Maa Omwati Degree College Hassanpur

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NOTES

Subject:- Sequence and Series

B.A./B.Sc. 4th Semester

Topology of real numbers

Sets — A Set is well defined Collection of distinct object

Finite — A Set Containing finite number of Elements

Infinite — A Set Containing infinite number

Interval — let a and b be two real No.

Closed Interval — The Set Containing Element a & b and all real b/w a and b

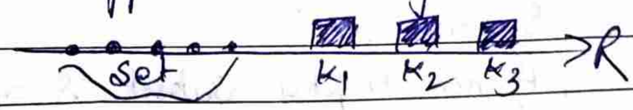
open Interval — The Set Containing all real b/w a and b.

Subset — If A and B be two set St. Every element of A is also an Element of B.

Bounded Above Set —

let S be non Empty subset of reals. Then S is said to be Bounded above if. ∃ a real number K_1 St $x \leq K_1$ $\forall x \in S$. K_1 is called an upper Bound of set S

If a set has an upper Bound K_1 then all real number greater then K_1 are also upper. Bounds of the Set.



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Unbounded Above set :-
A non Empty subset S, of reals is said to be unbounded above if we cannot find any real number k for which $x \leq k \; \forall x \in S$

Bounded Below Set :-
Let S be a non empty subset of reals. Then set S is said to be Bounded below if $\exists$ a real number k_1 s.t $x \geq k_1 \; \forall x \in S$. k_1 is called the lower Bound of the set S. All other real number less than k_1 are also lower Bounds of the set S.

Unbounded Below set :-
A non Empty subset S of R is said to be unbounded Below if we cannot find a real number k s.t $x \geq k \; \forall x \in S$.

Bounded Set :-
A non empty subset S of R is said to be Bounded if $\exists$ two real

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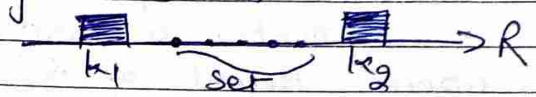
numbers k_1 and k_2 such that $k_1 \leq x \leq k_2 \; \forall x \in S$. k_1 is lower Bound and k_2 is an upper Bound of the set S.



Unbounded Set :-
A non Empty subset S of R is said to be an unbounded set if either S is unbounded above or unbounded Below or unbounded Below and above both.

Greatest Element :-
Let S be a non Empty subset of reals. A real number k is said to be greatest Element of the set S if $k \in S$. and k is greater than all other Elements of the set.

Least Element :-
Let S be a non Empty subset of reals. A real number k_1 is said to be least Element of the set S if $k_1 \in S$ and k_1 is less than all other Element of the set

Least upper Bound (Supremum).

Let S be a non Empty Subset of R. A real number u is called least upper Bound of set S,

(1) u is an upper Bound of set S and

(2) if u' is any other upper Bound of set S, then $u \le u'$.

Uniqueness of Supremum.

Th^m Least upper Bound of a Set, if it Exist is unique

Proof. If Possible Suppose that u and u' are two least upper Bound of Set S. $\therefore$ u and u' Both are upper Bounds of set S. As u is least upper Bound of set S.

$u \le u'$ ——— (1)

u' is least upper Bound of set S.

$u' \le u$ ——— (2)

$u = u'$ from (1) and (2).

The least upper Bound is unique

Th^m The greatest member of a Set if it Exist, must be Supremum of the Set

Proof Let S be a non Empty Subset of R and u be its greatest member.

$x \le u \quad \forall x \in S$.

$\therefore$ u is an upper Bound of set S.

Also if u' be any other upper Bound of set S. then $x \le u'$. $\forall x \in S$.

In Particular $u \le u'$.

Thus u is the least upper Bound of set S.

u must be the Supremum of the set S,

Greatest lower Bound OR Infimum!—

Let S be a non Empty subset of R. A real number l is called greatest lower Bound of Set S

(i) l is lower Bound of set S and

(2) If l' is any other lower Bound of S. then $l \ge l'$.

Th^m. Greatest lower Bound. of a Set, if it Exist, is unique

Proof let if Possible, l, l' be two greatest lower Bounds of Set S. clearly l and l' Both are lower Bounds of set S.

l is greatest lower Bound and l' is lower Bound. of set S.

$l \ge l'$

l' is greatest lower Bound and l is

lower Bound of set S, so.
$l' \geq l$.
from (1) and (2), we get
$l = l'$
greatest lower Bound of a set if it exist is unique

Th^m If S is a non Empty set of real numbers which is Bounded Below, then a real number l is the infimum
(1) $x \geq l$. (2) for each $\epsilon > 0$, $\exists$ atleast $x \in S$.

Proof Let u be supremum of set S.
u is an upper Bound of set S.
$x \leq u$, $\forall x \in S$.
for each $\epsilon > 0$ $u-\epsilon$ is less than u.
$u-\epsilon$ is not an upper Bound of set S So $\exists$ atleast one $x \in S$ where

Converse Let us suppose that u is an real number satisfying Condition (i) and (ii).
from Condition (1) $x \leq u$
u is an upper Bound of set S.
let u' be any real number less than u
$u - u' > 0$. $\epsilon = u - u'$ $x > u - \epsilon$
$x > u - (u - u')$ or $x > u'$.
u' cannot be an upper Bound of set S
Every real number less than u is not an upper Bound of set

70
98
700
9
7
9) 700
63
70

Greatest lower Bound of infimum.
Let S be a non Empty Subset of R.
A real number l called of set S.
(1) l is lower Bound of set S.
(2) If l' is any other Bound of set S then $l \geq l'$

uniqueness of infimum:–
Th^m Greatest lower Bound of a set, if it exist is unique

Proof Let if possible. l, l' be lower Bound of set S.
l and l' both are Bound of set S.
l is greatest lower Bound $l \geq l'$ and
l' is lower Bound of set S. $l' \geq l$ ... (2)
from (1) and (2). $l = l'$.
greatest lower Bound of a set is unique

Completeness of axiom:–
Every non Empty subset of real which is Bounded above has a real number as its supremum.

Th^m. Every non Empty Subset of real number which Bounded Below has a real number as its infimum.

Proof:– Let S be a non Empty Subset of real number which is Bounded Below by u.
$x \geq u$ $\forall x \in S$
$T = \{-x : x \in S\}$ let y be any Element

$y = -x$ for some $x \in S$.

$x \geq u \quad \forall x \in S$.

$-x \leq -u \quad \forall x \in S$.

$y \leq -u \quad \forall y \in T$

Set T is Bounded above by $-u$.

By Completeness axiom of reals, Set T has least upper Bound,

let $-l$ be the least upper Bound of set T.

$y \leq -l \quad \forall y \in T$.

$-x \leq -l \quad \forall x \in S$.

$x \geq l \quad \forall x \in S$,

l is lower Bound of set S.

let l' be any other lower Bound of set S.

$x \geq l'$

$-x \leq -l'$

$-l'$ is an upper Bound of set T.

