

MAA OMWATI DEGREE COLLEGE,
HASSANPUR (PALWAL)

NOTES

SUBJECT : Optics-I (Physics)

CLASS : B.SC 3rd Sem

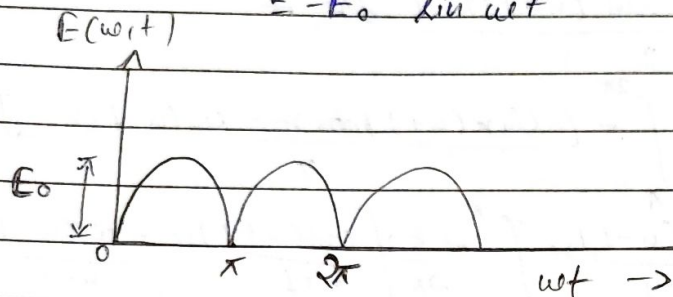
SESSION : 2021-2022

Q. Full wave Rectifier:-

A Full wave Rectifier passes the positive half-cycle and inverts the negative half cycle of the incoming waves as shown in fig:-

$$E(\omega t) = E_0 \sin \omega t \quad 0 \leq \omega t \leq \pi$$

$$= -E_0 \sin \omega t \quad \pi \leq \omega t \leq 2\pi$$



taking $\omega t = x$,

$$E(x) = E_0 \sin x \quad 0 \leq x \leq \pi$$

$$= -E_0 \sin x \quad \pi \leq x \leq 2\pi$$

$$\left[\because \omega t = \pi, t = \frac{\pi}{\omega} \right]$$

The Fourier series of $E(\omega t) = E(x)$.

$$E(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

evaluating the Fourier's coefficients, we get,

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} E(x) dx$$

$$= \frac{E_0}{2\pi} \int_0^{\pi} \sin x dx + \frac{E_0}{2\pi} \int_{\pi}^{2\pi} -\sin x dx$$

$$= \frac{E_0}{2\pi} (-\cos x) \Big|_0^{\pi} + \frac{E_0}{2\pi} (\cos x) \Big|_{\pi}^{2\pi}$$

$$= \frac{E_0}{2\pi} [-\cos \pi + \cos 0] + \frac{E_0}{2\pi} [\cos 2\pi - \cos \pi]$$

$$\Rightarrow \frac{E_0}{2\pi} [1+1] + \frac{E_0}{2\pi} [1+1] = \frac{2E_0}{2\pi} + \frac{2E_0}{2\pi} = \frac{2}{\pi} E_0$$

For $n > 1$,

$$\begin{aligned}
 A_n &= \frac{1}{\pi} \int_0^{2\pi} E(x) \cos nx \, dx \\
 &= \frac{E_0}{\pi} \int_0^{\pi} \sin x \cos nx \, dx + \frac{E_0}{\pi} \int_{\pi}^{2\pi} -\sin x \cos nx \, dx \\
 &= \frac{E_0}{\pi} \int_0^{\pi} [\sin (n+1)x - \sin (n-1)x] \, dx + \frac{E_0}{\pi} \int_{\pi}^{2\pi} [-\sin (n+1)x + \sin (n-1)x] \, dx \\
 &= \frac{E_0}{2\pi} \left[\frac{-\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1} \right]_0^{\pi} + \frac{E_0}{2\pi} \left[\frac{\cos (n+1)x}{n+1} - \frac{\cos (n-1)x}{n-1} \right]_{\pi}^{2\pi} \\
 &= \frac{E_0}{2\pi} \left[\frac{-\cos (n+1)\pi + 1}{n+1} + \frac{\cos (n-1)\pi - 1}{n-1} \right] + \frac{E_0}{2\pi} \left[\frac{\cos (n+1)2\pi - \cos (n+1)\pi}{n+1} - \frac{\cos (n-1)2\pi - \cos (n-1)\pi}{n-1} \right] \\
 &= \frac{E_0}{2\pi} \left[\frac{-\cos (n+1)\pi + 1}{n+1} + \frac{\cos (n-1)\pi - 1}{n-1} + \frac{1 + \cos (n-1)\pi}{n-1} - \frac{1 - \cos (n+1)\pi}{n+1} \right] \\
 &= \frac{E_0}{2\pi} \left[\frac{-\cos (n+1)\pi + \cos (n-1)\pi + 1}{n+1} - \frac{1}{n-1} \right] \\
 &= \frac{E_0}{\pi} \left[\frac{-\cos (n+1)\pi + \cos (n-1)\pi - 2}{n^2-1} \right] \\
 &= \frac{E_0}{\pi} \left[\frac{-\cos (n+1)\pi + \cos (n-1)\pi - 2}{n^2-1} \right] \\
 &\left[\because \cos (n+1)\pi = (-1)^{n+1} \text{ and } \cos (n-1)\pi = (-1)^{n-1} \right] \\
 &= \frac{E_0}{\pi} \left[\cos (n-1)\pi \left(\frac{1}{n-1} - \frac{1}{n+1} \right) - \frac{2}{n^2-1} \right] \\
 &= \frac{E_0}{\pi} \left[\cos (n-1)\pi \left(\frac{2}{n^2-1} \right) - \frac{2}{n^2-1} \right] \\
 &= \frac{2E_0}{(n^2-1)\pi} [\cos (n-1)\pi - 1]
 \end{aligned}$$

$$= \begin{cases} 0 & \text{if } n \text{ is an odd int eg } n, n > 1 \\ \frac{-4E_0}{(n^2-1)\pi} & \text{if } n \text{ is an even int eg } n, n > 1 \end{cases}$$

For $n=1$,

$$\begin{aligned}
 A_n &= A_1 = \frac{E_0}{\pi} \int_0^{\pi} \sin x \cos x \, dx + \frac{E_0}{\pi} \int_{\pi}^{2\pi} -\sin x \cos x \, dx \\
 &= \frac{E_0}{2\pi} \left[\int_0^{\pi} \sin 2x \, dx - \int_{\pi}^{2\pi} \sin 2x \, dx \right] \\
 &= \frac{E_0}{2\pi} \left[-\frac{\cos 2x}{2} \right]_0^{\pi} + \frac{\cos 2x}{2} \Big|_{\pi}^{2\pi} \\
 &= \frac{E_0}{4\pi} [-\cos 2\pi + \cos 0 + \cos 4\pi - \cos 2\pi] \\
 &= \frac{E_0}{4\pi} [-1 + 1 + 1 - 1] = 0
 \end{aligned}$$

Further, $n > 1$,

$$\begin{aligned}
 B_n &= \frac{E_0}{\pi} \int_0^{\pi} \sin nx \sin nx \, dx + \frac{E_0}{\pi} \int_{\pi}^{2\pi} -\sin 2x \sin nx \, dx \\
 &= \frac{E_0}{2\pi} \int_0^{\pi} [\cos (n-1)x - \cos (n+1)x] \, dx - \frac{E_0}{2\pi} \int_{\pi}^{2\pi} [\cos (n-1)x - \cos (n+1)x] \, dx \\
 &= \frac{E_0}{2\pi} \left[\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1} \right]_0^{\pi} - \frac{E_0}{2\pi} \left[\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1} \right]_{\pi}^{2\pi} \\
 &= \frac{E_0}{2\pi} \left[\frac{\sin (n-1)\pi}{n-1} - \frac{\sin 0}{n-1} - \frac{\sin (n+1)\pi}{n+1} + \frac{\sin 0}{n+1} + \frac{\sin (n+1)2\pi}{n+1} - \frac{\sin (n-1)2\pi}{n-1} \right] \\
 &= \frac{E_0}{2\pi} \left[\frac{\sin (n-1)\pi}{n-1} - \frac{\sin (n-1)2\pi}{n-1} - \frac{\sin (n+1)\pi}{n+1} + \frac{\sin (n+1)2\pi}{n+1} \right] = 0 \\
 &\left[\because \sin (n-1)\pi = \sin (n+1)\pi = \sin 0 = 0 \right. \\
 &\quad \left. \& \sin 2(n-1)\pi = \sin 2(n+1)\pi = 0 \right]
 \end{aligned}$$

For $n=1$,

$$\begin{aligned}
 B_n &= B_1 = \frac{E_0}{\pi} \int_0^\pi \sin \omega t \sin \omega t d(\omega t) + \frac{E_0}{\pi} \int_\pi^{2\pi} -\sin \omega t \sin \omega t d(\omega t) \\
 &= \frac{E_0}{2\pi} \left[\int_0^\pi 2 \sin^2 \omega t d(\omega t) - \int_\pi^{2\pi} 2 \sin^2 \omega t d(\omega t) \right] \\
 &= \frac{E_0}{2\pi} \left[\int_0^\pi (1 - \cos 2\omega t) d(\omega t) - \int_\pi^{2\pi} (1 - \cos 2\omega t) d(\omega t) \right] \\
 &= \frac{E_0}{2\pi} \left[(\omega t) \right]_0^\pi - \frac{\sin 2\omega t}{2} \left[-\omega t \right]_\pi^{2\pi} + \frac{\sin 2\omega t}{2} \left[-\omega t \right]_\pi^{2\pi} \\
 &\Rightarrow \frac{E_0}{2\pi} [\pi - 0 - \pi + 0] = 0
 \end{aligned}$$

Using the values of Fourier coefficients, the required Fourier output voltage of a full wave rectifier is,

$$E(x) = \frac{2E_0}{\pi} - \frac{4E_0}{\pi} \left[\frac{1}{3} \cos 2x + \frac{1}{15} \cos 4x + \frac{1}{35} \cos 6x + \dots \right]$$

Replacing x by ωt , so we get,

$$E(\omega t) = \frac{2E_0}{\pi} - \frac{4E_0}{\pi} \left[\frac{1}{3} \cos 2\omega t + \frac{1}{15} \cos 4\omega t + \frac{1}{35} \cos 6\omega t + \dots \right]$$

* Integration of Fourier series:-

$$f(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) \quad \text{--- (1)}$$

$$\text{where } A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \quad \text{--- (1a)}$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad \text{--- (1b)}$$

$$\begin{aligned}
 \text{where } n &= 1, 2, 3, \dots \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt
 \end{aligned}$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad \text{--- (1c)}$$

$$\begin{aligned}
 \text{where } n &= 1, 2, 3, \dots \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt
 \end{aligned}$$

At finite the parts of discontinuity,

$$\frac{1}{2} [f(x+0) + f(x-0)]$$

instead of $f(x)$ in eq. (1),

Integrating term by Fourier's series,

$$\int_a^b f(x) dx = (A_0 x) \Big|_a^b + \sum_{n=1}^{\infty} \left(\frac{A_n}{n} \sin nx - \frac{B_n}{n} \cos nx \right) \Big|_a^b$$

* Fourier integrals:-

If $f(x)$ satisfies the following conditions, i.e.,

- $f(x)$ is defined in the interval $-L < x < L$
- $f(x)$ and $f'(x)$ are piecewise continuous in interval $-L < x < L$
- $f(x)$ is periodic with period $2L$.
- $f(x)$ is absolutely integrable in interval $(-\infty < x < \infty)$.

Then, Fourier series integral theorem states that,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} \cos(\omega(x-u)) d\omega \right] du \quad \text{--- (2)}$$

Double integral right hand sides of eq. (2) is called the Fourier integrals.

First form:-

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} \cos(\omega(x-u)) d\omega \right] du$$

$$\text{Now, } \int_{-\infty}^{\infty} \cos [w(x-u)] du = 2 \int_{-\infty}^{\infty} \cos [w(x-u)] du$$

$\therefore \cos w(x-u)$ is even function of w

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[2 \int_{-\infty}^{\infty} \cos [w(x-u)] du \right] dw \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} f(u) \cos [w(x-u)] du \\ &= \int_{-\infty}^{\infty} [A(w) \cos wx + B(w) \sin wx] dw \quad \text{--- (2)} \end{aligned}$$

$$\text{where } A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos (wu) du$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin (wu) du$$

Second form:- If $f(x)$ is an even function of x then, $f(u) \cos (wu)$ is an even function and $f(u) \sin (wu)$ is an odd function, by using property of definite integrals, we get,

$$A(w) = \frac{2}{\pi} \int_0^{\infty} f(u) \cos (wu) du$$

$$B(w) = 0$$

Fourier integral formula,

$$f(x) = \int_{-\infty}^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

reduces to Fourier cosine integral formula,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos wx \left[\int_0^{\infty} f(u) \cos (wu) du \right] dw \quad \text{--- (3)}$$

Third form:- If $f(x)$ is an odd function of x , then $f(u) \cos (wu)$ is an odd function and $f(u) \sin (wu)$ is an even function,

then, using property of definite integrals, we get,

$$A(w) = 0$$

$$B(w) = \frac{2}{\pi} \int_0^{\infty} f(u) \sin (wu) du$$

then, Fourier integral formula given by eq. (2), reduces to Fourier series sine formula,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin wx \left[\int_0^{\infty} f(u) \sin (wu) du \right] dw \quad \text{--- (2)}$$

Fourth form:-

w.k. that, $\sin [w(x-u)]$ is an odd function of w ,

$$\int_{-\infty}^{\infty} \sin [w(x-u)] dw = 0$$

is an odd function of w and substituting the values of the limits $-\infty$ to $+\infty$, it becomes zero, i.e.,

$$\int_{-\infty}^{\infty} \sin [w(x-u)] dw = 0$$

using the above integrals,

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} \sin [w(x-u)] dw \right] du$$

Multiply above eq. by $i(-1) = i$ & adding to eq.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} \cos [w(x-u)] dw \right] du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} [\cos [w(x-u)] + i \sin [w(x-u)]] dw \right] du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{iw(x-u)} dw du \quad \text{--- (4)}$$

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★ Find the Fourier transform of Gaussian Function, i.e.,

$$f(x) = e^{-x^2/2}$$

Fourier transform of $f(x)$ is (by definition).

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \times I \quad \text{--- (i)}$$

$$I = \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{-ikx} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[x^2/2 + i \frac{k}{\sqrt{2}} \cdot \frac{x}{\sqrt{2}} + \left(\frac{ik}{\sqrt{2}}\right)^2\right]} \cdot e^{k^2/2} dx$$

$$= e^{-k^2/2} \int_{-\infty}^{\infty} e^{-\left(x/\sqrt{2} + ik/\sqrt{2}\right)^2} dx \quad \text{--- (ii)}$$

$$\text{Put } \frac{x}{\sqrt{2}} + \frac{ik}{\sqrt{2}} = y$$

Diffⁿ both sides, we get,

$$\frac{dx}{\sqrt{2}} = dy \text{ or } dx = \sqrt{2} dy$$

Also from eq. (ii), when $x \rightarrow 0$, $y \rightarrow \infty$ & when $x \rightarrow -\infty$, $y \rightarrow -\infty$.

$$I = e^{-k^2/2} \int_{-\infty}^{\infty} e^{-y^2} \cdot \sqrt{2} dy$$

$$= \sqrt{2} e^{-k^2/2} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \sqrt{2} e^{-k^2/2} \cdot 2 \int_0^{\infty} e^{-y^2} dy \quad \text{--- (iii)}$$

[∵ Integrand is an even function of y]

$$\text{Now, put } y^2 = z$$

Diffⁿ both sides,

$$2y dy = dz$$

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$$\text{or, } dy = \frac{dz}{2y} = \frac{dz}{2z^{1/2}} = 2z^{-1/2} dz$$

Also, when $y \rightarrow 0$, $z \rightarrow 0$ & $y \rightarrow \infty$, $z \rightarrow \infty$,

∴ Eq. (iii) becomes,

$$I = \sqrt{2} e^{-k^2/2} \int_0^{\infty} e^{-z} \left(\frac{1}{2} z^{-1/2} dz \right)$$

$$= \sqrt{2} e^{-k^2/2} \int_0^{\infty} e^{-z} \cdot z^{-1/2} dz$$

$$= \sqrt{2} e^{-k^2/2} \left(\frac{1}{2} \right)$$

∴ $\int_0^{\infty} e^{-z} z^{-1/2} dz = \text{Gamma Fun. of } 1/2$ & is denoted by $\Gamma\left(\frac{1}{2}\right)$ [∵ value of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$]

$$= \sqrt{2} e^{-k^2/2} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{2\pi} \cdot e^{-k^2/2}$$

$$\therefore \text{from eq. (i), } F(k) = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} e^{-k^2/2} = e^{-k^2/2}$$

★ Find the sine and cosine transform of e^{-x} .

