find the eqn of tangent plane and normal to the surface $xyz=4$ at the point $(1,2,2)$

Sol. the eqn of the surface is

$$F(x,y,z) = xyz-4=0$$

Diff partially wrt x, y, z

$$\frac{\partial f}{\partial x} = yz, \quad \frac{\partial f}{\partial y} = zx, \quad \frac{\partial f}{\partial z} = xy$$

At the point $(1,2,2)$

$$\frac{\partial f}{\partial x} = 4, \quad \frac{\partial f}{\partial y} = 2, \quad \frac{\partial f}{\partial z} = 2$$

∴ Eqn of the tangent plane at $(1,2,2)$ is

$$(X-1)\frac{\partial f}{\partial x} + (Y-2)\frac{\partial f}{\partial y} + (Z-2)\frac{\partial f}{\partial z} = 0$$

$(x-1)4 + (y-2)\,0 + (z-2)\frac{\partial f}{\partial z} = 0$

$2x + y + z - 6 = 0$

Eqn of the normal is

$$\frac{x-1}{\frac{\partial f}{\partial x}} = \frac{y-2}{\frac{\partial f}{\partial y}} = \frac{z-2}{\frac{\partial f}{\partial z}}$$

$$\frac{x-1}{4} = \frac{y-2}{0} = \frac{z-2}{2}$$

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}.$$

Exp. Find the envelope of the sphere

$$(x-a\cos\theta)^2 + (y-a\sin\theta)^2 + z^2 = b^2$$

Sol. Here $f(x, y, z, \theta)$

$$x^2+y^2+z^2-2ax\cos\theta-2ay\sin\theta+a^2-b^2=0 \quad -(1)$$

$$\frac{\partial f}{\partial \theta} = 2ax\sin\theta - 2ay\cos\theta = 0$$

$-(2)$

To obtain the eqn of the envelope, we eliminate θ b/w (1) and (2)

From (2), $\tan\theta = \frac{y}{x}$

$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$ and $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

Substituting these valu in (1), the eqn of the envelope is

$$x^2+y^2+z^2-2ax.\frac{x}{\sqrt{x^2+y^2}}-2ay\frac{y}{\sqrt{x^2+y^2}}+a^2-b^2=0$$

$$(x^2+y^2+z^2)-2a\left(\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right)+a^2-b^2=0$$

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$$x^2+y^2+z^2+a^2-b^2 = 2a\sqrt{x^2+y^2}$$

Squaring both sides

$$\left(x^2+y^2+z^2+a^2-b^2\right)^2$$

$$= 4a^2\left(x^2+y^2\right)$$