$-l$ is LUB of set T

$-l \leq -l'$

$l \geq l'$

l is a lower Bound of set S and l' is any other lower Bound of set S, $l \geq l'$.

High Court of Punjab and Haryana at Chandigarh

Archimedean Property of reals:

If x and y are two positive real number then $\exists$ a natural number n s.t. $nx > y$

Th^m — Between two distinct real numbers, there are infinitely many rational no.

Proof — let x and y be two real number & $y > x$

$y - x > 0$.

$y - x$ and 1 are two positive real numbers

$\therefore$ By Archimedean property to reals.

$n(y-x) > 1$.

$ny - nx > 1$

$ny > nx + 1$,

$nx + 1 \leq ny$

$nx+1$ is real number,

$m \leq nx + 1 < m+1$

we ① $m \leq nx+1 < ny$ —— ②

$nx + 1 < m+1 \quad nx < m$

$nx < m \leq nx+1 < ny$

$nx < m < ny$

$x < \frac{m}{n} < y$ or $x < r_1 < y$

$\frac{m}{n} = r_1$

Q1. If S and T are non empty Bounded subset of $\mathbb{R}$, then prove that $S \cup T$ is also Bounded and $\sup(S \cup T) = \max(\sup S, \sup T)$

Proof — let $\sup S = u_1$ and $\sup T = u_2$

let $u = \max(u_1, u_2)$

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$x \le u_1 \le u$

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$x \le u_2 \le u \quad \forall x \in T.$

$x \le u \quad \forall x \in S \cup T$

$S \cup T$ is bounded above and u is upper Bound of $S \cup T$.

Further, for any $\epsilon > 0$, $u - \epsilon < u$

either $u - \epsilon < u_1$ or $u - \epsilon < u_2$.

If $u - \epsilon < u_1$ then $\exists\ x \in S$ and hence $x \in S \cup T$ such that $x > u - \epsilon$.

If $u - \epsilon < u_2$, then $\exists\ x \in T$ and hence $x \in S \cup T$ such that $x > u - \epsilon$

Thus for $\epsilon > 0$, $\exists\ x \in S \cup T$ such that $x > u - \epsilon$, where u is an upper Bound of $S \cup T$.

$$\sup(S \cup T) = u = \max\{u_1, u_2\}$$
$$= \max\{\sup S, \sup T\}.$$

Q 5 Prove that set of rationals is not order Complete

Proof To Establish the result it is Enough to show that $\exists$ a non Empty subset of $\mathbb{Q}$ which is Bounded above But doesnot have supremum i.e $\nexists$ no rational number which is the least of all its upper Bounds

Let $\mathbb{Q}$ is set of rational

$S = \{x : x \in \mathbb{Q},\ x > 0$ and $x^2 < 2\}$ i.e the set of all those positive rational. Square are less than 2

$1 \in S$ so $S \ne \phi$ 2 is an upper Bound of sets

S is non Empty Bounded above subset of $\mathbb{Q}$
let p be the supremum of S and $p \in \mathbb{Q}$
By Trichotomy law
$p^2 = 2$ or $p^2 < 2$ or $p^2 > 0$ where $p > 0$

Case I when $p^2 = 2$
$p = \sqrt{2}$, which is not a rational no.
$p^2 = 2$ is not possible.

Case II $p^2 < 2$

$$y = \frac{4+3p}{3+2p}$$

As $p > 0$ so $y > 0$ and $y \in \mathbb{Q}$

$$y^2 - 2 = \left(\frac{4+3p}{3+2p}\right)^2 - 2 = \frac{(4+3p)^2 - 2(3+2p)^2}{(3+2p)^2}$$

$$= \frac{16 + 9p^2 + 24p - 2(9 + 4p^2 + 12p)}{(3+2p)^2}$$

$$= \frac{p^2 - 2}{(3+2p)^2} < 0$$

$y^2 - 2 < 0$
$y^2 < 2$ and $y > 0$. $y \in S$.

$$y - p = \frac{4+3p}{3+2p} - p = \frac{2(2-p^2)}{3+2p} > 0$$

$y - p > 0 \Rightarrow y > p$

Case III when $p^2 > 2$

$$y = \frac{4+3p}{3+2p}$$

$$y^2 - 2 = \frac{(4+3p)^2}{(3+2p)^2} - 2 = \frac{p^2 - 2}{(3+2p)^2} > 0$$

$y^2 - 2 > 0$
$y^2 > 2$

$$y - p = \frac{4+3p}{3+2p} - p = \frac{2(2-p^2)}{3+2p} < 0.$$

$y - p < 0 \Rightarrow y < p.$
y is an upper Bound of S and y is less than least upper Bound of set S which is a contradiction, $p^2 > 2$ is not possible
$p \in \mathbb{Q}$ Cannot be supremum of set S.
$\mathbb{Q}$ is not order Complete.

Neighbourhood of a point :-
A set $S \subseteq \mathbb{R}$ is said to be a neighbourhood of a point $p \in \mathbb{R}$. if $\exists$ an open interval $(p-\epsilon, p+\epsilon)$ for some $\epsilon > 0$ $p \in (p-\epsilon, p+\epsilon) \subseteq S$.
The neighbourhood of point p is called a symmetric nbd of p with radius ϵ.

Deleted neighbourhood :-
Deleted neighbourhood of p with radius ϵ
$= (p-\epsilon, p+\epsilon) - p. = (p-\epsilon, p) \cup (p, p+\epsilon)$
$= \{x : 0 < |x-p| < \epsilon\}$

Th^m. Superset of a neighbourhood of a point p is also a neighbourhood of p.

Proof. Let A be neighbourhood of point p.

$\therefore \exists\ \varepsilon > 0$ s.t.

$p \in (p-\varepsilon, p+\varepsilon) \subseteq A$. — (1)

B is superset of A. Then $A \subseteq B$. — (2)

from (1) and (2).

$p \in (p-\varepsilon, p+\varepsilon) \subseteq B$.

B is a neighbourhood of point p.

Interior point of a set :-

If A is neighbourhood of a point x, then point x.

Interior of a set :-

The set of all interior points of set A.

Open sets :-

A set $S \subseteq R$ is said to be open if it is nhd of each of its points.

Set $S \subseteq R$ is said to be open set if for each $p \in S$.

Th^m Prove that the union of an arbitrary family of open set is an open set.

Proof Let $(A_\lambda : \lambda \in \Lambda)$ be an arbitrary family of open set.

$$A = \bigcup_{\lambda \in \Lambda} A_\lambda .$$

A is an open set.

Let x be an arbitrary element of A.

$x \in \bigcup_{\lambda \in \Lambda} A_\lambda$

$x \in A_\lambda$

A_λ is an open set and so atleast one A_λ is nhd of x

$x \in (x-\varepsilon, x+\varepsilon) \subseteq A_\lambda$.

$x \in (x-\varepsilon, x+\varepsilon) \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$.

$x \in (x-\varepsilon, x+\varepsilon) \subseteq A$

A is nhd of x.