(i) Sine transform:-

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin kx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin kx dx \quad \int_0^{\infty} \int_0^{\infty} e^{-x} \sin kx dx \quad (\text{Integrate by parts})$$

$$= \frac{e^{-x}}{1+k^2} (-\sin kx - k \cos kx)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+k^2} (-\sin kx - k \cos kx) \right]_0^{\infty}$$

$$\text{Put limits, } = \frac{k}{1+k^2} \times \sqrt{\frac{2}{\pi}}$$

(ii) Cosine transform:-

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-x}}{1+k^2} (k \sin kx - \cos kx) \, dx$$

$$\int_0^{\infty} e^{-x} \cos kx \, dx \quad (\text{integrating by parts})$$

$$= \frac{e^{-x}}{1+k^2} (k \sin kx - \cos kx) \Big|_0^{\infty}$$

put limits, $= \frac{\sqrt{2}}{\pi} \frac{1}{(1+k^2)} = \frac{\sqrt{2}}{\pi} k \cdot \frac{1}{(1+k^2)}$

* Show that Fourier's transform of Gaussian's distribution function $f(x) = Ne^{-\alpha x^2}$ is also a Gaussian function.

The given Gaussian's functions are,

$$f(x) = Ne^{-\alpha x^2}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx - \alpha x^2} \, dx$$

$$= \frac{N}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-\alpha x^2} \cos kx \, dx + i \int_{-\infty}^{\infty} e^{-\alpha x^2} \sin kx \, dx \right]$$

the 2nd integral is zero, as it is the integral of an odd function from $-\infty$ to ∞ .

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} \cos kx \, dx = \sqrt{\frac{\pi}{\alpha}} e^{-k^2/4\alpha}$$

$$F(k) = \frac{N}{\sqrt{2\pi}} \sqrt{\frac{\pi}{\alpha}} e^{-k^2/4\alpha}$$

or, $F(k) = \frac{N}{\sqrt{2\alpha}} e^{-k^2/4\alpha}$

* Application of Fourier transform:-

(1) Evaluation of integrals:- Some typical integrals can be solved easily by the use of Fourier transform.

For example,

$$\text{let } I_1 = \int_0^{\infty} e^{-ax} \cos kx \, dx$$

$$\text{or } I_2 = \int_0^{\infty} e^{-ax} \sin kx \, dx$$

Integrating by parts, we get,

$$I_1 = \int_0^{\infty} \left[-\frac{1}{a} e^{-ax} \cos kx \right]_0^{\infty} - \int_0^{\infty} \frac{k}{a} e^{-ax} \sin kx \, dx$$

as, $\left(-\frac{1}{a} e^{-ax} \cos kx \right)$ is zero when $x = \infty$ and it is $-\frac{1}{a}$ when $x = 0$, so

$$I_1 = \frac{1}{a} - \frac{k}{a} \int_0^{\infty} e^{-ax} \sin kx \, dx = \frac{1}{a} - \frac{k}{a} I_2 \quad \text{--- (1)}$$

$$\text{Similarly, } I_2 = \int_0^{\infty} \left[-\frac{1}{a} e^{-ax} \sin kx \right]_0^{\infty} + \frac{k}{a} \int_0^{\infty} e^{-ax} \cos kx \, dx$$

As, $\left(-\frac{1}{a} e^{-ax} \sin kx \right)$ is zero when $x = \infty$ or $x = 0$, so that

$$I_2 = \frac{k}{a} I_1 \quad \text{--- (2)}$$

Substituting the values of I_2 in eqⁿ (1).

$$I_1 = \frac{1}{a} - \frac{k}{a^2} I_1$$

$$I_1 + \frac{k^2}{a^2} I_1 = \frac{1}{a} \text{ or } I_1 \left[\frac{a^2 + k^2}{a^2} \right] = \frac{1}{a}$$

$$I_1 = \frac{a}{(a^2 + k^2)} \quad \text{--- (3)}$$

From eq. ②, so we get,

$$I_2 = \frac{a}{k^2 + a^2} \cdot \frac{k}{a} = \frac{k}{k^2 + a^2} \quad \text{--- (4)}$$

Hence, if $F(x)$ is taken as $\sqrt{\frac{\pi}{2}} e^{-ax}$, with cosine and sine transforms are $\frac{a}{a^2 + k^2}$ and $\frac{k}{a^2 + k^2}$ respectively.

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos kt \, dk$$

$$\sqrt{\frac{2}{\pi}} e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a}{a^2 + k^2} \cos kn \, dk \quad \text{--- (5)}$$

$$\text{and } F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin(k) \sin kn \, dk$$

$$\text{or } \sqrt{\frac{\pi}{2}} e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{k}{a^2 + k^2} \sin kn \, dk \quad \text{--- (6)}$$

$$\therefore F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{\pi}{2}} e^{-ax} \cos kt \, dt = \frac{a}{k^2 + a^2}$$

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{\pi}{2}} e^{-ax} \sin kt \, dt = \frac{k}{k^2 + a^2}$$

$$\text{From eq. ⑤, ⑥, so we get,} \quad \int_0^\infty \frac{\cos kn}{a^2 + k^2} \, dk = \frac{\pi}{2a} e^{-ax} \quad \text{--- (7)}$$

$$\text{and } \int_0^\infty \frac{k \sin kn}{a^2 + k^2} \, dk = \frac{\pi}{2} e^{-ax} \quad \text{--- (8)}$$

(iii) Solution of ordinary differential equation:-

Let us consider a differential equation of n^{th} order is given by,

$$a_n \frac{d^2 y}{dx^2} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad \text{--- (1)}$$

where $a_0, a_1, a_2, \dots$ are the const. coefficients.

From eq. ①, can be written as,

$$L(y) = f(x)$$

$$\text{where } L(y) = a_n \frac{d^2 y}{dx^2} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_1 \frac{dy}{dx} + a_0 y$$

Taking Fourier's transform of both sides of eq. ②, we get,

$$T[L(y)] = a_n T\left[\frac{d^2 y}{dx^2}\right] + a_{n-1} T\left[\frac{d^{n-1} y}{dx^{n-1}}\right] + \dots + a_1 T\left[\frac{dy}{dx}\right] + a_0 T(y) \quad \text{--- (3)}$$

$$\text{Now, } T\left[\frac{d^2 y}{dx^2}\right] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{-ikx} \frac{d^2 y}{dx^2} \, dx \quad \text{--- (4)}$$

[∵ by definition of Fourier transform]

Integrating by R.H.S of above by parts, so we get,

$$T\left[\frac{d^2 y}{dx^2}\right] = \sqrt{\frac{2}{\pi}} \left[e^{-ikx} \frac{d^2 y}{dx^2} \right] + (ik) \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{-ikx} \frac{d^2 y}{dx^2} \, dx$$

First term vanishes at boundaries $x = \pm \infty$ by assuming $\frac{d^2 y}{dx^2} = 0$.

$$T\left[\frac{d^2 y}{dx^2}\right] = ik \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{-ikx} y \frac{d^2 y}{dx^2} \, dx$$

Repeating the above process $(n-1)$ times, so we get,

$$T\left[\frac{d^2 y}{dx^2}\right] = (ik)^n \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{-ikx} y \, dx = (ik)^n T(y)$$

Following this process, eq. ③, may be written as,

$$\begin{aligned} T[L(y)] &= a_n (ik)^n T(y) + a_{n-1} (ik)^{n-1} T(y) + \dots + a_1 (ik) T(y) + a_0 T(y) \\ &= [a_n (ik)^n + a_{n-1} (ik)^{n-1} + \dots + a_1 (ik) + a_0] T(y) \\ &= P(ik) T(y) \end{aligned}$$

where $P(ik) = a_n(ik)^n + a_{n-1}(ik)^{n-1} + \dots + a_1(ik) + a_0$

$$\therefore T[y] = T[L(y)] [P(ik)]^{-1} \\ = T[f(x)] [P(ik)]^{-1}$$

$$\text{or } T(y) = F(k) G(k) \quad \text{--- (5)}$$

where $F(k)$ is Fourier transform of $f(x) = T[f(x)]$

$$G(k) = [P(ik)]^{-1}$$

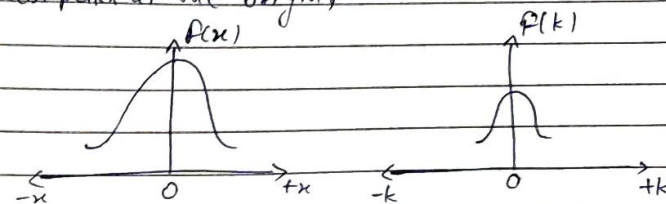
Hence y is written as a function of x by inverse transform of (5), as

$$y(x) = T^{-1}[F(k) G(k)] \quad \text{--- (6)}$$

R.H.S of the above equation is inverse Fourier transform of $[F(k) G(k)]$. But the product $[F(k) G(k)]$ is Fourier transform of $f(x)$ and $g(x)$ using convolution theorem. Hence $T^{-1}[F(k) G(k)]$ is the convolution function $f(x)g(x)$.

$$\therefore y(x) = T^{-1}[F(x) G(k)] = f(x)g(x) \\ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) g(x-u) du \text{ is the required solution.}$$

Both $f(x)$ & $g(x)$ are Gaussian distribution functions with their peaks at the origin,



INTERFERENCE OF DIVISION OF WAVEFRONT

* Interference of light:-

This redistribution of light energy obtained by the superposition of light waves from two coherent sources of light is called the interference of light.

At such points the intensity of light is maximum and this is called the constructive interference.

Similarly, at points where the crest of one wave falls over the trough of the other waves, the resultant amplitude of the resulting waves is minimum. At such points the intensity of light is minimum, and this is called destructive interference.

* Types of Interference:-

Interference can be divided in the following two classes:-

(i) Interference by division of wavefront:- A wavefront is divided into two parts either by reflection or by refraction. Some examples of this class of interference are:-

- Fresnel's biprism.
- Fresnel's double mirror
- My Lloyd's mirror
- Billet's split lens.

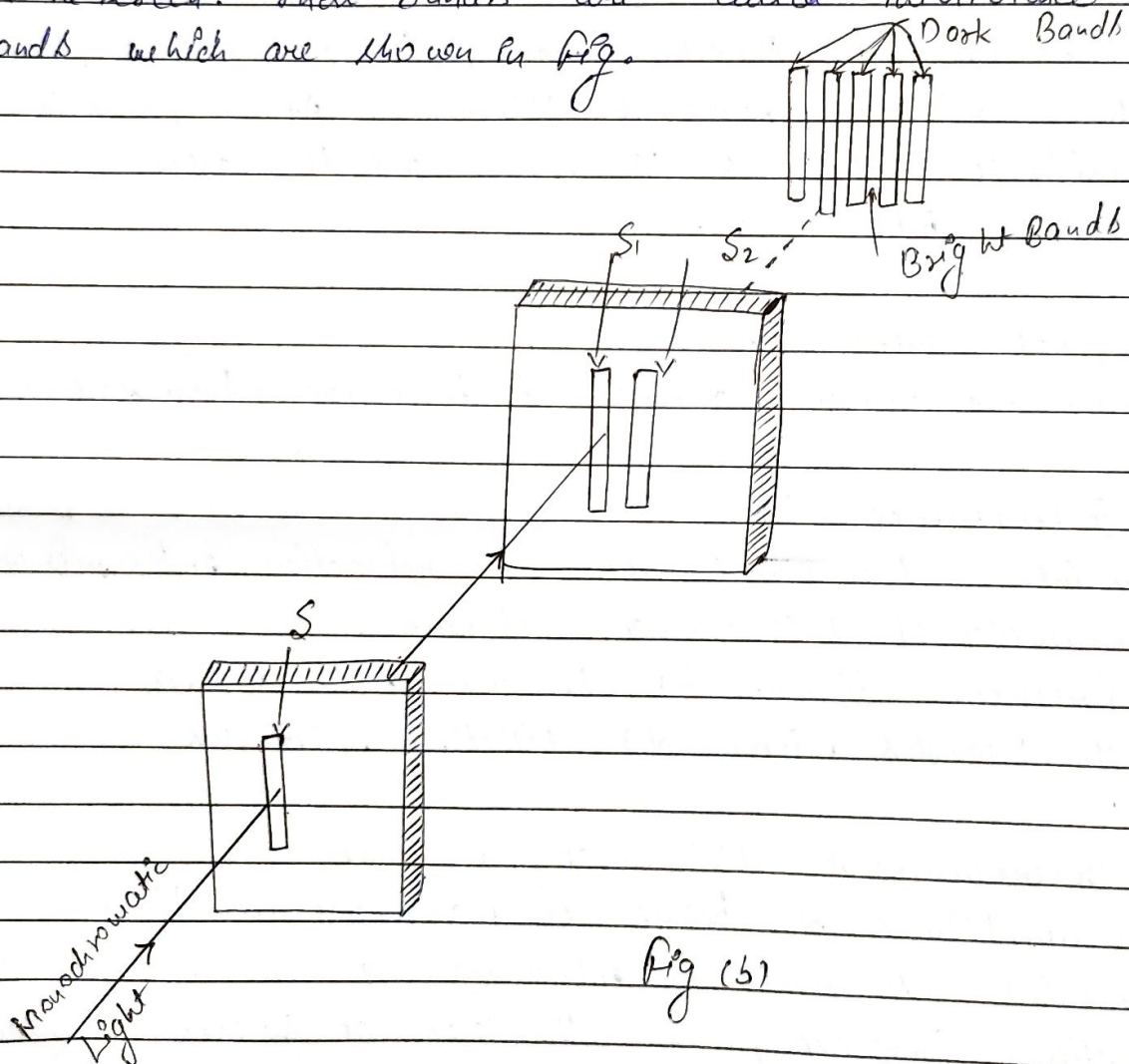
(ii) Interference of division of amplitude:-

The amplitude of beams of monochromatic light is divided into two parts by either by partial reflection or refraction. Some examples of this class of interference are:-

- Newton's rings
- Machelson's interferometer
- Parallel and wedge shaped - film.