A is nhd of all its points and A is an open set.

Th^m — The interior of set A is the largest open subset of A.

Proof Let A^0 be the interior set of A.

Also, let x be any element of A^0.

$x \in A^0$.

x is an interior point of A

There exist $\varepsilon > 0$ $x \in (x-\varepsilon, x+\varepsilon) \subseteq A$.

$x \in A$.

$x \in A^0 \Rightarrow x \in A$

$A^0 \subseteq A$.

A^0 is an open set.

A^0 is an open subset of A.

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B be any other open subset of A

Case I when $B = \phi$.
$B \subseteq A^o$.

Case II when $B \neq \phi$.
let x be any element of B.
B is an open set, B is a nbd of x.
$\exists\ \epsilon > 0$. st $x \in (x-\epsilon, x+\epsilon) \subseteq B$
$B \subseteq A$
$x \in (x-\epsilon, x+\epsilon) \subseteq A$
A is nbd of x
x is an interior point of A.
$x \in A^o$
$x \in B \Rightarrow x \in A^o$,
$B \subseteq A^o$
A^o is largest open subset of A

Closed Sets :-
Let A be any subset of R. Then set
A is said to be a closed set iff.
Complement of A is an open set.
A is closed iff $A^c = R - A$ is open

Thm The union of finite number of closed sets is a closed set.

Proof let $F_1, F_2 \text{---} F_n$ be a finite number of closed sets
$F = \bigcup_{\lambda=1} F_\lambda$ is closed set

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$F_1, F_2 \text{---} F_n$ be n closed sets.
$F_1^c, F_2^c \text{---} F_n^c$ are n open sets.
finite intersection of open set is open.
$\bigcap_{\lambda=1} F_\lambda^c$ is an open set.
By De Morgan's law.
$\left(\bigcup_{\lambda=1} F_\lambda\right)^c$ is an open set
$\bigcup_{\lambda=1} F_\lambda$ is a closed set.
The union of a finite number of closed set is a closed sets.

Q5. Is every infinite set open? Justify your ans

A5 Infinite set many or may not be an open set. open interval (a, b) is an infinite set and it is an open set.
But set N of natural number is also an infinite set and it is not an open set.
$a \in N$ $\quad \epsilon > 0 \quad (a-\epsilon, a+\epsilon)$ is not contained in N.
N is not an open set.

Limit point of a Set :-
A real number p is said to be a limit point or accumulation point of a Set $S \subseteq R$ if every nbd of p contains at least one point of S different from p.

Derived Set :-
The Set of all limit points of a Set $A \subseteq R$.

Isolated point → Let A be a Subset of R. Then $x \in R$ is called an isolated point of Set A if $x \in A$ But x is not limit point of A

Adherent point :- Let A be a Subset of R. Then $x \in R$ is called an adherent point of A if every nbd of x contains atleast one point of A
$x \in (x-\epsilon, x+\epsilon) \cap A \neq \phi$.

Closure of a Set :- Let A be Subset of R. Then closure of A is Set containing all points of A and all limit point of A
$\bar{A} = A \cup d(A)$.

Thm If A and B are Subset of R, then
(1) $(A \cup B)' = A' \cup B'$.

Proof $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$
$A' \subseteq (A \cup B)'$ and $B' \subseteq (A \cup B)'$
$A' \cup B' \subseteq (A \cup B)'$ — (1)

Converse Let x be any Element of $(A \cup B)'$
$x \in (A \cup B)'$
x is a limit point of $(A \cup B)$
Every nbd of x contains a point $y \in (A \cup B)$ $y \neq x$
Every nbd of x contain a point $y \in A$ or $y \in B$ $y \neq x$
x is a limit point of A or
$x \in A'$ or $x \in B'$
$x \in (A' \cup B')$
$x \in (A \cup B)'$ $x \in A' \cup B'$
$(A \cup B)' \subseteq (A' \cup B')$ — (2)
from (1) and (2)
$(A \cup B)' = A' \cup B'$

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Bolzano Weierstrass Theorem

Every infinite Bounded Subset of real number has a limit point.

Proof Let A be an infinite Bounded Subset of real number with Bounds $m = \inf A$ and $M = \sup A$.

$m \le x \le M$. —— (1)

a Set of real number

$T = \{x : x$ Exceeds finite number of Elent of $A\}$ —— (2)

from (2)

$x \in T \Rightarrow \begin{cases} x \ge \text{only finite no} & \text{—— (3)} \\ x \le \text{infinite number} & \text{—— (4)} \end{cases}$

from (1) and (4) $m \in T$

$M \ge x$

$y \in T$ $y \ge$ only finite no' of Elent

$M \ge y$

M is an upper Bound of Set T.

T is non Empty Bonded above Subset of R

let u be least upper Bound of Set T $u+\epsilon \notin T$

$u+\epsilon \ge$ infinite no' of Elent of A

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u is l.u.b of Set T and $u-\epsilon$ Bound on upper Bound of Set T.

$x_1 \in T$ $u-\epsilon < x_1$

$x_1 \le$ infinite number of Elent of A

$u-\epsilon < x_1 \le$ infinite number of Elent of A —— (6)

(5) and (6)

$(u-\epsilon, u+\epsilon)$ Contains infinite no' of Element of A

u is limit point of Set A

Th^m Prove that the Supremum of a non Empty Bounded Set is Either the greatest member of the set or is a limit point of the set.

Proof Let A be non Empty Bounded Subset of R. By Completeness property of reals Set A has Supremum u.

$x \le u$.

If $u \in A$ then u is greatest member of A

If $u \notin A$ then for Each $\epsilon > 0$

$u-\epsilon$ is not an upper Bound of A.

$\exists$ Some $x_1 \in A$ $u-\epsilon < x_1 < u < u+\epsilon$

$(u-\epsilon, u+\epsilon)$ is a nbd of u and Contains a point $x_1 \in A$, $x_1 \ne u$.

u is a limit point of Set A

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Thm The closure of a set $A \subseteq R$ is the smallest closed superset of A

Proof let $\bar{A}$ be closure of A.
$\bar{A}$ is closed.
let x be any element of A.
$x \in A \Rightarrow x \in A \cup A'$
$x \in \bar{A}$
$x \in A \Rightarrow x \in \bar{A}$
$A \subseteq \bar{A}$.

Prove $\bar{A}$ is smallest closed superset of A

let B be any other closed superset of A $A \subseteq B$,
$\bar{A} \subseteq \bar{B}$,
But $\bar{B} = B$ B is closed set
$\bar{A} \subseteq B$.
$\bar{A}$ is smallest closed superset of A.

Thm If A and B are arbitrary subset of R

(1) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$
$\bar{A} \subseteq \bar{B}$.

Proof let $x \in \bar{A}$
x is adherent point of A
for each $\epsilon > 0$ $(x-\epsilon, x+\epsilon) \cap A \neq \phi$
$(x-\epsilon, x+\epsilon) \cap B \neq \phi$

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x is an adherent point of A
$(x-\epsilon, x+\epsilon) \cap A \neq \phi$.
$(x \in \bar{B}$
$x \in \bar{A} \Rightarrow x \in \bar{B}$.
$\bar{A} \subseteq \bar{B}$.

Q If x is a limit point of A and $A \subseteq B$, then x is also a limit point of B.

Ans. $A \subseteq B$ $A' \subseteq B'$
x is limit point of A
$x \in A'$
$x \in B'$
x is limit point of B.

Compact Set:-
A non empty set $A \subseteq R$ is said to be a compact set if it is closed and bounded.

Cover and open Cover :-
let A be a subset of R. The family $(A_\lambda : \lambda \in \Lambda)$ of subset of R is said to be a cover of set A if $A \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$

open Cover :-
let A be a subset of R. The family $(A_\lambda : \lambda \in \Lambda)$ of open subset of R

is said to be open Cover of A if
$A \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$.

Open Cover :-
let A be a subset of R. The family $(A_\lambda : \lambda \in \Lambda)$ of open subset of R is said to be an open cover of A if
$A \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$

Subcover :-
let $(A_\lambda : \lambda \in \Lambda)$ be a cover of set $A \subseteq R$. If $\exists$ a subset T of Λ s.t. $\{A_\lambda : \lambda \in T\}$ also covers A, then family $\{A_\lambda : \lambda \in T\}$

Heine Borel Property :-
Every open cover of a compact set has a finite subcover.

Proof Lemma : (Statement).
If a set $A \subseteq R$ satisfies the Heine Borel property, then any closed subset of A also satisfies Heine Borel property.

Proof of Lemma :-
Let A satisfies the Heine Borel property and then any closed subset of A also satisfies

Heine Borel property
Proof Lemma :- let A satisfies Heine Borel property and B is a closed subset of A. B also satisfies Heine Borel property.
let $(A_\lambda)_{\lambda \in \Lambda}$ is an open cover of B
$B \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$ — (1)
B has a finite sub-cover
$B \subseteq A$ $\therefore A - B \subseteq B^c$ — (2)
from (1) and (2).
$B \cup (A-B) \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda \cup B^c$.
$A \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda \cup B^c$.

B is a closed set, B^c is an open set.
$\bigcup_{\lambda \in \Lambda} A_\lambda \cup B^c$ is an open cover of A. But A Heine Borel property, this open cover of A reduces to a finite subcover
$A_{\lambda_1} \subseteq A_{\lambda_2}$ — A_{λ_n}
$A \subseteq A_{\lambda_1}$ — $A_{\lambda_n} \cup B^c$.
$B \subseteq A_{\lambda_1} \cup$ — $A_{\lambda_n} \cup B^c$.
$B \subseteq A_{\lambda_1} \cup$ — $\cup A_{\lambda_n}$
$(A_\lambda : \lambda \in \Lambda)$ of B reduces to finite subcover

Proof of Theorem :-
let $A \subseteq R$ be a Compact Set.
$\therefore$ A is closed and Bounded.
Let l and u be g.l.b and l.u.b of set A $l \le x \le u$

$A \subseteq (l, u)$.

Case I when $l = u$.
Set A is singleton set and every open cover of A has a finite subcover

Case II. when $l \neq u$.
If possible (l,u) does not satisfy Heine Borel Property
$\exists$ a family T of open sets which covers $I = (l, u)$ but no finite sub family of T covers I.
$I' = \left(l, \frac{l+u}{2}\right)$ and $I'' = \left(\frac{l+u}{2}, u\right)$, I' ~~or~~ I''
is not covered by finitely many members of T.
a seq^n of closed interval.
$I \supset I_1 \supset I_2$ ——— $I_{n+1} \subset I_n$
length of interval $I = (u-l)$
$I_1 = \frac{1}{2}(u-l)$

$I_n = \frac{1}{2^n}(u-l)$
$\bigcap I_n$ = singleton set (ξ_p)
net $\xi_p \in I$

$(\xi_p - \epsilon, \xi_p + \epsilon) \subseteq B$.
$I_n \subseteq (\xi_p + \epsilon, \xi_p - \epsilon) \subseteq B$.

Converse of Heine Borel Th^m :–
Every set satisfying Heine Borel Property is a compact set.

Proof let us suppose that set $A \subset R$ Heine Borel property.
The family $\{(-n, n) : n \in N\}$ is a family of open sets and is a open cover of $A \subseteq R$.
$A \subseteq \bigcup_{i=1}^{k} (-n_i, n_i)$.

$A \subseteq (-n_k, n_k)$.
A is Bounded set.
A is closed set.
let y be any element of A^c, and x be any element of A If $\epsilon = \left|\frac{x-y}{2}\right|$.
$(x-\epsilon, x+\epsilon)$ $(y-\epsilon, y+\epsilon)$.
$(x-\epsilon, x+\epsilon) \cap (y-\epsilon, y+\epsilon) = \phi$.
A satisfies Heine Borel property.
$A \subseteq \bigcup_{i=1}^{n} (x_i - \epsilon, x_i + \epsilon)$
$(x_i - \epsilon, x_i + \epsilon) \bigcap_{i=1}^{n} (y_i - \epsilon, y_i + \epsilon) = \phi$
$\bigcap_{i=1}^{n} (y_i - \epsilon, y_i + \epsilon) = V$
$V \cap A = \phi$
$V \subseteq A^c$ or $y \in V \subseteq A^c$
A^c is nhd of

Ques. Show that a finite set is a Compact set.

An. Let $A = \{a_1 — a_n\}$ be a finite set.
Suppose that the family $(O_\lambda : \lambda \in N)$ is an open cover of A.

$$A \subseteq \bigcup_{\lambda \in n} A_\lambda$$

Let $a_1 \in A_{k_1}$, $a_2 \in A_{k_2}$ —— $a_n \in A_{k_n}$.

$$A \subseteq \bigcup_{i=1}^{n} A_{k_i}$$

A is a Compact set.

Q. Show that union of a finite family of Compact set is Compact.

Ans. Let $\{A_\lambda : 1 \le \lambda \le n\}$ be a family of Compact set. Each A_λ is closed and Bounded. finite union of closed set is closed

$\bigcup_{i=1}^{n} A_\lambda$ is closed set.

Each A_λ is Bounded

$A_\lambda \subseteq (a_\lambda, b_\lambda)$

$l = \min(a_1 — a_n)$ $u = \max(b_1 — b_n)$

$A_\lambda \subseteq (l, u)$

$\bigcup_{\lambda=1}^{n} A_\lambda \subseteq (l, u)$

$\bigcup_{\lambda=1}^{n} A_\lambda$ is a Bounded set

$\bigcup_{\lambda=1}^{n} A_\lambda$ is a Compact set.

Chapter
Sequence

Defn A sequence is a function whose domain is the set of natural number and range can be any set.

Range of Sequence: The set of all distinct term of a sequence $\langle a_n \rangle$.