* Experimental Demonstration of Interference - Young's Double Slit experiment :-

Dr. Thomas Young was the first who demonstrated the phenomenon of interference of light with the help of the experiment set up by him. It is a narrow slit which is illustrated by a source of monochromatic light (sodium lamp). Two more fine slits of S_1 and S_2 about 0.5 mm apart are placed at a suitable equidistant from S , alternate dark and bright bands are obtained on the screen. These bands are called interference bands which are shown in Fig.



These waves are of the same wavelength, same amplitude or same time period. These waves superimpose over each other. At points where the crest of one wave falls over the crest of the other waves, or trough of one wave falls over the trough of the other waves, the resultant amplitude is Maximum and light intensity is also max^m. These points are dotted at A, C, E on the screen. These are the regions of constructive interference. At points where the crest of one wave falls over the trough of the other wave, the resultant amplitude is Minimum and light intensity is also Minimum. These points are marked as cross and are obtained at B, D etc. on the screen. These are the regions of destructive interference.

It should be noted that the bright and dark bands are obtained alternatively and are equally spaced.

* Coherent sources:-

To produce continuous and stationary interference pattern it's essential that the intensity of light at any point in the superposition region should not change with time.

The two sources should be:-

- (i) light waves continuously of the same wavelength or frequency
- (ii) The two waves are either in the same phase or have a constant phase difference.

"Such like sources are called coherent sources.

"If both the sources of light are obtained from a single parent source by same device, then all the random phase changes occurring in the original parent sources are repeated in the secondary sources and the

relative phase remains const. These are called the coherent sources."

The coherent sources can be obtained experimentally by the following methods:-

- (i) Young's Lloyd's single mirror:- A narrow source and its image by reflection will act as a coherent source.
- (ii) Fresnel's double mirror:- Two virtual images produced by reflection will act as coherent sources.
- (iii) Fresnel's Biprism:- Two virtual images produced by refraction will act as coherent sources.
- (iv) Billet's split lens:- Two real images produced by refraction will act as coherent sources.

* Conditions for getting good interference fringes:-

- (i) The sources should be coherent. i.e. they should be continuously emits waves of same wavelength or frequency and should be have either the same phase or a constant phase difference.
- (ii) The amplitudes of two waves should be equal or nearly equal. With this we get good contrast b/w the bright and dark fringes.
- (iii) The two sources should be narrow.
- (iv) The sources of light should be strong.
- (v) The sources should be of monochromatic light.

* Phase difference & Path difference:-

We know that, if the path difference b/w two waves is λ

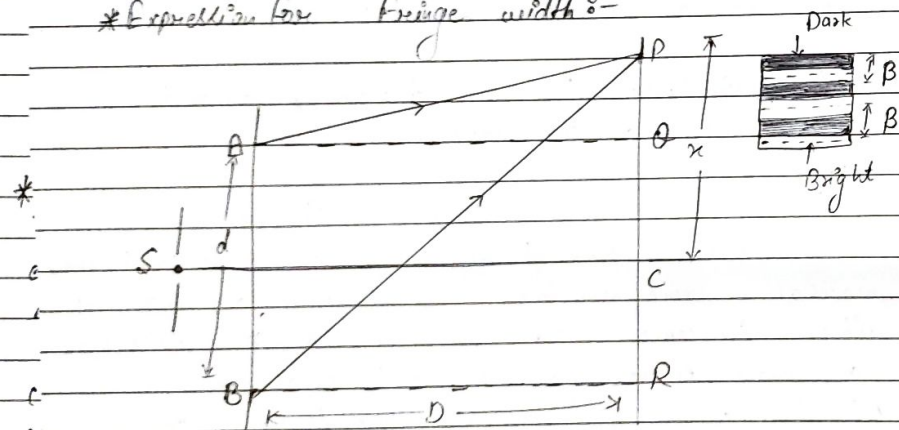
The phase difference is 2π . Therefore, a path difference x , the distance phase difference δ is calculated as follows:-
For a path difference λ , the phase difference $= 2\pi$.

$\therefore$ For a path difference x , the phase difference $= \frac{2\pi}{\lambda} x$

Hence, the phase difference, $\delta = \frac{2\pi}{\lambda} x$

or phase difference $= \frac{2\pi}{\lambda} x$ path difference.

* Expression for fringe width:-



Consider a slit S illuminated by a source of monochromatic light of wavelength λ and two pinholes A and B equidistant from S. Let A and B be the two coherent sources separated by distance d . Let the screen be placed at a distance D from the coherent sources. The point C is equidistant from A to B. The path difference, between the two waves reaching C is zero.

Consider a point P at a distance x from C. The path difference from Fig. is as follows:-

$$PB = x - \frac{d}{2} \text{ and } PR = x + \frac{d}{2}$$

From $\triangle PRB$,

$$(BP)^2 = D^2 + \left(x + \frac{d}{2}\right)^2$$

From $\triangle APD$,

$$(AP)^2 = D^2 + \left(x - \frac{d}{2}\right)^2$$

$$\therefore (BP)^2 - (AP)^2 = \left[D^2 + \left(x + \frac{d}{2}\right)^2\right] - \left[D^2 + \left(x - \frac{d}{2}\right)^2\right]$$

$$= D^2 + x^2 + \frac{d^2}{4} + xd - D^2 - x^2 + \frac{d^2}{4} + xd$$

$$(BP)^2 - (AP)^2 = 2xd$$

$$(BP - AP)(BP + AP) = 2xd$$

$$(BP - AP) = \frac{2xd}{(BP + AP)}$$

But $BP - AP = D$ (approximately)

$$\therefore \text{The path diff.} = BP - AP = \frac{2xd}{2D} = \frac{xd}{D} \quad \text{--- (1)}$$

$$\text{and phase diff.} = \frac{2\pi}{\lambda} \left(\frac{xd}{D}\right) \quad \text{--- (2)}$$

(i) Bright fringes:-

If the path diff. is a whole No. Multiple of λ , the point is bright, i.e. for point P to be bright.

$$\frac{xd}{D} = n\lambda \text{ where } n = 0, 1, 2, 3, \dots$$

$$\text{or } x = \frac{n\lambda D}{d} \quad \text{--- (3)}$$

where $n=0$, $x_0=0$, the point P lies at C and is bright.

For, $n=1$, $x_1 = \frac{1D}{d}$

$n=2$, $x_2 = \frac{2D}{d}$

$n=3$, $x_3 = \frac{3D}{d}$

$x_n = \frac{nD}{d}$

Therefore, the distance b/w two consecutive bright fringes which is also known as width of bright fringes is,

$x_1 - x_0 = \frac{1D}{d} - 0 = \frac{1D}{d}$

$x_2 - x_1 = \frac{2D}{d} - \frac{1D}{d} = \frac{1D}{d}$

$x_3 - x_2 = \frac{3D}{d} - \frac{2D}{d} = \frac{1D}{d}$ and so on.

Hence, width of all bright fringes are equal.

(ii) Dark fringes:- If the path difference is an odd number multiple of half wavelength, the point P is dark, i.e., the point P to be dark.

$\frac{nD}{d} = (2n+1) \frac{\lambda}{2}$, where $n=0, 1, 2, 3, \dots$

$x = \frac{(2n+1)D\lambda}{2d}$ — (4)

when, $n=0$, $x_0 = \frac{1D}{2d}$

$n=1$, $x_1 = \frac{3D}{2d}$

$n=2$, $x_2 = \frac{5D}{2d}$

$n=3$, $x_3 = \frac{7D}{2d}$

$x_n = \frac{(2n+1)D}{2d}$

Therefore, the distance b/w any two consecutive dark fringes which is also known as width of the dark fringes is,

$x_1 - x_0 = \frac{3D}{2d} - \frac{1D}{2d} = \frac{1D}{d}$

$x_2 - x_1 = \frac{5D}{2d} - \frac{3D}{2d} = \frac{1D}{d}$

$x_3 - x_2 = \frac{7D}{2d} - \frac{5D}{2d} = \frac{1D}{d}$ and so on.

Hence, width of all dark fringes are equal.

The breadth of bright or dark fringes is however equal to half the fringe width. The fringe width is shown in fig.

$\beta = \frac{1D}{d}$ — (5)

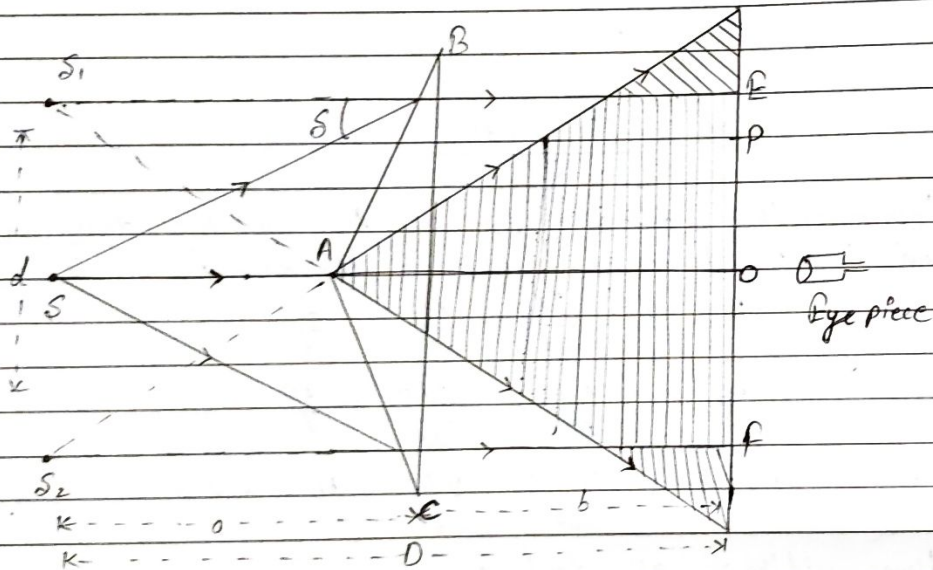
The fringe width β depends upon the following factors:-

- (i) the fringe width β is directly proportional to wavelength of light
- (ii) fringe width is directly proportional to the distance of the screen from the two sources.

* Fresnel's Biprism :-

Fresnel's biprism is one of the most accurate method to produce interference pattern and to determine the wavelength of monochromatic light.

The biprism ABC is a combination of two acute-angled prisms placed base to base. The acute angle is about 1° on both sides as shown in fig.



When the light from the source falls on the lower part of the biprism, it appears to come from virtual source S_2 due to refraction. Similarly, when the light from source S falls on the upper part of the biprism, it appears to come from the virtual source S_1 . S_1 and S_2 are two coherent sources. By and E and F, fringes of large width are produced which are due to the diffraction.

The point O on the screen is equidistant from the sources S_1 and S_2 , hence it is maximum intensity.

On both sides of O, alternate dark and bright fringes are produced. The intensity at any point P on the screen is maximum if the path difference,

$$S_2P - S_1P = n\lambda$$

where $n = 0, 1, 2, 3, \dots$ and the intensity of point P will be minimum if the path difference is,

$$S_2P - S_1P = \frac{(2n+1)\lambda}{2}$$

where $n = 0, 1, 2, 3, \dots$

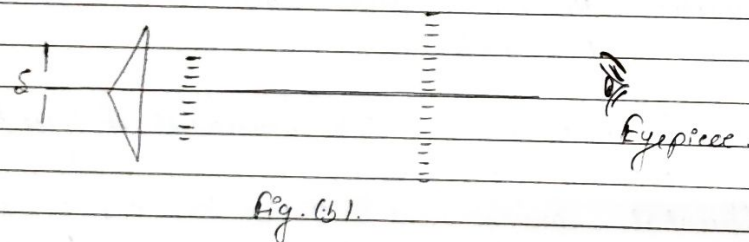
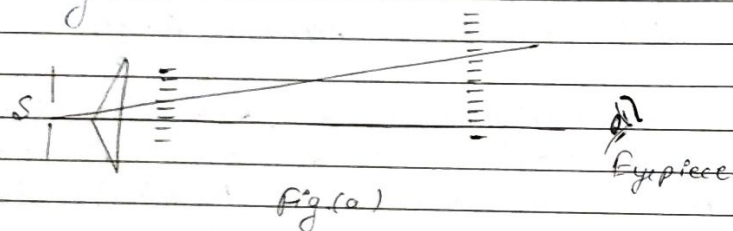
The fringe width is given by the expression $\beta = \frac{\lambda D}{d}$

where λ is the wavelength of light used, d is distance b/w the coherent sources and D is the distance b/w source and the screen. If a is the distance b/w the slit S and biprism and b is distance b/w the biprism and the screen. Then, $D = a + b$.

Determination of wavelength λ :- The wavelength of light λ is determined by biprism as follows :-

- Experimental arrangement :- Taken long optical bench with four uprights. These uprights are used to hold a slit of adjustable width, biprism, a lens and a micrometer eyepiece. Each upright can be adjusted :-
 (i) along the breadth of the optical bench and its position can be read by a vernier scale provided with the uprights.
 (ii) along the length of the optical bench and its position may be read on the scale of the bench.
 (iii) along the height of the optical bench to adjust the slit, biprism etc. at the same height.

- b) Setting of the experiment :- The following adjustments are made before making observation :-
- Mount the slit, biprism and eyepiece of uprights. Adjust the height of all the uprights approximately same.
 - Make the slit vertical and narrow, illuminate by a source of monochromatic light.
 - Bring the biprism close to the slit and look through the eyepiece. Fringes are seen in the eyepiece but not very clear.



- c) Measurements :- After setting the experiment, the biprism is placed at any position on the optical bench and observations are made as follows :-
- Note the positions of slit and eyepiece and the distance D .
 - Set the cross-wire of the eyepiece on one particular fringe and note the reading with the help of micrometer. Find the difference of the two readings & calculate the average fringe width β .

(iii) Now, take a convex lens of short focal length f and introduce it between the biprism and the eyepiece. Measure the distance b/w the images of S_1 and S_2 as seen through the eyepiece. Let it be d_1 as shown in fig.

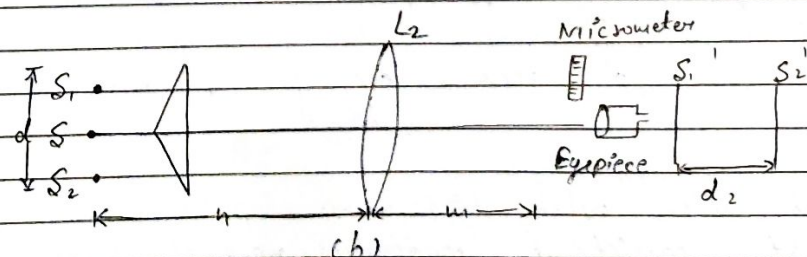
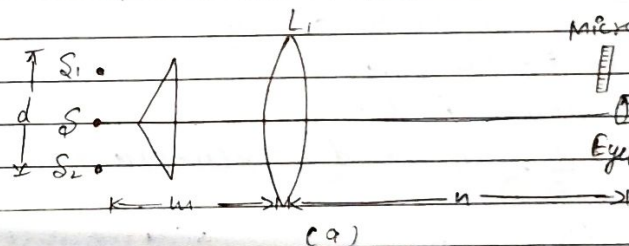
$$\frac{h_1}{h_0} = \frac{I}{O} = \frac{d_1}{d} = \frac{v}{u} = \frac{u}{u} \Rightarrow md_1 = ud \quad \text{--- (i)}$$

Now, move the lens towards the eyepiece and let it be at some other position L_2 so that the images of S_1 and S_2 are seen in the eyepiece again. Let it be d_2 as shown in fig.

$$\frac{d_2}{d} = \frac{v}{u} = \frac{u}{u} \Rightarrow nd_2 = ud \quad \left[\begin{matrix} v=u \\ u=u \end{matrix} \right] \quad \text{--- (ii)}$$

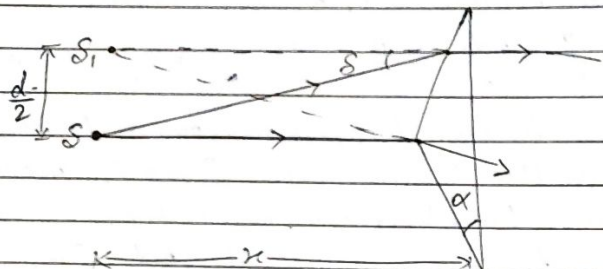
Multiplying eq. (i) by (ii), we get,

$$(md_1)(nd_2) = (ud)(ud) \text{ or } d_1d_2 = d^2$$



Substituting the values of D , β and d in the eq.

$$\beta = \frac{AD}{d} \text{ or } d = \frac{BD}{d}$$



The value of d , the distance b/w the two sources S_1, S_2 may also be calculated directly if the refractive index of the material of the prism and the refracting angle of prism are known. As the refracting angle α is very small, so the deviation δ produced by prism is given by,

$$\delta = (\mu - 1)\alpha$$

The action of the prism is shown in fig.

$$S_1 S_2 = \frac{d}{2} = \mu \delta \text{ or } \frac{d}{2} = \mu(\mu - 1)\alpha$$

$$d = 2\mu(\mu - 1)\alpha$$

If α is given in degrees, then,

$$d = 2\mu(\mu - 1)\alpha \frac{\pi}{180^\circ} \quad (11)$$

* Fringes with white light using biprism:-
White light consists of continuous range of wavelength from $4000 \text{ \AA}$ to $7500 \text{ \AA}$ in the visible region.

$$\beta = \frac{AD}{d}$$

In biprism the two coherent sources are produced by refraction and the distance b/w the two coherent sources depends upon the refractive index of the prism which in turn depends upon the wavelength.

The distance of the n^{th} bright fringes from the centre with monochromatic light is given by,

$$x_n = \frac{n\lambda D}{d}$$

$$\text{where } d = 2\mu(\mu - 1)\alpha$$

$$\text{or } x_n = \frac{n\lambda D}{2\mu(\mu - 1)\alpha}$$

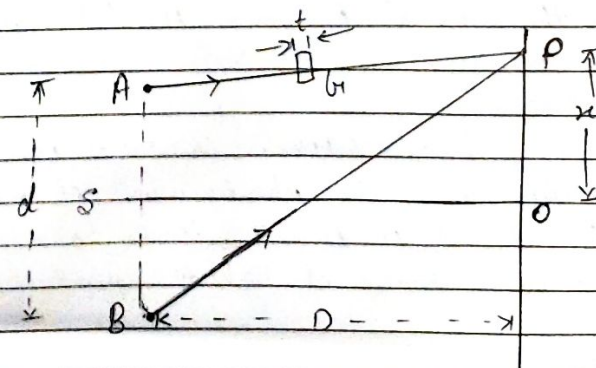
The distance of the n^{th} fringes from the centre of blue and red rays will be,

$$x_b = \frac{n\lambda_b D}{2\mu(\mu_b - 1)\alpha} \text{ or } x_r = \frac{n\lambda_r D}{2\mu(\mu_r - 1)\alpha}$$

The zero order fringes lies at the centre for $n=0$ for all wavelengths.

* Determination of thickness of thin sheet of transparent materials:-

The biprism experiment can be used to determine the thickness of thin sheet of transparent materials like glass or mica.



Let A and B be two virtual coherent sources. Let a point O be equidistant from A and B on the screen. When a transparent sheet of μ of thickness t and refractive index μ is introduced in the path of one of the beams, the fringes which was originally at O shifts to P as shown in Fig. The time taken by the waves from B to P in air is same as the time taken by the waves from A to P partly in air and partly through the transparent sheet. Let c_0 be the velocity of light in air and c be the velocity of light in the sheet.

$$\therefore \frac{BP}{c_0} = \frac{AP - t}{c_0} + \frac{t}{c}$$

$$\text{or, } BP = AP - t + \frac{c_0}{c} t$$

$$\text{But } \frac{c_0}{c} = \mu$$

$$\therefore BP - AP = \mu t - t = (\mu - 1)t$$

If point P is originally occupied by n^{th} fringes then the path difference,

$$BP - AP = n\lambda$$

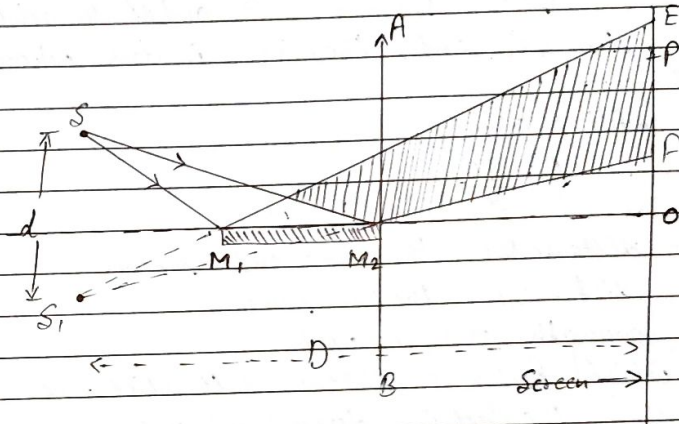
$$\therefore (\mu - 1)t = n\lambda$$

where n is the no. of fringes in the distance n .

* Lloyd mirror:-

Lloyd's mirror is used to find the wavelength of light using the phenomenon of interference. The mirror devised by Dr. Lloyd is an arrangement to produce two coherent sources of light to produce interference

fringes. It consists of a highly polished plane mirror M_1, M_2 on surface and blackened to back at the avoid multiple reflections. S is a source of monochromatic light.



Light from the source S falls on the mirror at almost grazing incidence so that incidence light is reflected. This also produces a good contrast in the fringes. Incident and reflected beams of light which produce a good interference have nearly same amplitudes. The sources S and S_1 act as two coherent sources.

The central fringe is obtained at a point O on the screen where the geometrical path difference b/w the wave coming from S and S_1 is zero. The central fringe at O is not available. If the screen is shifted in the new position touching M_1 shown by line AB, the central fringe can be seen by M_2 . A path difference of π b/w the incident and reflected rays is produced at M_2 . Since no change in phase takes place in the incident light coming from S.

- This produces that a light beam after reflection from an optically denser medium undergoes a phase change of π .
 If the path difference $S_2P - S_1P = 2(n+1)\lambda$, the point P is bright where $n=0, 1, 2, 3, \dots$ and if the path difference $S_2P - S_1P = 2n\lambda$, the point P is dark, where $n=0, 1, 2, 3, \dots$
 The fringe width is given by,

$$\beta = \frac{\lambda D}{d}$$

where D = distance b/w the source and screen.

d = distance b/w the two sources.

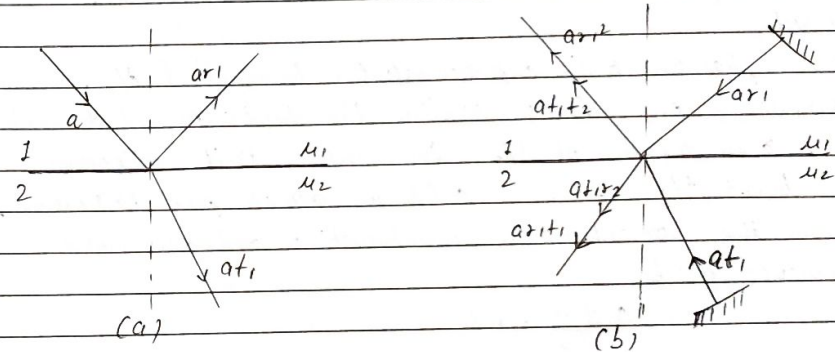
λ = wavelength of light used.

- The central white fringe is shifted and the shift is related to the thickness of the sheet is determined by using relation of thin sheet,

$$[(\mu - 1)t = n\lambda] \quad \text{--- (1)}$$

- * Phase change of Reflection (Stoke's law):-
 The fact about light waves is that when it is reflected at the surface of a denser medium, it suffers a phase change of π or path of $\lambda/2$ is introduced in it using a principle of optical reversibility. According to the principle, in the absence of any absorption if a light ray that is reflected and refracted will its original path if its direction is reversed.
 Let us consider a wave of amplitude 'a' incident on surface separating the two media 1 and 2 having refractive index μ_1 and μ_2 . Let the medium 2 be denser than the medium 1.

Let r_1 and t_1 be the amplitudes of reflecting and transmitting coefficients when the ray goes from medium 1 to medium 2. The amplitudes of reflected and refracted beam will be ar_1 and at_1 respectively.



Now, we reverse the two rays ar_1 and at_1 . The ray ar_1 is incident on medium 1 and the ray at_1 is incident on medium 2 as shown in fig.

The amplitudes of reflected and refracted waves due to waves ar_1 , therefore will be ar_1^2 and ar_1t_1 respectively. Similarly, the amplitudes of reflected and refracted waves due to waves at_1 will be at_1r_2 and at_1t_2 respectively.

$\therefore$

$$ar_1^2 + at_1t_2 = a$$

$$t_1t_2 = 1 - r_1^2 \quad \text{--- (1)}$$

Further, the two rays of amplitudes at_1r_2 and ar_1t_1 must cancel each other. i.e.,

$$at_1r_2 + ar_1t_1 = 0 \quad \text{or} \quad r_2 + r_1 = 0$$

$$[r_2 = -r_1]$$

By performing experiments, it is found that the ray reflected from optically denser medium suffers a phase change of π and no change in phase occurs when reflection takes place from a rarer medium. The ray at will behave a phase change of π on reflection while the ray at, shall not.

Since a phase of π is equivalent to a path of $\lambda/2$. The surface of denser medium then due to introduction of phase π or a path of $\lambda/2$. The effective path difference will be $\Delta + \lambda/2$ where Δ is an actual path difference. Therefore, the conditions of constructive and destructive interference will be,

$$\Delta + \frac{\lambda}{2} = n\lambda$$

$$\Delta = n\lambda - \frac{\lambda}{2}$$

$$\Delta = (2n-1) \frac{\lambda}{2} \quad \text{for brightness where } n=0, 1, 2, 3 \dots$$

$$\Delta = (2n+1) \frac{\lambda}{2} \quad \text{for brightness where } n=0, 1, 2, 3 \dots$$

$$\Delta = 2n \cdot \frac{\lambda}{2}$$

$$\Delta = n\lambda \quad \text{for darkness where } n=0, 1, 2, 3 \dots$$

* Difference b/w Fresnel's Biprism and Lloyd's Mirror:-
Fresnel's Biprism

1. Large no. of fringes on both sides of central fringes are obtained.

2. Central fringes is visible.

3. Central fringes is always bright, i.e., it is a point of maxima.

4. If the width of the slit is increased, zeroth order fringes becomes blurred and when slit of width is quite large fringes disappear.

5. The condition for bright and dark fringes are:-

(i) For bright fringes:-
Path difference, $S = n\lambda$

where $n=0, 1, 2, 3 \dots$

(ii) For dark fringes:-
Path difference

$$S = (2n-1) \frac{\lambda}{2} \text{ where}$$

$n=1, 2, 3, \dots$

Lloyd's Mirror

1. Less than half pattern of fringes is obtained.

2. Central fringes is not ordinary visible.

3. Central fringes is always dark, i.e., a point of minima and is usually not visible.

4. Until the slit is of finite size, zeroth order fringes remains sharp and fringes pattern does not appear for wide slits.

5. Conditions for bright and dark fringes are reversed as:-

(i) For bright fringes:-
Path difference $S = (2n-1) \frac{\lambda}{2}$

where $n=1, 2, 3 \dots$

(ii) For dark fringes:-
 $S = n\lambda$ where $n=0, 1, 2, 3 \dots$

Matrix Method in Paraxial Optics

* Matrices:-

A system of $m \times n$ numbers of elements arranged in m rows and n columns is called a $m \times n$ matrix of the order of $m \times n$ matrix. Thus the Matrix,

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

has $m = 2$ rows, $n = 3$ columns and has $2 \times 3 = 6$ elements.

* Types of Matrices:-

(i) Row Matrix:- A Matrix is said to be row Matrix if $m = 1$ row and $n \geq 1$ columns.

For examples, $(1, 2, 3)$ is a row Matrix of order of 1×3 .

(ii) Column Matrix:- A Matrix is said to be column Matrix if $m \geq 1$ and $n = 1$. For examples, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column matrix of order 3×1 .

(iii) Rectangular Matrix:- A Matrix is said to be rectangular if $m \neq n$. For example, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ are rectangular Matrix of order 2×3 and 3×2 respectively.

(iv) Square Matrix:- A Matrix is said to be square if $m = n$. For example, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a square matrix of order 2×2 .

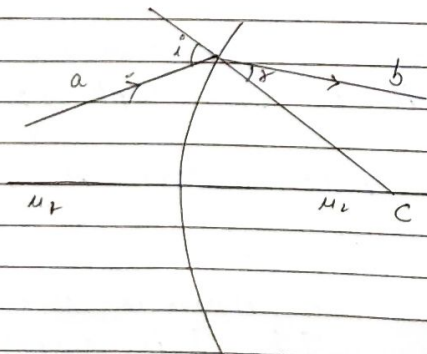
(v) Diagonal Matrix:- A Matrix is said to be diagonal Matrix having all the elements equal to zero except elements (having same rows and same columns).
For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix of order 3×3 .

(vi) Identity Matrix:- A Matrix is said to be identity Matrix having each diagonal elements equal to 1.
For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity Matrix of order 3×3 .

(vii) Null Matrix:- A Matrix is said to be null Matrix having each elements equal to zero $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a Null Matrix or zero Matrix.

* Refraction:- When a beam of light travels from one medium to another, it deviates from its path.

This phenomenon of change in the path of light, when it travels from one medium to another is called the refraction.



When the ray of light travels from an optically rarer medium to an optical denser medium, it bends towards the

Normal and when it travels from denser to a rarer medium, it bends away from the normal.

(i) The incident ray, the refracted ray and normal at the point of incidence lie in the same plane.

(ii) The ratio of the sine of the angle of incidence to the sine of refraction is constant for the given set of two media, i.e.,

$$\frac{\sin i}{\sin r} = \text{constant}$$

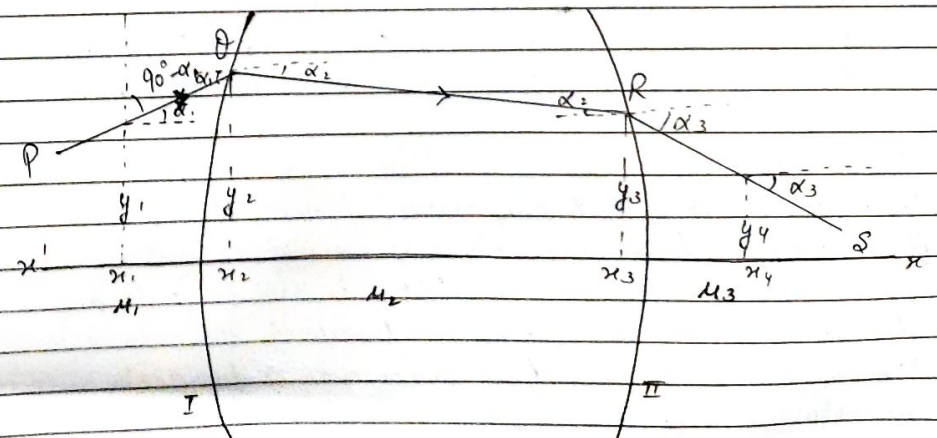
This is called the refractive index.