Constant Sequence: A Sequence $\langle a_n \rangle$ defined by $a_n = l$ $\forall n \in N$

Bounded Sequence: A Sequence $\langle a_n \rangle$ is said to be Bounded if $\exists$ real number k and K st $k \le a_n \le K$ $\forall n \in N$.

Unbounded Seqn: A Sequence a_n is said to be unbounded if either it is unbounded above or unbounded below or both

Least upper Bound:
A real number
(1) $a_n \le u$ (2) for each $\epsilon > 0$ $\exists$ some positive num

Greatest lower Bound,
A real number l
(1) $a_n \ge l$
(2) for each $\epsilon > 0$ $\exists$ some positive integer m.

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Convergent Sequence

A Sequence $\langle a_n \rangle$ is said to be Converge to a real number. l iff for any given $\epsilon > 0$. Small it may be $\exists$ a positive integer m. $|a_n - l| < \epsilon$.

The real number l is called limit of the sequence.

Divergent →

If Sequence is not Convergent then. Series is divergent.

Th^m. Prove that Every Convergent Seq^n is Bounded But not Conversely

Proof Let $\langle a_n \rangle$ be a Seq^n which is Convergent to l. $\langle a_n \rangle$ is a Bounded Sequence. Let $\epsilon > 0$ be any given number $\langle a_n \rangle$ is Convergent to l, $\exists$ a positive integer m. $|a_n - l| < \epsilon$

$l - \epsilon < a_n < l + \epsilon$

$t = \min(a_1 \text{———} a_{m-1}, l-\epsilon)$

$T = \max(a_1 \text{———} a_{m-1}, l+\epsilon)$

Seq^n a_n is Bounded

But the Converse is not True

The Seq^n $\langle a_n \rangle$ $a_n = (-1)^{n-1}$.

The Seq^n is $(1, -1, 1, -1, \text{—})$

and $(-1 \leq a_n \leq 1\ \forall n)$

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$= 1, -1, 1, -1, \ldots$

Th^m A Seq^n $\langle a_n \rangle$ is Convergent if given $\epsilon > 0$. $\exists$ a positive integer m $|a_n - a_m| < \epsilon$.

Proof the Sequence $\langle a_n \rangle$ Converging to l.

$|a_n - l| < \frac{\epsilon}{2}$.

$|a_m - l| < \frac{\epsilon}{2}$.

$|a_n - a_m| = |a_n - l + a_m - l|$

$\leq |a_n - l| + |l - a_m|$.

$\leq |a_n - l| + |a_m - l|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Divergent Seq^n →

A Seq^n $\langle a_n \rangle$ is said to be diverge to ∞ if for each positive number K.

$a_n > K$

$\lim_{n \to \infty} a_n = \infty$

Oscillatory seq^n →

(a) A Sequence $\langle a_n \rangle$ is said to be oscillatory

(b) A Seq^n $\langle a_n \rangle$ is said to be oscillatory finitely if it is Bounded and oscillatory

(c) A Seq^n $\langle a_n \rangle$ is said to be oscillatory infinitely if it is unbounded and oscillatory

Null Seq^n :-

A Seq^n $\langle a_n \rangle$ is said to be a null seq^n if a_n converges to zero.

Th^m :- If b_n is a null sequence and $|a_n| \le |b_n|$ $\forall n$ then $\langle a_n \rangle$ converges to zero

Proof $\langle b_n \rangle$ is null seq^n

$|b_n - 0| < \epsilon$

$|b_n| < \epsilon$

$|a_n| \le b_n$

$|a_n| < \epsilon$

$|a_n - 0| < \epsilon$

a_n converges to 0

Ques List first five terms of the following Seq^n

(a) $a_n = 1 + (-1)^n$

Sol^n $n = 1, 2, 3, 4$ ——

$a_1 = 1 + (-1) = 0$ $a_2 = 2$

$a_3 = 1 + (-1)^3 = 1 - 1 = 0$ $a_4 = 2$

$a_5 = 0$

$a_n = \langle 0\ 2\ 0\ 2\ 0 \rangle$

Ques Discuss the Boundedness of following

Qu $a_n = 7$ $\forall$

Sol $a_n = 7$ $\forall n \in N$

It is a constant seq^n with range $\{7\}$ range is a singleton finite set and hence is a Bounded set $\langle a_n \rangle$ is Bounded

(b) $a_n = \dfrac{2n+3}{3n+4} = \dfrac{3n+4-(n+1)}{(3n+4)} = 1 - \dfrac{n+1}{3n+4}$.

$0 < a_n < 1$ $\forall n$

$\langle a_n \rangle$ is a Bounded Sequence

Ques Show that the seq^n $\left\langle \frac{1}{n^2} \right\rangle$ converges to 0.

A— $a_n = \frac{1}{n^2}$

$|a_n - 0| = \left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2}$

$|a_n - 0| < \epsilon$ if $\frac{1}{n^2} < \epsilon$

$|a_n - 0| < \epsilon$ if $n^2 > \frac{1}{\epsilon}$

$|a_n - 0| < \epsilon$ if $n > \frac{1}{\sqrt{\epsilon}}$

$|a_n - 0| < \epsilon$

$|a_n - 0| < \epsilon$

$\lim_{n\to\infty} a_n = 0$

$\langle a_n \rangle = \frac{1}{n^2}$ converges to 0

Ques Show that the Seq^n $(-1)^n$ does not converge.

Sol^n Let us assume that the seq^n $(-1)^n$ Converges to l

Let $0 < \epsilon < 1$. Then $\exists$ a positive integer n

$|(-1)^n - l| < \frac{\epsilon}{2}$

$2 = |(-1)^{n+1} - (-1)^n| = |(-1)^{n+1} - l + l - (-1)^n|$

$\leq |(-1)^{n+1} - l| + |(-1)^n - l|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \not< \epsilon$.

$(-1)^n$ doesnot Converge

Some Basic Th^m on limits.

Th^m If a seq^n $\langle a_n \rangle$ Converge to $a \neq 0$ then $\exists$ a positive number k and a positive integer m $|a_n| > k$ $\forall n \geq m$

Pf $\langle a_n \rangle$ is Convergent to a

for any given $\epsilon > 0$,

$|a_n - a| < \frac{|a|}{2}$.

$|a| = |a - a_n + a_n|$

$\leq |a - a_n| + |a_n|$

$= |a_n - a| + |a_n|$

$< \frac{|a|}{2} + |a_n|$

$|a| - \frac{|a|}{2} < |a_n|$

$|a_n| > \frac{|a|}{2}$

$k = \frac{|a|}{2}$ $|a_n| > k$ $\forall n \geq m$.