If μ_1 and μ_2 are the absolute refractive index of two media.

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}$$

This relation is called as Snell's law.

* Co-ordinates of a ray in paraxial rays:-



Let μ_1 be the refractive index of the medium to the left of the lens, μ_2 be the refractive index of the lens and μ_3 be the refractive index of the medium to the right of the lens.

Consider a ray PQ incident on surface I. Let QR, RS be refracted and emergent rays respectively. Point A can be represented by quantities (y_1, α_1) . In angle α made by the ray with X-axis, another quantities λ is used as the second co-ordinate for the ray. This quantity λ is defined as,

$$\lambda = \mu \cos(90^\circ - \alpha) = \mu \sin \alpha.$$

For paraxial rays, angle α is very small, $\sin \alpha \approx \alpha$ and therefore, $\lambda = \mu \alpha$.

where μ is the refractive index.

The quantity λ is the optical directional cosine of rays.

Point of the rays	Co-ordinates of ray in medium of R.I.			
	μ_1	μ_2	μ_3	
P	(y_1, μ_1, α_1)	-	-	
Q	(y_2, μ_1, α_1)	(y_2, μ_2, α_2)	-	
R	-	(y_3, μ_2, α_2)	(y_3, μ_3, α_3)	
S	-	-	(y_4, μ_3, α_3)	

Translation and refraction:-

When a ray of light propagates in an optical system, it undergoes the following two operations:-

1) Translation:- A ray of light is said to undergo translation operation when it travels in the same medium. The ray PQ has undergone a translation operation from P to Q.

2) Refraction:- A ray of light is said to undergo refraction operation when it travels from one medium to another medium.

* Effect of translation and translation matrix:-

Suppose on a cylindrically symmetric matrix, a ray of light travels from P to Q in a medium of refractive index μ .

Suppose α_1 and α_2 are the angles of inclination of the ray at P and Q respectively.

In case of translation, ray of light travels in a straight line in a medium of refractive index μ .

$\therefore \alpha_1 = \alpha_2 = \alpha$ (ray)
The co-ordinates of travelling ray at P and Q are $(y_1, \mu \alpha_1)$ and $(y_2, \mu \alpha_2)$ respectively.

$$y_2 = y_1 + ON \quad \text{--- (i)}$$

in $\triangle PNO$, $\frac{ON}{PN} = \tan \alpha_1$

$$\text{or } ON = PN \tan \alpha_1$$

Since, the ray under consideration is paraxial ray, then angles α_1 and α_2 will be very small.

$$\tan \alpha_1 \approx \alpha_1$$

$$ON = PN \alpha_1 = D \alpha_1$$

where D is the linear distance travelled by the ray from P to Q.

Eq. (i) can be written as,

$$y_2 = y_1 + D \alpha_1 \quad \text{--- (ii)}$$

Second coordinate is at points P and Q is,

$$x_1 = \mu x, \text{ and } x_2 = \mu x,$$

But $x_1 = x_2$

$$\therefore \mu x_1 = \mu x_2$$

$$\text{or } x_1 = x_2 \quad \text{--- (iii)}$$

we can rewrite eq. (iii) as,

$$y_2 = y_1 + \frac{D}{\mu} \mu x,$$

$$y_2 = y_1 + \left(\frac{D}{\mu}\right) x_1 \quad \text{--- (iv)}$$

Eq. (iii) & (iv) can be written as,

$$x_2 = (1) x_1 + (0) y_1,$$

$$y_2 = \left(\frac{D}{\mu}\right) x_1 + (1) y_1 \quad \text{--- (v)}$$

In matrix form, eq. (v) can be written as,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{D}{\mu} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Here $\begin{bmatrix} 1 & 0 \\ \frac{D}{\mu} & 1 \end{bmatrix}$ is called the translational matrix and is denoted by T.

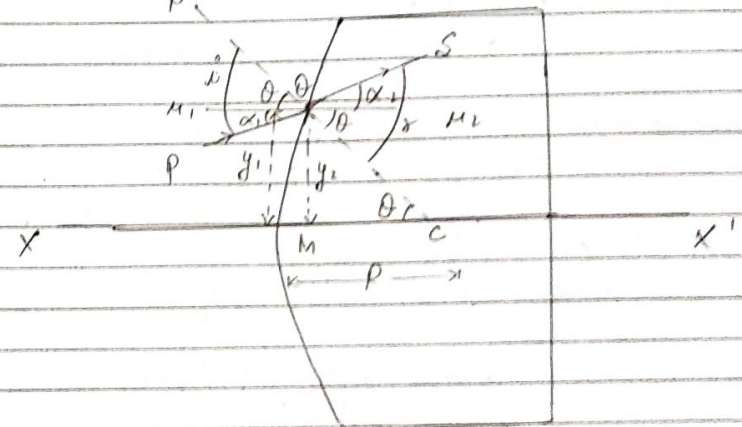
Determinant of translation matrix is, 1,

$$\det T = \begin{vmatrix} 1 & 0 \\ \frac{D}{\mu} & 1 \end{vmatrix} = 1.$$

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ are output and input matrix.

Output Matrix = (Translation Matrix) $\times$ (Input Matrix).

* Effect of Refraction and Refraction Matrix :-
Let us consider a spherical refracting surface of a radius of curvature R. Let μ_1 and μ_2 be the refractive index of the two media.



The incident ray PQ, is refracted along QS. Suppose $i = i_1$ and $r = r_2$ are the angles of incidence and angle of refraction.

Using Snell's law,

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} \quad \text{--- (i)}$$

if rays are paraxial, then angles are small.

i.e., $\sin i \approx i$ and $\sin r \approx r$, then eq. (i) may be,

$$\frac{i}{r} = \frac{\mu_2}{\mu_1}$$

$$i \mu_1 = r \mu_2 \quad \text{--- (ii)}$$

But from the fig.,

$$i = \alpha_1 + \theta \quad \text{--- (iii)}$$

$$r = \alpha_2 + \theta \quad \text{--- (iv)}$$

$\alpha_1, \alpha_2, \theta$ are the angles made by incident ray, refracted

ray and normal to the surface respectively.

In $\triangle OMC$,

$$\tan \theta = \frac{y_1}{R}$$

$$\tan \theta \approx \theta$$

$$\theta = \frac{y_1}{R} \quad \text{--- (v)} \quad [\because \text{Angle is small}]$$

Using eq. (ii), (iii), (iv), so we get,

$$\mu_1(\alpha_1 + \theta) = \mu_2(\alpha_2 + \theta)$$

$$\mu_2 \alpha_2 = \mu_1 \alpha_1 - (\mu_2 - \mu_1) \theta \quad \text{--- (vi)}$$

From eq. (v), (vi), so we get,

$$\mu_2 \alpha_2 = \mu_1 \alpha_1 - (\mu_2 - \mu_1) \frac{y_1}{R}$$

$$= \mu_1 \alpha_1 - \left(\frac{\mu_2 - \mu_1}{R} \right) y_1$$

But $\mu_1 \alpha_1 = A_1$

so $\mu_2 \alpha_2 = A_2$

$$A_2 = A_1 - p y_1$$

where $p = \frac{\mu_2 - \mu_1}{R} \quad \text{--- (vii)}$

This p is known as power of refracting surface.

Since the height of ray before and after refraction are same,

$$y_1 \approx y_2 \quad \text{--- (viii)}$$

From eq. (vii), (viii), so we get,

$$A_2 = (1)A_1 + (-p)y_1$$

$$y_2 = (0)A_1 + (1)y_1$$

In matrix form,

$$\begin{bmatrix} A_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ y_1 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} A_2 \\ y_2 \end{bmatrix} = R \begin{bmatrix} A_1 \\ y_1 \end{bmatrix}$$

Here R is called the refraction matrix,

$$R = \begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix}$$

Determinant of refraction matrix is 1.

$$\det \cdot R = \begin{vmatrix} 1 & -p \\ 0 & 1 \end{vmatrix} = 1$$

* System Matrix:-

An optical matrix system can be consist of a series of lens, is characterised by translation and refraction matrices.

When a ray of light entering such an optical system is specified by its coordinates (A_1, y_1) then, the coordinates of the ray leaving the system is given by (A_2, y_2) .

$$\begin{bmatrix} A_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix} \begin{bmatrix} A_1 \\ y_1 \end{bmatrix} = S \begin{bmatrix} A_1 \\ y_1 \end{bmatrix}$$

where $S = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix}$

is called as system matrix.

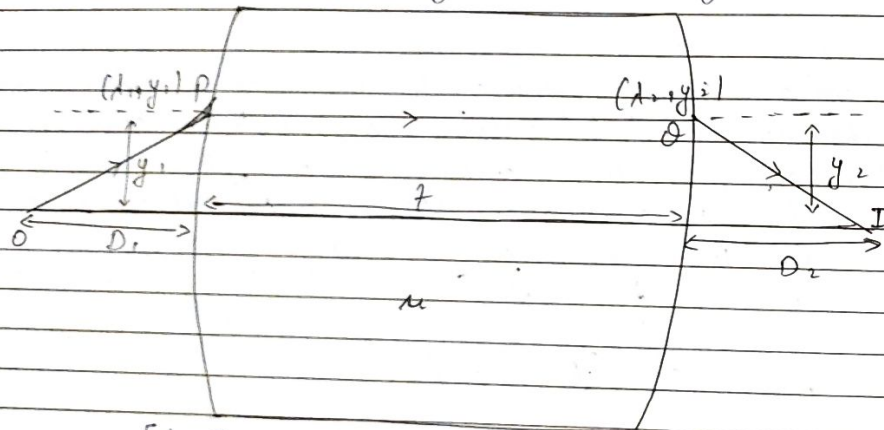
Also using the property that determinant of the product of matrices is equal to the product of determinants of the matrices, so we get,

$$\det \cdot S = 1$$

$$bc - ad = 1$$

* System Matrix for a thick lens - Derivation of lens formula for thin and thick lenses:-

Consider a thick lens as shown in fig. Let R_1 and R_2 are radii of curvature of its left and right surfaces. Suppose a ray of light is incident at left surface of the lens and emerges out as ray OT . Suppose coordinates of the ray is at point P and Q are (x_1, y_1) and (x_2, y_2) respectively.



$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{t}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -P_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

where $P_1 = \frac{(n-1)}{(R_1)}$ — (i)

$P_2 = \frac{(n-1)}{(-R_2)}$ — (ii)

are the powers of left and right refracting surfaces of lens.

The system matrix for this optical system is given by,

$$S = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix} = \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{t}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -P_2 \\ 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 - \frac{P_1 t}{n} & -P_1 - P_2 \left(1 - \frac{t}{n} P_1\right) \\ \frac{t}{n} & 1 - \frac{t}{n} P_2 \end{bmatrix}$$

This is the required system matrix for a thick lens.

Lens formula for thick lens:-

$t \rightarrow 0$, hence the above system matrix reduces to,

$$S = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix} = \begin{bmatrix} 1 & -P_1 - P_2 \\ 0 & 1 \end{bmatrix}$$

$a = P_1 + P_2$, $b = 1$, $c = 1$ and $d = 0$. — (iii)

Image plane of an optical system is given by,

$bD_2 + aD_1D_2 - cD_1 - d = 0$ — (iv)

Hence D_1 and D_2 are the distances of object plane and image plane from refracting surfaces.

From eq. (iii), (iv) we get,

$$D_2 + (P_1 + P_2) D_1 D_2 - D_1 = 0$$

$$D_1 - D_2 = (P_1 + P_2) D_1 D_2$$

Dividing both sides by $D_1 D_2$, we get,

$$\frac{1}{D_2} - \frac{1}{D_1} = P_1 + P_2$$

Using values of P_1 and P_2 , we get,

$$\frac{1}{D_2} - \frac{1}{D_1} = \frac{n-1}{R_1} - \frac{n-1}{R_2}$$

$$\frac{1}{D_2} - \frac{1}{D_1} = (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} \right]$$

$D_1 = u$ = distance of object from refracting surface

$D_2 = v$ = distance of image from refracting surface.

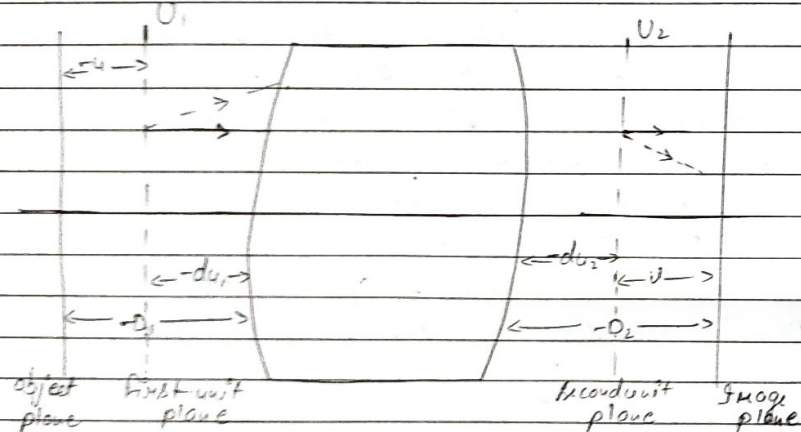
$$\frac{1}{v} - \frac{1}{u} = (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} \right]$$

But $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$, where f is the focal length of the lens,

$$\frac{1}{f} = (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} \right]$$

* UNIT PLANES :-

There are two planes, one in object space and the other in image space, defined such that lateral magnification is unity and positive.



U_1 and U_2 represent unit planes. Let du_1 and du_2 are distances of unit planes from the refracting surfaces as shown in fig.

The elements of the system matrix will be related with du_1 and du_2 as,

$$b + a du_1 = \frac{1}{c - a du_2} = 1 \quad (\because u = 1)$$

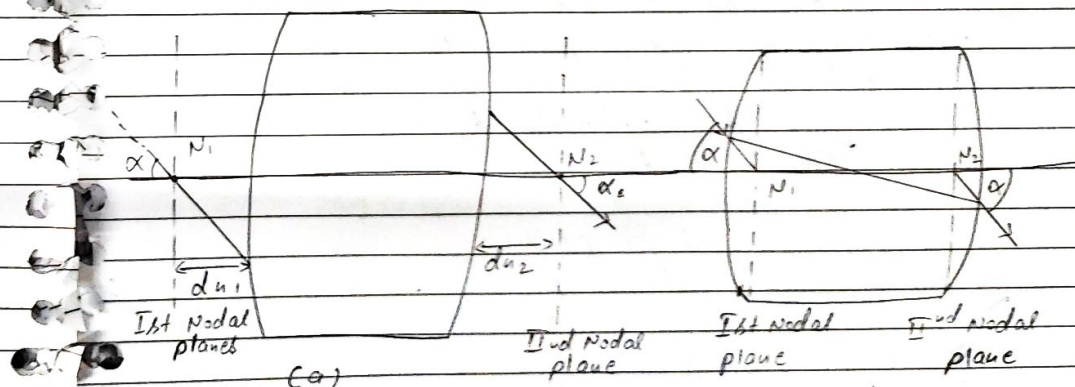
$$du_1 = \frac{1-b}{a}$$

$$du_2 = \frac{c-1}{a}$$

* Nodal planes :-

Two points on the principal axis of an optical system having unit true angular magnification are called as Nodal points.

An incident ray directed towards the first nodal point, N_1 , or emerging from the optical system, or will appear to proceed from the second nodal point N_2 in a direction $\parallel$ to the incident rays.



Suppose du_1 and du_2 are the distances of Ist and IInd nodal points N_1 and N_2 from the refracting surfaces respectively. According to the definition of Nodal points, a ray incident at an angle α on Ist Nodal point N_1 emerges from the IInd Nodal point N_2 at same angle.

If d_1 and d_2 are optical direction cosine, then,

$$d_1 = \mu_1 \sin \alpha_1 = \mu_2 \sin \alpha_2 = d_2 = \mu_2 \sin \alpha_2$$

For paraxial rays, $\alpha_1 = \alpha_2 = \alpha$.

the medium of refractive index on both sides is same,

$$\mu_1 = \mu_2$$

$$S = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix}$$

The system matrix for optical system then, emergent ray at any point is given by,

$$\begin{bmatrix} d_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} b + adu_1 & -a \\ 0 & -adu_2 + c \end{bmatrix} \begin{bmatrix} d_1 \\ y_1 \end{bmatrix}$$

$$d_2 = (b + adu_1)d_1 - ay_1 \quad \text{--- (i)}$$

If object is on the axial point, then $y = 0$

Eq. (i) can be written as,

$$d_2 = (b + adu_1)d_1$$

Since, $d_1 = d_2$

$$b + adu_1 = 1$$

$$adu_1 = (1 - b)$$

$$du_1 = \frac{1 - b}{a} \quad \text{--- (ii)}$$

we k. that, also,

$$du_1 = \frac{1 - b}{a} \quad \text{--- (iii)}$$

$$du_2 = \frac{c - 1}{a}$$

where du_1 = distance of unit planes from refracting

From eq. (ii), (iii), we get,

$$du_2 = du_1$$

But $b + adu_1 = \frac{1}{c - adu_2}$

$$c - adu_2 = 1 \Rightarrow adu_2 = c - 1$$

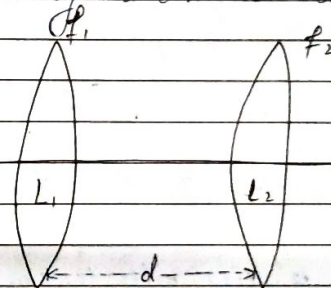
$$du_2 = \frac{c - 1}{a} \quad \text{--- (iv)}$$

Now, from eq. (ii), (iv) we get,

$$du_2 = du_1$$

* System of two thin lenses:-

Suppose f_1 and f_2 are two focal lengths of two thin lenses L_1 and L_2 respectively. Suppose these two thin lenses are separated by a distance d as shown in fig.



System matrix of a thin lenses is given as,

$$S = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix} = \begin{bmatrix} 1 & -p_1 - p_2 \\ 0 & 1 \end{bmatrix} \quad \text{--- (i)}$$

where p_1 and p_2 are the powers of the two refracting surfaces of a thin lens, i.e.,

$$-a = -p_1 - p_2 = -\left(\frac{\mu - 1}{R_1}\right) - \left(\frac{\mu - 1}{-R_2}\right)$$

$$= -(\mu - 1) \left[\frac{1}{R_1} - \frac{1}{R_2} \right] = -\frac{1}{f}$$

i.e., a is reciprocal of the focal length of the lens.

$$S = \begin{bmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix} \text{ and } S_2 = \begin{bmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{bmatrix}$$

Translation matrix for a light ray travelling distance d b/w two lenses is given as,

$$T = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}$$

Hence, system matrix for a combination of two lenses is,

$$S = S_1 T S_2 = \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_2} \\ d & -\frac{d}{f_2} + 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f_1} - \frac{1}{f_1} + \frac{d}{f_1 f_2} & -\frac{1}{f_1} + \frac{1}{f_1 f_2} \\ d & -\frac{d}{f_2} + 1 \end{bmatrix}$$

Comp^y this matrix eqⁿ (i^o) with the general matrix given by eqⁿ (i^o), we get

$$-a = -\frac{1}{f_1} + \frac{d}{f_1 f_2} - \frac{1}{f_2} \text{ or } a = \frac{1}{f_1} - \frac{d}{f_1 f_2} + \frac{1}{f_2}$$

But we know that, a is reciprocal of focal length,

$$a = \frac{1}{f}$$

$$\frac{1}{f} = \frac{1}{f_1} - \frac{d}{f_1 f_2} + \frac{1}{f_2}$$

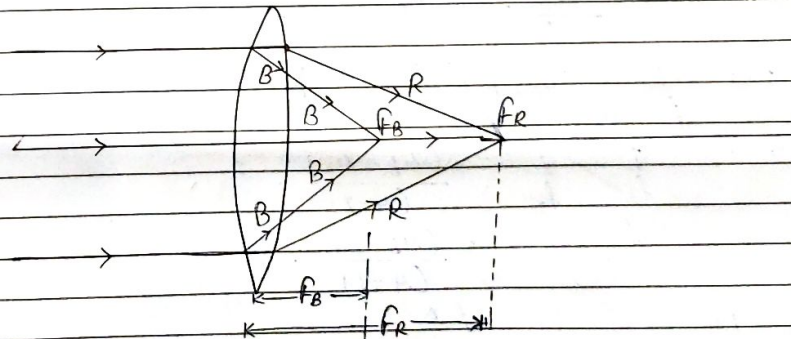
where f = focal length of combination of two lenses separated by the distance d .

$$\text{i.e., } \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

* CHROMATIC ABERRATIONS :-

If a parallel beam of white light from an object is incident on a thin convex lens, then the image of the object is usually coloured and blurred. This defect arises due to the fact that refractive index of the glass is different for different colours. This defect of the image is called chromatic aberration.

$$\frac{1}{f} = (n-1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$



Longitudinal Chromatic Aberration :-

The formation of image when different colours meet at different axis of the lens, then aberration is called longitudinal aberration. If the object is infinity, then difference b/w the focal lengths, f_r and f_v is a measure of the longitudinal chromatic aberration.

Expression for longitudinal chromatic aberration:-

Suppose μ_b , μ_r , f_b and f_r represents the refractive indices and focal lengths for blue and red colours respectively.

Suppose μ and f are focal length and refractive index respectively.

From the lens formula, we know that

$$\frac{1}{f_r} = (\mu_r - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad \text{--- (i)}$$

$$\frac{1}{f_b} = (\mu_b - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad \text{--- (ii)}$$

From eq. (i), (ii) we get,

$$\frac{1}{f_b} - \frac{1}{f_r} = (\mu_b - \mu_r) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$= \frac{(\mu_b - \mu_r)}{(\mu - 1)} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\text{But } \frac{1}{f} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\text{Hence, } \frac{1}{f_b} - \frac{1}{f_r} = \frac{(\mu_b - \mu_r)}{(\mu - 1)} \cdot \frac{1}{f}$$

$$\frac{f_r - f_b}{f_b f_r} = \frac{(\mu_b - \mu_r)}{(\mu - 1)} \cdot \frac{1}{f}$$

$$\text{But } f_r \cdot f_b = f^2$$

$$\frac{f_r - f_b}{f^2} = \frac{(\mu_b - \mu_r)}{(\mu - 1)} \times \frac{1}{f}$$

$$\text{or, } f_r - f_b = f \frac{(\mu_b - \mu_r)}{(\mu - 1)}$$

$$f_r - f_b = u \cdot f$$

where $u = \frac{\mu_b - \mu_r}{\mu - 1}$ is dispersive power of material of the lens.

Alternative Method:-

We know that focal length of the lens is given by,

$$\frac{1}{f} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad \text{--- (i)}$$

If μ changes by $\delta\mu$ it results in change in f by δf , then differentiating eq. (i), we get,

$$-\frac{\delta f}{f^2} = \delta\mu \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$= \frac{\delta\mu}{\mu - 1} \cdot \frac{1}{f}$$

$$\delta f = -f \frac{\delta\mu}{\mu - 1}$$

$$f_r - f_b = f \left(\frac{\mu_b - \mu_r}{\mu - 1} \right)$$

$$f_r - f_b = u \cdot f$$

$$\left[\frac{1}{f} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right]$$

$$\text{or } \frac{1}{f} \times \frac{1}{(\mu - 1)} = \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

* Two thin lenses of same material and mounted co-axially have minimum spherical and chromatic aberration. Find the ratio of their focal lengths.

Suppose f_1 and f_2 are the focal lengths of the two lenses and suppose they are separated by distance d .

Then, condition for minimum spherical aberrations,

$$d = f_1 - f_2 \quad \text{--- (i)}$$

or condition of minimum chromatic aberration is,

$$d = \frac{f_1 + f_2}{2}$$

$$2d = f_1 + f_2 \quad \text{--- (ii)}$$

From eq. (i), (ii), we get,

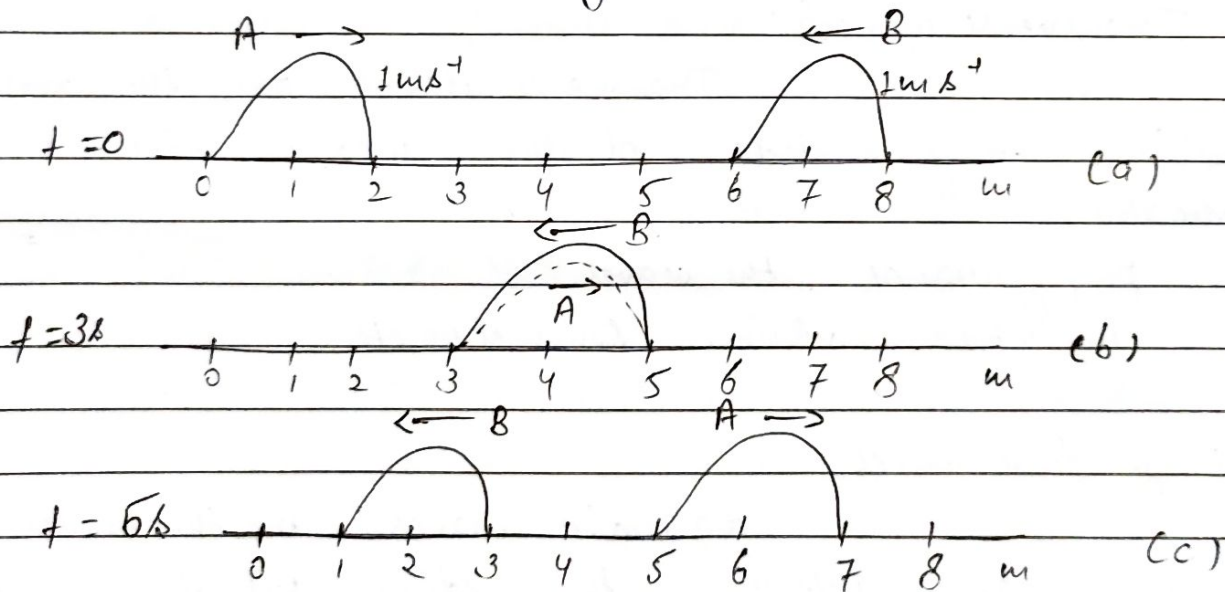
Chapter - 1st

VIBRATIONS OF CONTINUOUS SYSTEMS

* Superposition of waves:-

The displacement at any time due to any number of waves meeting simultaneously at a point of a medium, is the vector sum of the individual displacements due to each one of waves at that point at the same time.

If $\vec{y} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \dots$
If y_1, y_2, y_3 represents the displacements produced by various waves at a particular time at a particular position and y will be resultant displacement.



Superposition of two waves of identical shape travelling in opposite directions

The superposition principle is not applicable for waves when amplitude of mechanical waves is too large.
eg:- radio waves.

(i) When two waves of same frequency are moving in same direction through in a medium in the same phase superpose upon each other. Then, redistribution of energy takes place which is called interference of waves.

(ii) When two waves of same frequency are moving in opposite direction superpose on each other resultant waves has vibrations between two fixed points, it is called stationary wave.

(iii) When two waves of slightly different frequency are moving in same direction superpose on each other then, periodic variations takes place which is called phenomenon of beats.

* Transverse waves:-

Transverse waves are the waves in which every particle of the medium vibrates periodically in a direction normal to the direction of propagation of the waves.

Ex:- in string, violin, sonometer etc.

* Longitudinal waves:-

Longitudinal waves are the waves in which every particle of the medium vibrates periodically in the direction of propagation of the waves.

Ex:- sound waves, waves set up in a spring.

* Principle of Superposition of waves:- This principle states that if two or more waves travel in a medium then, the resultant disp. at any point at any instant is vector

sum of instantaneous disp. produced at that point by individual individual waves if each wave were present alone.

Chapter-2nd

Fourier's series And its applications-

* Fourier theorem and Fourier series:-

Fourier's theorem states that "any periodic waves, however complex of wavelength λ may be considered as made of simple harmonic waves of definite amplitude and phase and periods, the frequencies of components are in the ratio $n:2n:3n:4n \dots$

This theorem is general and it applicable to all types of complex periodic wave enables to split them into sine and cosine vibrations whose frequencies increasing in the ratio of natural No. with time period T ,

$$\text{Thus, } y = f(t+nT) = f(t), n=1, 2, \dots \quad (1)$$

$$y = f(t) = A_0 + A_1 \cos \omega t + A_2 \cos 2\omega t + \dots + A_n \cos n\omega t + B_1 \sin \omega t + B_2 \sin 2\omega t + \dots + B_n \sin n\omega t \quad (2)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos n\omega t + \sum_{n=1}^{\infty} B_n \sin n\omega t$$

$$\therefore \text{where } \omega = \frac{2\pi}{T}$$

The series represents by eq. (2), it is called Fourier series. A_0 , A_n and B_n are the Fourier coefficients.