Ques Give Ex to show that Sequence $\langle a_n + b_n \rangle$ $\langle a_n - b_n \rangle$ $\langle a_n \cdot b_n \rangle$ and $\langle a_n / b_n \rangle$ are Convergent But $\langle a_n \rangle$ and $\langle b_n \rangle$ Both are not Convergent

Ans (A) Let $a_n = (-1)^n$ and $b_n = (-1)^n$.

neither $\langle a_n \rangle$ nor $\langle b_n \rangle$ $a_n + b_n = 0$

$\langle a_n + b_n \rangle$ is Convergent to 0 But $\langle a_n \rangle$ and $\langle b_n \rangle$ Both are not Convergent.

(ii) Let $a_n = (-1)^n$ and $b_n = (-1)^n$.

$a_n - b_n = 0$

$\langle a_n - b_n \rangle$ is Convergent and Convergent to 0.

(B) $a_n = (-1)^n$ and $b_n = (-1)^n$ $\forall n$.

$a_n b_n = (-1)^{2n} = 1$.

$\langle a_n b_n \rangle$ is Convergent and Converge to 1

But neither of the Seq^n $\langle a_n \rangle$ and $\langle b_n \rangle$ is Convergent

(C) $a_n = (-1)^n$ and $b_n = (-1)^n$

$\frac{a_n}{b_n} = 1$ $\forall n$

$\langle a_n / b_n \rangle$ is Cgt and Converge to 1

$\langle a_n \rangle$ and $\langle b_n \rangle$ Both are Cgt

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Thm If a seqⁿ $\langle a_n \rangle$ of non-negative term converge to a, then prove that a is always non negative.

Proof If possible suppose $a<0$. $\langle a_n \rangle$ is convergent to a $\therefore$ for given $\epsilon>0$ $\exists$ m positive integer so

$|a_n - a| < \epsilon$

$a-\epsilon < a_n < a+\epsilon$

$\epsilon = \frac{-a}{2} > 0$

$a + \frac{a}{2} < a_n < a - \frac{a}{2}$

$\frac{3a}{2} < a_n < a/2$

$a<0$ given then $a_n < 0$.

Thm Let a_n be seqⁿ s.t. $a_n \neq 0$ $\forall n \in N$ and $\frac{a_{n+1}}{a_n} \to l$. If $|l|<1$, $\lim_{n\to\infty} a_n = 0$

Proof $|l|<1$ $\epsilon_0 > 0$ $|l| + \epsilon_0 < 1$

$h = |l| + \epsilon_0$

$0<h<1$

$\frac{a_{n+1}}{a_n} \to l$ $\epsilon_0 > 0$

$\left| \frac{a_{n+1}}{a_n} - l \right| < \epsilon_0$

$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{a_{n+1}}{a_n} - l + l \right| \leq \left| \frac{a_{n+1}}{a_n} - l \right|$

$+ |l| < \epsilon_0 + |l| = h$

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$\left| \frac{a_{m+1}}{a_m} \right| < h$

$n = m, m+1 \text{ — } n-1$

$\left| \frac{a_{m+2}}{a_{m+1}} \right| < h$.

$\vdots$

$\left| \frac{a_n}{a_{n-1}} \right| < h$

multiply all $(n-m)$ equalities

$\left| \frac{a_n}{a_m} \right| < h^{n-m}$

$|a_n| < \frac{h^n}{h^m} |a_m|$

$\frac{|a_m|}{h^m} = K$

$|a_n| < K |h^n|$. $h^n \to 0$

$|h^n - 0| < \frac{\epsilon}{K}$

$|h^n| < \frac{\epsilon}{K}$

$|a_n| < K \cdot \frac{\epsilon}{K}$

$|a_n - 0| < \epsilon$

$|a_n|$ converge to 0 $\lim_{n\to\infty} a_n = 0$.

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Th^m Squeeze principle :- If a_n, b_n and c_n are three sequence such that

(1) $a_n \le b_n \le c_n \ \forall n$ and (2) seq^n $\langle a_n \rangle$ and $\langle c_n \rangle$ Both Converge to a then b_n also Converges to a

Proof Let $\epsilon > 0$ be any given number.

Since $a_n \to a$ and $c_n \to a$ $\therefore \exists$ a positive integer m_1 and m_2

$|a_n - a| < \epsilon \ \forall n \ge m_1$

$|c_n - a| < \epsilon \ \forall n \ge m_2$.

Let $m = \max(m_1, m_2)$.

$|a_n - a| < \epsilon$ and $|c_n - a| < \epsilon$.

$a - \epsilon < a_n < a + \epsilon \quad a - \epsilon < c_n < a + \epsilon$

$a_n \le b_n \le c_n \ \forall n$

$a - \epsilon \le a_n \le b_n \le c_n$.

$a - \epsilon < b_n < a + \epsilon$

$|b_n - a| < \epsilon$

b_n Converges to a

Cauchy's first Th^m on limits.

If a seq^n $\langle a_n \rangle$ Converges to a and $b_n = a_1 + a_2 + \dots + a_n / n$, then seq^n b_n also Converges to a

Proof. Let $c_n = a_n - a$.

$\lim_{n \to \infty} c_n = \lim_{n \to \infty} (a_n - a) = a - a = 0$

$b_n = \frac{e_1 + a_2 + \dots + e_n}{n} = \frac{(c_1 + a) + \dots}{n}$

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$= \frac{na + \dots + c_n}{n} = a + \frac{c_1 + \dots + c_n}{n}$.

Let $\epsilon > 0$ be any given number

$c_n \to 0 \quad |c_n - 0| < \epsilon$

$\langle c_n \rangle$ is Bounded $\quad \forall n \ge m$.

$\left| \frac{c_1 + c_2 + \dots + c_n}{n} - 0 \right| = \left| \frac{c_1 + c_2 + \dots + c_n}{n} \right|$

$\le \frac{1}{n} |c_1| + \dots + |c_n|$.

$= \frac{1}{n} (|c_1| + \dots + |c_m|) + (|c_{m+1}| + \dots + |c_n|)$)

$< \frac{1}{n} (k + \dots m) + \frac{\epsilon}{2} \dots (n-m)$).

$= \frac{km}{n} + \frac{\epsilon(n-m)}{2n}$.

$\frac{km}{n} < \frac{\epsilon}{2}$ if $\frac{n}{km} > \frac{2}{\epsilon} \quad n > \frac{2km}{\epsilon}$.

$\frac{\epsilon(n-m)}{2n} < \frac{\epsilon}{2}$ if $\frac{n-m}{n} < 1 \quad n - m < n$ if $m > 0$

$M = \max \{ m, \frac{2km}{\epsilon} \}$.