Evaluation of Fourier Coefficients:-

To evaluate the coefficients A_0 :-Integrating both sides of eq. (1) w.r to 't' over the complete cycle from $0 \rightarrow T$

$$\int_0^T f(t) dt = A_0 \int_0^T dt + \sum_{n=1}^{\infty} A_n \int_0^T \cos n\omega t dt + \sum_{n=1}^{\infty} B_n \int_0^T \sin n\omega t dt$$

$$= A_0 \cdot T$$

where as the integrals of all the sines and cosines are zero.

$$A_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{T} \int_0^T y dt \quad \text{--- (3)}$$

To evaluate the coefficients of A_n :-Multiply both the sides of eq. (1) by $\cos n\omega t$ and integrate for a complete cycle b/w limits $0 \rightarrow T$.

$$\text{then, } \int_0^T f(t) \cos n\omega t dt = A_0 \int_0^T \cos n\omega t dt + \sum_{n=1}^{\infty} A_n \int_0^T \cos n\omega t \cdot \cos n\omega t dt + \sum_{n=1}^{\infty} B_n \int_0^T \cos n\omega t \cdot \sin n\omega t dt \quad \text{--- (4)}$$

$$\int_0^T \sin n\omega t dt = \int_0^{2\pi/\omega} \sin n\omega t dt = \left[-\frac{\cos n\omega t}{n\omega} \right]_0^{2\pi/\omega}$$

$$\Rightarrow -(\cos 2n\pi - \cos 0) \cdot \frac{1}{n\omega} = (-1+1) \cdot \frac{1}{n\omega} = 0$$

$$\text{and } \int_0^T \cos n\omega t dt = \int_0^{2\pi/\omega} \cos n\omega t dt = \left[\frac{\sin n\omega t}{n\omega} \right]_0^{2\pi/\omega}$$

$$\Rightarrow \frac{\sin 2n\pi - \sin 0}{n\omega} = 0$$

Making use of results,

$$\frac{2}{T} \int_0^T \cos n\omega t \cdot \cos m\omega t dt = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$$\int_0^T \sin n\omega t \cdot \cos m\omega t dt = 0$$

Eq. (4) reduces to,

$$\int_0^T f(t) \cos n\omega t dt = A_n \frac{T}{2}$$

$$\text{or, } A_n = \frac{2}{T} \int_0^T f(t) \cos n\omega t dt \quad \text{--- (5)}$$

Thus, A_n represents twice the average value of $f(t) \cos n\omega t$ over the interval T .For evaluating B_n :-Multiply both sides of the eq. (1) by $\sin n\omega t$ and integrate b/w the limits $t=0$ to $t=T$

$$\int_0^T f(t) \sin n\omega t dt = A_0 \int_0^T \sin n\omega t dt + \sum_{n=1}^{\infty} A_n \int_0^T \sin n\omega t \cdot \cos n\omega t dt + \sum_{n=1}^{\infty} B_n \int_0^T \sin n\omega t \cdot \sin n\omega t dt \quad \text{--- (6)}$$

Making use of results,

$$\frac{2}{T} \int_0^T \sin n\omega t \cdot \sin m\omega t dt = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$$\int_0^T \cos n\omega t \cdot \cos m\omega t dt = \frac{1}{2} \int_0^T [\cos (n-m)\omega t + \cos (n+m)\omega t] dt$$

For $n \neq m$,

$$\Rightarrow \frac{1}{2} \left[\frac{\sin (n-m)\omega t}{(n-m)\omega} + \frac{\sin (n+m)\omega t}{(n+m)\omega} \right]_0^{2\pi/\omega} = 0$$

$$\text{and for } n=m, \int_0^T \cos n\omega t \cdot \cos n\omega t dt = \int_0^T \cos^2 n\omega t dt = \frac{1}{2} \int_0^T (1 + \cos 2n\omega t) dt$$

$$\Rightarrow \frac{1}{2} \left[t + \frac{\sin 2n\omega t}{2n\omega} \right]_0^{2\pi/\omega} = \frac{1}{2} \left[T + \frac{\sin 4n\pi - 0}{2n\omega} \right] = \frac{T}{2}$$

$$\text{and, } \int_0^T \sin n\omega t \cdot \cos n\omega t dt = 0 \quad \left[\because t=T=\frac{2\pi}{\omega} \right]$$

From eq. (6), so we get,

$$\int_0^T f(t) \sin n\omega t dt = B_n \frac{T}{2}$$

$$\therefore B_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t) dt \quad \text{--- (7)}$$

The function $f(t)$ can be written as,

$$f(t) = \frac{1}{T} \int_0^T f(t) dt + \frac{2}{T} \sum_{n=1}^{\infty} \cos(n\omega t) \int_0^T f(t) \cos(n\omega t) dt + \frac{2}{T} \sum_{n=1}^{\infty} \sin(n\omega t) \int_0^T f(t) \sin(n\omega t) dt$$

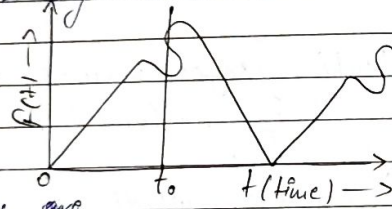
This expression is called the Fourier series.

* Limitations of Fourier's theorem (Dirichlet conditions):-

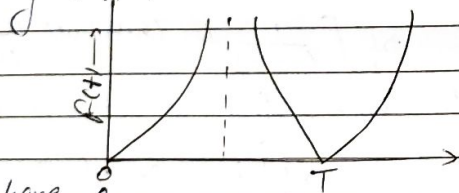
The function should satisfy the following conditions called Dirichlet conditions which may be stated as follows:-

① The function $f(t)$ should be single valued.

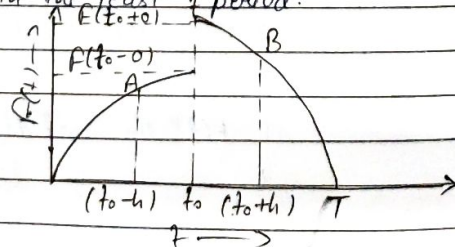
It should have only one value for a given value of t .



② The function should have a finite value, it should have only one infinite value at any place, such as at $t = t_0$.



③ The function $f(t)$ should have a finite no. of discontinuities and a finite no. of maxima and minima in the least period.



* Fourier series of function $f(x)$ b/w limits $-\pi$ to π (Expansion of function of x in a series of sine and cosine of multiple of x):-

Let us suppose that function $f(x)$ satisfies the Dirichlet conditions

$$f(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) \quad \text{--- (1)}$$

Function $f(x)$ expanded in the above form has the period 2π in the interval $(-\pi$ to $\pi)$.

For finding A_0 :-

we integrate both sides of eq. (1) b/w the limits $-\pi$ to π ,

$$\int_{-\pi}^{+\pi} f(x) dx = A_0 \int_{-\pi}^{+\pi} dx + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{+\pi} \cos nx dx + \sum_{n=1}^{\infty} B_n \int_{-\pi}^{+\pi} \sin nx dx$$

$$\Rightarrow \int_{-\pi}^{+\pi} dx = (x)_{-\pi}^{+\pi} = 2\pi.$$

$$\int_{-\pi}^{+\pi} \cos nx dx = \frac{1}{n} \left[\sin nx \right]_{-\pi}^{+\pi} = 0$$

$$\int_{-\pi}^{+\pi} \sin nx dx \rightarrow \frac{-1}{n} \left[\cos nx \right]_{-\pi}^{+\pi} = \frac{-1}{n} \cos n\pi - \cos(-n\pi) = 0$$

$$\int_{-\pi}^{+\pi} f(x) dx = 2\pi A_0 \quad \text{or} \quad A_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

Hence, A_0 is the average value of $f(x)$ over the interval $(-\pi, \pi)$.

For finding the value of A_n :-

we multiply the both sides of eq. (1) by $\cos nx$ and integrate b/w the limits $-\pi$ to π .

$$\int_{-\pi}^{+\pi} f(x) \cos nx dx = A_0 \int_{-\pi}^{+\pi} \cos nx dx + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{+\pi} \cos nx \cdot \cos nx dx + \sum_{n=1}^{\infty} B_n \int_{-\pi}^{+\pi} \sin nx \cdot \cos nx dx \quad \text{--- (2)}$$

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$$\begin{aligned}\int_{-\pi}^{+\pi} \cos nx \, dx &= \left[\frac{\sin nx}{n} \right]_{-\pi}^{+\pi} = \frac{1}{n} [\sin(n\pi) - \sin(-n\pi)] = 0 \\ \int_{-\pi}^{+\pi} \cos^2 nx \, dx &= \frac{1}{2} \int_{-\pi}^{+\pi} (1 + \cos 2nx) \, dx \\ &= \frac{1}{2} \int_{-\pi}^{+\pi} dx + \frac{1}{2} \int_{-\pi}^{+\pi} \cos 2nx \, dx \\ &= \frac{1}{2} \left[x + \frac{\sin 2nx}{2n} \right]_{-\pi}^{+\pi} \\ &= \frac{1}{2} (\pi - (-\pi)) + \frac{1}{4n} [\sin 2n\pi - \sin(-2n\pi)] = \pi + 0\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{+\pi} \cos nx \cdot \sin nx \, dx &= \frac{1}{2} \int_{-\pi}^{+\pi} 2 \sin nx \cdot \cos nx \, dx \\ &= \frac{1}{2} \int_{-\pi}^{+\pi} \sin 2nx \, dx = -\frac{1}{2} \left(\frac{\cos 2nx}{2n} \right)_{-\pi}^{+\pi} \\ &= -\frac{1}{4n} [\cos 2n\pi - \cos 2n(-\pi)] = -\frac{1}{4n} [1 - 1] = 0\end{aligned}$$

Substituting these values in eq. (2), we get,

$$\begin{aligned}\int_{-\pi}^{+\pi} f(x) \cos nx \, dx &= A_n \pi \\ A_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(u) \cos nu \, du. \quad \text{--- (3)}\end{aligned}$$

To find the values of B_n , we multiply eq. (1) by $\sin nx$ and integrate b/w $-\pi$ to $+\pi$.

$$\begin{aligned}\therefore \int_{-\pi}^{+\pi} f(x) \sin nx \, dx &= A_0 \int_{-\pi}^{+\pi} \sin nx \, dx + \int_{-\pi}^{+\pi} \sum A_n \sin nx \cdot \\ &\quad \cos nx \, dx + \sum B_n \int_{-\pi}^{+\pi} \sin^2 nx \, dx \quad \text{--- (4)}\end{aligned}$$

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$$\begin{aligned}\text{Here, } \int_{-\pi}^{+\pi} \sin nx \, dx &= \left[-\frac{\cos nx}{n} \right]_{-\pi}^{+\pi} = -\frac{1}{n} [\cos n\pi - \cos(-n\pi)] \\ &= -\frac{1}{n} [\cos n\pi - \cos n\pi] = 0\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{+\pi} \sin nx \cdot \cos nx \, dx &= 0 \\ \int_{-\pi}^{+\pi} \sin^2 nx \, dx &= \frac{1}{2} \int_{-\pi}^{+\pi} (1 - \cos 2nx) \, dx \\ &= \frac{1}{2} (x)_{-\pi}^{+\pi} - \frac{1}{2} \int_{-\pi}^{+\pi} \cos 2nx \, dx \\ &= \frac{1}{2} (2\pi) - 0 = \pi\end{aligned}$$

Substituting these values in eq. (4), we get,

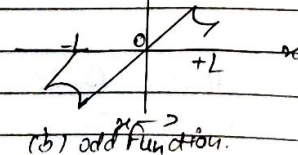
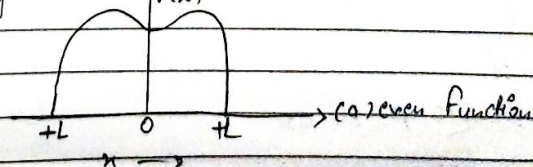
$$\begin{aligned}\int_{-\pi}^{+\pi} f(x) \sin nx \, dx &= B_n \pi \\ B_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(u) \sin nu \, du \quad \text{--- (5)}\end{aligned}$$

A function $f(x)$ satisfying Dirichlet conditions,

$$\begin{aligned}f(x) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(u) \, du + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos nx \int_{-\pi}^{+\pi} f(u) \cos nu \, du \\ &\quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \sin nx \int_{-\pi}^{+\pi} f(u) \sin nu \, du.\end{aligned}$$

* Even and odd functions :-

A function $f(x)$ is said to be an even function if $f(x) = f(-x)$, e.g. - $\cos x, x^2, x^4, \sin^2 x$.
A function is said to be an odd function if $f(-x) = -f(x)$, e.g. - $\sin x, x^3, \cos x, x^3, x^2 \sin x$.



* Fourier series for an even function in the intervals $-\pi$ to π :-

Let $f(x)$ be an even function of x , i.e. $f(x) = f(-x)$ in the intervals $(-\pi$ to $\pi)$.

As $f(x)$ is an even function, so that

$$\int_{-\pi}^{+\pi} f(x) dx = 2 \int_0^{\pi} f(x) dx$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx = \frac{2}{2\pi} \int_0^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} f(x) dx \quad \text{--- (1)}$$

Since, $\cos nx$ ($n=0, 1, 2, 3, \dots$) is an even function.

So, the product $f(x) \cos nx$ is also an even function.

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = 2 \int_0^{\pi} f(x) \cos nx dx$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \quad \text{--- (2)}$$

As, $\sin nx$ ($n=0, 1, 2, 3, \dots$) is an odd function, so product $f(x) \sin nx$ is also an odd function

$$\int_{-\pi}^{\pi} f(x) \sin nx dx = 0$$

because the integral of an odd function of x over symmetric limits gives zero. On substituting the limits, it becomes zero.

$$\text{But, } B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$\therefore B_n = 0$$

Thus, Fourier series of an even function contains only cosines, i.e.,

$$f(x) = \sum_{n=1}^{\infty} A_n \cos nx$$

$$A_0 + \sum_{n=1}^{\infty} A_n \cos nx$$

where A_0, A_1, A_2 etc. are the given by eq's (1), (2),

$$f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) du + \sum_{n=1}^{\infty} \left[\frac{2}{\pi} \int_0^{\pi} f(u) \cos nu du \right] \cdot \pi \cos nx \quad \text{--- (3)}$$

* Fourier series for an odd function in the interval $(-\pi$ to $\pi)$ - sine series :-

Let $f(x)$ be an odd function i.e. $f(x) = -f(-x)$ defined in the intervals $(-\pi$ to $\pi)$. In this case, it is impossible to express $f(x)$ as a sine series because A_0 & A_n (coefficients of cosine terms) will vanish as proved below :-

As $f(x)$ is an odd function, so that,

$$\int_{-\pi}^{\pi} f(x) dx = 0$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$A_0 = 0$$

Since, $\cos nx$ ($n=0, 1, 2, 3, \dots$) is an even function, so product $f(x) \cos nx$ is an odd function because it is a product of an odd function & of an even function.

$$\therefore \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$$

The, function $f(x) \sin nx$ is an even function, as it is a product of two odd functions $f(x)$ and $\sin nx$.

$$\therefore \int_{-\pi}^{\pi} f(x) \sin nx dx = 2 \int_0^{\pi} f(x) \sin nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad (1)$$

Thus, Fourier series of an odd function contains only sine terms.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where coefficients b_n are given by eq. (1)

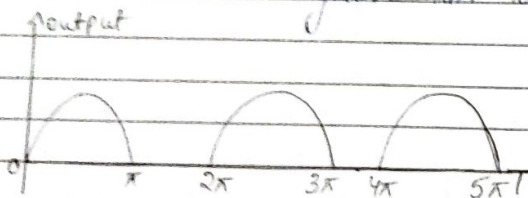
$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \int_0^{\pi} f(x) \sin nx \, dx \quad (2)$$

Since, the function $f(x)$ is a series of sines, it is called sine series in the interval $(0, \pi)$.

* Illustrative examples of Fourier series:-

a) Half-wave Rectifiers:-

A half-wave rectifier passes the positive half cycle and does not allow the negative half to pass as shown in fig.



$$f(\omega t) = E_0 \sin \omega t \quad \text{for } 0 \leq \omega t \leq \pi$$

$$f(\omega t) = 0 \quad \text{for } \pi \leq \omega t \leq 2\pi$$

where E_0 is the amplitude of the rectified voltage.