$\left| \frac{c_1 + c_2 + \dots + c_n}{n} - 0 \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$\left| \frac{c_1 + \dots + c_n}{n} - 0 \right| < \epsilon$

$\left\langle \frac{c_1 + \dots + c_n}{n} \right\rangle$ Converges to 0

$\left\langle a_1 + \frac{\dots + e_n}{n} \right\rangle$ Converges to a

Cauchy's Second Th^m

If $\langle a_n \rangle$ is a seq^n of positive terms and $\lim_{n\to\infty} (a_n)^{1/n}$ and $\lim_{n\to\infty} a_{n+1}/a_n$ Both exist finitely or infinitely, then $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

Proof $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

I when l is finite

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$$

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \frac{\epsilon}{2}$$

$$-\frac{\epsilon}{2} < \frac{a_{n+1}}{a_n} - l < \frac{\epsilon}{2}.$$

$$l - \frac{\epsilon}{2} < \frac{a_{n+1}}{a_n} < l + \frac{\epsilon}{2}.$$

$n = m, m+1$ —— $(n-1)$ in (1)

$$l - \frac{\epsilon}{2} < \frac{a_{m+1}}{a_m} < l + \frac{\epsilon}{2}$$

$$l - \frac{\epsilon}{2} < \frac{a_{m+2}}{a_{m+1}} < l + \frac{\epsilon}{2}$$

$$l - \frac{\epsilon}{2} < \frac{a_n}{a_{n-1}} < l + \frac{\epsilon}{2}$$

multiplying above $(n-m)$ inequalities

$$\left(l - \frac{\epsilon}{2}\right)^{n-m} < \frac{a_n}{a_m} < \left(l + \frac{\epsilon}{2}\right)^{n-m}.$$

$$a_m \left(l - \frac{\epsilon}{2}\right)^{n-m} < a_n < \left(l + \frac{\epsilon}{2}\right)^{n-m} \cdot a_m.$$

$$\lim_{n\to\infty} (a_m)^{1/n} \left(l - \frac{\epsilon}{2}\right)^{1-\frac{m}{n}} = 1 \cdot \left(l - \frac{\epsilon}{2}\right).$$

$$\lim_{n\to\infty} (a_m)^{1/n} \left(l + \frac{\epsilon}{2}\right)^{1-\frac{m}{n}} = 1 \cdot \left(l + \frac{\epsilon}{2}\right).$$

for given $\epsilon > 0$

$$\left| a_m^{1/n} \left(l - \frac{\epsilon}{2}\right)^{1-\frac{m}{n}} - \left(l - \frac{\epsilon}{2}\right) \right| < \frac{\epsilon}{2}.$$

$$\left| (a_m)^{1/n} \left(l + \frac{\epsilon}{2}\right)^{1-\frac{m}{n}} - \left(l + \frac{\epsilon}{2}\right) \right| < \frac{\epsilon}{2}.$$

$m = \max(m_1, m_2)$

$$\left(l - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} < (a_m)^{1/n} \left(l - \frac{\epsilon}{2}\right)^{1-m/n} < \left(l - \frac{\epsilon}{2}\right) + \frac{\epsilon}{2}$$

$$\left(l + \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} < (a_m)^{1/n} \left(l + \frac{\epsilon}{2}\right)^{1-m/n} < \left(l + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2}$$

$$l - \epsilon < (a_n)^{1/n} < l + \epsilon$$

$$-\epsilon < (a_n)^{1/n} - l < \epsilon$$

$$|(a_n)^{1/n} - l| < \epsilon$$

$$-\epsilon < (a_n)^{1/n} - l < \epsilon.$$

$$|(a_n)^{1/n} - l| < \epsilon$$

$$\lim_{n\to\infty} (a_n)^{1/n} = l$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$$

Case II lim infinite

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty \quad \lim_{n\to\infty} \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} = 0$$

Proceeding in case I

$$\lim_{n\to\infty} \left(\frac{1}{a_n}\right)^{1/n} = 0$$

$$\frac{1}{(a_n)^{1/n}} = 0 \qquad \lim_{n\to\infty} (a_n)^{1/n} = \infty$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Ques Show that $\lim_{n\to\infty} \left[\frac{1}{n^2} - \frac{1}{(2n)^2}\right] = 0$

A. $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{1}{(2n)^2}$

$= \frac{1}{(n+0)^2} + \text{———} \frac{1}{(n+n)^2}$

$< \frac{1}{n^2} + \text{———} \frac{1}{n^2}$

$= \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2}$

$a_n < \frac{1}{n} + \frac{1}{n^2}$

$a_n = \frac{1}{(n+0)^2} + \text{———} \frac{1}{(n+n)^2}$

$> \frac{1}{(2n)^2} \text{———} \frac{1}{(2n)^2}$

$= \frac{n+1}{(2n)^2} = \frac{n}{4n^2} + \frac{1}{4n^2} = \frac{1}{4n} + \frac{1}{4n^2}$

$a_n > \frac{1}{4n} + \frac{1}{4n^2}$ — (2)

from $\frac{1}{4n} + \frac{1}{4n^2} < a_n < \frac{1}{n} + \frac{1}{n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

By Sequence principle $\lim_{n\to\infty} a_n = 0.$

$\lim_{n\to\infty} \left[\frac{1}{n^2} - \frac{1}{(2n)^2}\right] = 0.$

Ques. Show that the sequence $n^{1/n}$ converges to the limit 1

A. let $a_n = n$

$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n}.$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

a_n is seqn of positive term

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

By Cauchy's Second Th^m.

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$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$,

$\lim_{n\to\infty} (a_n)^{1/n} = 1$.

$(n)^{1/n}$ Converges to $\lim_{n\to\infty} 1$.

Ques Show that $\lim_{n\to\infty} \left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3} \cdots \frac{n}{n+1}\right)^{1/n} = 1$

A — $a_n = \frac{2}{1} \cdots \frac{n}{n+1}$

$a_{n+1} = \frac{2}{1} \cdots \frac{n}{n+1} \frac{n+1}{n}$

$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = \left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

By Cauchy Second Th^m

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (a_n)^{1/n} = 1$

$\lim_{n\to\infty} \left(\frac{2}{1}\cdot\frac{3}{2} \cdots \frac{n}{n+1}\right)^{1/n} = 1$

Ques If a_n Converges to l_1 and b_n Converges to the seq^n c_n/n Converges to $l_1 l_2$

$c_n = a_1 b_n \cdots a_n b_1$

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Sol^n $a_n = l_1 + \alpha_n$.

$\lim_{n\to\infty} a_n = l_1 + \lim_{n\to\infty} \alpha_n$.

$l_1 = l_1 + \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n = 0$

$\lim_{n\to\infty} |\alpha_n| = 0$.

By Cauchy first Th^m

$\lim_{n\to\infty} \frac{|\alpha_1| \cdots |\alpha_n|}{n} = 0$.

Every Cgt seq^n

By Cauchy's first Th^m.

$\lim_{n\to\infty} \frac{b_1 \cdots b_n}{n} = l_2$

$\frac{c_n}{n} = \frac{a_1 b_n \cdots a_n b_1}{n}$

$= \frac{(l_1 + \alpha_1) b_n \cdots (l_1 + \alpha_n) b_1}{n}$

$l_1 (b_n \cdots b_1) + (\alpha_1 b_n \cdots \alpha_n b_1) / n$.