The Fourier series of $f(\omega t)$ is written as:-

$$f(\omega t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \quad (1)$$

Calculation of Fourier coefficients:-

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\omega t) \, d\omega t = \int_0^{\pi} E_0 \sin \omega t \, d(\omega t)$$

$$= \frac{1}{2\pi} \left[\int_0^{\pi} E_0 \sin \omega t \, d(\omega t) + E_0 \int_{\pi}^{2\pi} 0 \, d(\omega t) \right]$$

$$= \frac{1}{2\pi} \int_0^{\pi} E_0 \sin \omega t \, d(\omega t) + 0 = \frac{E_0}{2\pi} [-\cos \omega t]_0^{\pi}$$

$$= \frac{E_0}{2\pi} [1 + 1] = \frac{E_0}{\pi} \quad (2)$$

$$(P_1) \quad a_n = \frac{1}{\pi} \int_0^{2\pi} E(\omega t) \cos n\omega t \, d(\omega t) = \frac{1}{\pi} \int_0^{\pi} E_0 \sin \omega t \cos n\omega t \, d(\omega t) \\ = \frac{1}{\pi} \left[\int_0^{\pi} E_0 \sin \omega t \cos n\omega t \, d(\omega t) + E_0 \int_{\pi}^{2\pi} 0 \cos n\omega t \, d(\omega t) \right]$$

$$\therefore \cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

$$= \frac{E_0}{2\pi} \left[\int_0^{\pi} [\sin(n+1)\omega t - \sin(n-1)\omega t] \, d(\omega t) + 0 \right]$$

$$= \frac{E_0}{2\pi} \left[\frac{-\cos(n+1)\omega t}{n+1} + \frac{\cos(n-1)\omega t}{n-1} \right]_0^{\pi}$$

$$= \frac{E_0}{2\pi} \left[\frac{-\cos(n+1)\pi}{n+1} + \frac{1}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n-1} \right]$$

$$= \frac{E_0}{2\pi} \left[\frac{-\cos(n-1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{E_0}{2\pi} \left[\cos(n-1)\pi \left(\frac{1}{n-1} - \frac{1}{n+1} \right) - \frac{2}{n^2-1} \right] \quad \left(\because \cos(n-1)\pi = \cos(n+1)\pi \right)$$

$$= \frac{E_0}{2\pi} \left[\cos(n-1)\pi \left(\frac{2}{n^2-1} \right) - \frac{2}{n^2-1} \right]$$

$$= \frac{2E_0}{2\pi(n^2-1)} [\cos(n-1)\pi - 1]$$

$$= \begin{cases} -2E_0/(n^2-1)\pi & \text{if } n \text{ is an even integer} \\ 0 & \text{if } n \text{ is an odd integer} \end{cases} \quad (3)$$

$$(iii) b_n = \frac{1}{\pi} \int_0^{2\pi} E(\omega t) \sin n\omega t d(\omega t) = \int_{-\pi}^{\pi} E_0 \sin \omega t \sin n\omega t d(\omega t) \\ - \frac{1}{\pi} \int_0^{\pi} E_0 \sin \omega t \sin n\omega t d(\omega t) + E_0 \int_{\pi}^{2\pi} 0 \sin(n\omega t) d(\omega t)$$

$$\text{or, } b_n = \frac{E_0}{\pi} \int_{-\pi}^{\pi} \sin \omega t \sin n\omega t d(\omega t) + 0 \quad \text{--- (4)}$$

$$\therefore \sin A + \sin B = \frac{\cos(A-B) + \cos(A+B)}{2}$$

$$\text{If } b > 1, \\ b_n = \frac{E_0}{2\pi} \int_0^{2\pi} [\cos(n-1)\omega t - \cos(n+1)\omega t] d(\omega t)$$

$$b_n = \frac{E_0}{2\pi} \int_0^{2\pi} \left[\frac{\sin(n-1)\omega t}{n-1} - \frac{\sin(n+1)\omega t}{n+1} \right] d(\omega t) \\ = \frac{E_0}{2\pi} \left[\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1} - \frac{\sin 0}{n-1} + \frac{\sin 0}{n+1} \right] \\ = \frac{E_0}{2\pi} \times 0 = 0$$

for $n=1$, in eq. (4), we get,

$$b_1 = b_1 = \frac{E_0}{\pi} \int_0^{2\pi} \sin^2 \omega t d(\omega t)$$

$$= \frac{E_0}{2\pi} \int_0^{2\pi} (-\cos 2\omega t + 1) d(\omega t) = \frac{E_0}{2\pi} \left[-\frac{\sin 2\omega t}{2} + \omega t \right]_0^{2\pi} \\ = \frac{E_0}{2\pi} \left[-\frac{\sin 2\pi}{2} + \pi + \frac{\sin 0}{2} - 0 \right] = \frac{E_0}{2} \quad \text{--- (5)}$$

Substituting the values of $a_0, a_1, b_1, \dots$ in eq. (1),

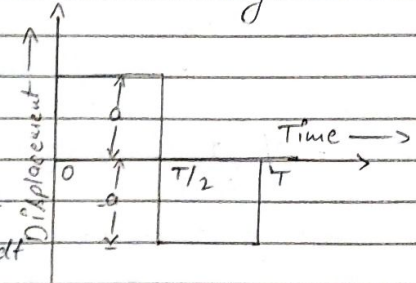
$$E(\omega t) = \frac{E_0}{\pi} + \frac{E_0}{2} \sin \omega t = \frac{2E_0}{\pi} \left[\frac{1}{3} \cos 2\omega t + \frac{1}{15} \cos 4\omega t + \dots \right] \quad \text{--- (6)}$$

(d) Rectangular waves:-

The analysis of a rectangular wave by Fourier theorem can be illustrated as shown in fig.

$$y = +a \text{ from } t=0 \text{ to } t=\frac{T}{2}$$

$$\text{and } y = -a \text{ from } t=\frac{T}{2} \text{ to } t=T.$$



$$A_0 = \frac{1}{T} \int_0^T f(\omega t) dt = \frac{1}{T} \int_0^{T/2} y dt + \frac{1}{T} \int_{T/2}^T y dt$$

$$= \frac{1}{T} \int_0^{T/2} a dt - \frac{1}{T} \int_{T/2}^T a dt \\ = \frac{a}{T} \left[t \right]_0^{T/2} - \frac{a}{T} \left[t \right]_{T/2}^T$$

$$A_0 = 0$$

$$A_n = \frac{2}{T} \int_0^T f(\omega t) \cos(n\omega t) dt$$

$$= \frac{2}{T} \int_0^{T/2} y \cos(n\omega t) dt + \frac{2}{T} \int_{T/2}^T y \cos(n\omega t) dt$$

$$= \frac{2a}{T} \int_0^{T/2} \cos(n\omega t) dt - \frac{2a}{T} \int_{T/2}^T \cos(n\omega t) dt$$

$$\because y = a \text{ for } t=0 \text{ to } T/2, y = -a \text{ for } t=T/2 \text{ to } T$$

$$= \frac{2a}{T} \int_0^{T/2} \cos \frac{2\pi n t}{T} dt - \frac{2a}{T} \int_{T/2}^T \cos \frac{2\pi n t}{T} dt \quad \left[\because \omega = \frac{2\pi}{T} \right]$$

$$= \frac{2a}{T} \cdot \frac{T}{2\pi n} \left[\sin \frac{2\pi n t}{T} \right]_0^{T/2} - \frac{2a}{T} \cdot \frac{T}{2\pi n} \left[\sin \frac{2\pi n t}{T} \right]_{T/2}^T$$

$$= \frac{a}{\pi n} [\sin n\pi - 0] - \frac{a}{\pi n} [\sin 2n\pi - \sin n\pi]$$

$$= \frac{a}{\pi n} [2 \sin n\pi - \sin 2n\pi] = 0$$

$$\begin{aligned} B_n &= \frac{2}{T} \int_0^{T/2} a \sin n\omega t dt - \frac{2}{T} \int_{T/2}^T a \sin n\omega t dt \\ &= \frac{2a}{T} \int_0^{T/2} \sin \frac{2\pi n t}{T} dt - \frac{2a}{T} \int_{T/2}^T \sin \frac{2\pi n t}{T} dt \\ &= \frac{2a}{T} \cdot \frac{T}{2\pi n} \left[-\cos \frac{2\pi n t}{T} \right]_0^{T/2} - \frac{2a}{T} \cdot \frac{T}{2\pi n} \left[-\cos \frac{2\pi n t}{T} \right]_{T/2}^T \\ &= \frac{a}{\pi n} [-\cos 4\pi + \cos 0] - \frac{a}{\pi n} [-\cos 2\pi + \cos 6\pi] \end{aligned}$$

$$= \frac{a}{\pi n} [-\cos 4\pi + 1 + \cos 2\pi - \cos 6\pi]$$

$$= \frac{a}{\pi n} [-\cos 4\pi + 1 + 1 - \cos 6\pi] = \frac{a}{\pi n} [2 - 2\cos 4\pi]$$

$$= \frac{2a}{\pi n} [1 - \cos 4\pi] = \frac{2a}{\pi n} [1 - (-1)^n] \quad \left[\because \cos 4\pi = (-1)^n \right]$$

For this expression putting $n=1, 2, 3, 4, \dots$

$$B_1 = \frac{4a}{\pi}, B_2 = 0, B_3 = \frac{4a}{3\pi}, B_4 = 0, \dots$$

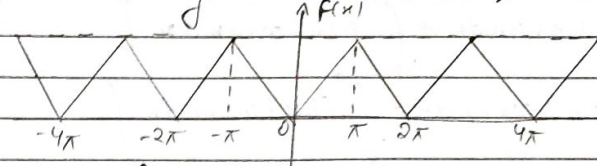
Hence, the complete Fourier series representing an exact rectangular wave with no constant term is given by,

$$y = \frac{4a}{\pi} \sin \omega t + \frac{4a}{3\pi} \sin 3\omega t + \frac{4a}{5\pi} \sin 5\omega t + \dots + \frac{4a}{n\pi} \sin(n\omega t)$$

$$y = \frac{4a}{\pi} \left[\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \dots + \frac{1}{n} \sin(n\omega t) \right]$$

(b) Triangular wave:-

Suppose the triangular wave is as shown in fig:-



$$f(x) = \begin{cases} kx & \text{for } 0 \leq x \leq \pi \\ -kx & \text{for } -\pi \leq x \leq 0 \end{cases}$$

where k is constant (+ve).

The Fourier expansion of the function $f(x)$ is:-

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$\therefore a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^0 (-kx) dx + \int_0^{\pi} kx dx \right]$$

$$= \frac{1}{2\pi} \left[-k \left[\frac{x^2}{2} \right]_{-\pi}^0 + k \left[\frac{x^2}{2} \right]_0^{\pi} \right]$$

$$= \frac{k}{2\pi} \left[-\frac{1}{2} [0^2 - (-\pi)^2] + \frac{1}{2} (\pi^2 - 0^2) \right]$$

$$= \frac{k}{2\pi} \left[0 + \frac{\pi^2}{2} + \frac{\pi^2}{2} \right] = \frac{k}{2\pi} \cdot \frac{2\pi^2}{2} = \frac{k\pi}{2} \quad \text{--- (2)}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \cos nx \, dx + \int_0^{\pi} f(x) \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 (-kx) \cos nx \, dx + \int_0^{\pi} (kx) \cos nx \, dx \right]$$

$$= \frac{k}{\pi} \left[- \int_{-\pi}^0 x \cos nx \, dx + \int_0^{\pi} x \cos nx \, dx \right]$$

Integrating by parts, we get,

$$a_n = \frac{k}{\pi} \left[- \left\{ \frac{x \sin nx}{n} \right\}_{-\pi}^0 - \int_{-\pi}^0 \frac{\sin nx}{n} \, dx \right] + \left\{ \frac{x \sin nx}{n} \right\}_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} \, dx$$

$$= \frac{k}{\pi} \left[- \left\{ 0 - \frac{1}{n} \int_{-\pi}^0 \sin nx \, dx \right\} + \left\{ 0 - \frac{1}{n} \int_0^{\pi} \sin nx \, dx \right\} \right]$$

$$= \frac{k}{\pi} \left[\frac{1}{n} \int_{-\pi}^0 \sin nx \, dx - \frac{1}{n} \int_0^{\pi} \sin nx \, dx \right]$$

$$= \frac{k}{n\pi} \left[\left\{ -\frac{\cos nx}{n} \right\}_{-\pi}^0 - \left\{ -\frac{\cos nx}{n} \right\}_0^{\pi} \right]$$

$$= \frac{k}{n^2\pi} \left[-[\cos 0 - \cos(-n\pi)] + (\cos n\pi - \cos 0) \right]$$

$$= \frac{2k}{n^2\pi} [\cos n\pi - 1] = \frac{2k}{n^2\pi} [1 - \cos n\pi]$$

$$= \frac{-2k}{n^2\pi} [1 - (-1)^n]$$

or, $a_n = \begin{cases} 0 & \text{when } n \text{ is an even integer,} \\ -\frac{4k}{n^2\pi} & \text{when } n \text{ is an odd integer.} \end{cases} \quad (3)$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \sin nx \, dx + \int_0^{\pi} f(x) \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 (-kx) \sin nx \, dx + \int_0^{\pi} (kx) \sin nx \, dx \right]$$

$$= \frac{k}{\pi} \left[\int_0^{\pi} x \sin nx \, dx - \int_{-\pi}^0 x \sin nx \, dx \right]$$

Integrating by parts,

$$b_n = \frac{k}{\pi} \left[\left\{ -\frac{x \cos nx}{n} \right\}_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} \, dx \right] - \frac{k}{\pi} \left[\left\{ -\frac{x \cos nx}{n} \right\}_{-\pi}^0 - \int_{-\pi}^0 \frac{-\cos nx}{n} \, dx \right]$$

$$= \frac{k}{\pi} \left[- \left(\frac{\pi \cos n\pi}{n} - 0 \right) + \int_0^{\pi} \frac{\cos nx}{n} \, dx \right]$$

$$= \frac{k}{\pi} \left[- \frac{0 - \pi \cos(-n\pi)}{n} + \int_{-\pi}^0 \frac{\cos nx}{n} \, dx \right]$$

$$= \frac{k}{\pi} \left[\frac{0 - \pi \cos n\pi}{n} + \int_0^{\pi} \frac{\cos nx}{n} \, dx + \frac{\pi \cos(-n\pi)}{n} + \int_{-\pi}^0 \frac{\cos nx}{n} \, dx \right]$$

$$= \frac{k}{\pi} \left[\frac{-\pi \cos n\pi}{n} + \left\{ \frac{\sin nx}{n^2} \right\}_0^{\pi} + \frac{\pi \cos(-n\pi)}{n} + \left\{ \frac{\sin nx}{n^2} \right\}_{-\pi}^0 \right]$$

$$= \frac{k}{\pi} \left[-\frac{\pi}{n} \cos n\pi + 0 + \frac{\pi}{n} \cos n\pi - 0 \right] = 0 \quad (4)$$

Substituting the values of (1), (3), (4), in eq. (2), we get,

$$f(x) = \frac{k\pi}{2} - \frac{4k}{\pi} \left(\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right) \quad (5)$$