$\lim_{n\to\infty} \frac{c_n}{n} = \lim_{n\to\infty} l_1 \left(\frac{b_1 \cdots b_n}{n}\right) + \lim_{n\to\infty} \frac{a_1 b_n \cdots a_n b_1}{n}$

$\lim_{n\to\infty} \frac{c_n}{n} = l_1 l_2 + \lim_{n\to\infty} \frac{a_1 b_n \cdots a_n b_1}{n}$.

$\left|\frac{\alpha_1 b_n \cdots \alpha_n b_1}{n}\right| \leq \frac{|\alpha_1| |b_n| \cdots |\alpha_n| |b_1|}{n}$

$\leq K \, |\alpha_1| \cdots |\alpha_n| / n$.

$[\alpha_1 b_n \cdots \alpha_n b_1] / n \to 0$

$[\alpha_1 b_n \cdots \alpha_n b_1] / n \to 0$.

$\lim_{n\to\infty} \frac{c_n}{n} = l_1 l_2$

$\frac{c_n}{n}$ Cgverge to $l_1 l_2$

Monotonic Seq^n.

(1) A Seq^n a_n is said to be monotonically increasing if $a_{n+1} \geq a_n$

(2) A Seq^n a_n is said to be strictly monotonically decreasing if $a_{n+1} < a_n$.

(3) A Seq^n $\langle a_n \rangle$ is said to be monotonic if it is any one of the type (1) to (4).

Monotone Convergence Th^m.

(1) Prove that a monotonically increasing seq^n $\langle a_n \rangle$ which is Bounded above Converges to its least upper Bound (lub)

An Let $S = \{a_1, a_2 \text{——} a_n \text{——}\}$

a_n is Bonded above. S is a non Empty and Bounded above subset

the set S has least upperBound

If μ = lub of set S to show $\langle a_n \rangle$ Converges to μ

Let $\epsilon > 0$ be any given number

μ is l.u.b of set S. $\mu - \epsilon$ is not an upper Bound of set S

$\mu - \epsilon < a_m$

a_n is monotonically increasing seq

$a_m \leq a_n$

$\mu - \epsilon < a_n$

μ is upper Bond of set S.

$\mu + \epsilon$ is also an upper Bond of set S

$a_n < \mu + \epsilon$

μ is upper Bond of set S

$\mu + \epsilon$ is also an upper Bond of set S

$a_n < \mu + \epsilon$

$\mu - \epsilon < a_n < \mu + \epsilon$

$|a_n - \mu| < \epsilon$

a_n Converges to μ

(ii) Let $S = \{a_1 \text{——} a_n\}$

a_n is Bounded Below. S is a Non Empty and Bounded Below of R

By Completeness property of reals the set S has greatest lower Bond

If l is glb of set S

Let $\epsilon > 0$ be any given number

l is glb of set S.

$a_m < l + \epsilon$

a_m is monotonically decreasing
$a_n \le a_m$
l is lower Bound of set S
$l-\epsilon < a_m < l+\epsilon$
$|a_m - l| < \epsilon$

Nested Seq^n :—
A Seq^n of interval I_n is said to be nested if Either
$I_1 \subseteq I_2 \subseteq I_3 \subseteq$ —
$I_1 \supseteq I_2 \supseteq I_3 \supseteq$ —

Th^m Nested Interval property
(1) $I_n = [a_n, b_n]$ is a Seq^n of closed intervals st
(2) $I_{n+1} \subseteq I_n$
(3) $\lim_{n\to\infty} I_n = 0$

$\bigcap_{n=1}^{\infty} I_n$ is a Singleton

Proof $I_{n+1} \subseteq I_n$
$(a_{n+1}, b_{n+1}) \subseteq (a_n, b_n)$

$a_1 \le a_2$ —— $\le a_n \le$ — $\le b_1$
$b_1 \ge b_2$ —— $\ge b_n \ge$ — $\ge a_1$
a_n is monotonically increasing and Bounded

$\langle a_n \rangle$ and $\langle b_n \rangle$ are cgt seq^n.
$\lim_{n\to\infty} a_n = a$ $\lim_{n\to\infty} b_n = b$.
$\lim_{n\to\infty} (b_n - a_n) = 0$
$b - a = 0$ $b = a$
$a_n \le a \le b_n$
$a \in I_n$ $\forall n \in N$.
uniqueness → Suppose that a and a' are two distinct real numbers.
$a < a'$ and $a, a' \in I_n$
$a_n \le a \le a' \le b_n$
$|b_n - a_n| \ge |a' - a|$.
$\lim_{n\to\infty} (b_n - a_n) = 0$

$\epsilon = |a' - a|$ $|b_n - a_n| < |a' - a|$.
a is only number $a \in I_n$
$\bigcap_{n=1}^{\infty} I_n = a$ which is singleton.

Ques Discuss the cgt of the seq^n $\langle a_n \rangle$
(1) $a_n = 2n-7 / 3n+2$
A $a_n = 2n-7 / 3n+2$ $a_{n+1} = 2n-5/3n+5$
$a_{n+1} - a_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$
$= \frac{(2n-5)(3n+2) - (2n-7)(3n+5)}{(3n+5)(3n+2)}$

$$\frac{(6n^2+4n-15n-10)-(6n^2+10n-2n-x)}{(3n+5)(3n+2)}$$

$$\frac{25}{(3n+5)(3n+2)} > 0$$

$a_{n+1} > a_n$

a_n is monotonically increasing

$\langle a_n \rangle = \{-1, -\frac{3}{8}, -\frac{1}{11}, \frac{1}{14}, \frac{3}{17}\}$

$a_n \geq -1$

$$a_n - 1 = \frac{2n-7}{3n+2} - 1 = \frac{2n-7-3n-2}{3n+2}$$

$$\frac{-n-9}{3n+2} = -\frac{n+9}{3n+2} < 0$$

$a_n < 1$

$-1 \leq a_n < 1$

a_n is Bounded, a_n is monotonic increasing and Bounded above

$\langle a_n \rangle$ is Cgt

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-7}{3n+2} = \lim_{n\to\infty} \frac{2-\frac{7}{n}}{3+\frac{2}{n}} = \frac{2}{3}$$

Ques Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ Exist and lies b/w 2 and 3

Let $a_n = \left(1+\frac{1}{n}\right)^n$

Expanding by Binomial Th^m

$$a_n = 1 + n\cdot\frac{1}{n} + \text{———} \frac{n(n-1)\text{———}(n-(n-1))}{n!}\frac{1}{n^n}$$

$$a_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) \text{———} \text{———}\left(1-\frac{n-1}{n}\right) \text{②}$$

$$a_n < 1 + 1 \text{———} \frac{1}{n!}$$

$$< 1 + 1 \text{———} \frac{1}{2^{n-1}}$$

$$\leq 1 + \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right) < 3$$

$a_n < 3$

$$a_n = 1 + 1 + \frac{1}{2!} \text{———} \left(1 - \frac{n-1}{n}\right)$$

$a_n > 2$

$2 < a_n < 3$

$$a_n = 1 + 1 + \text{———} \left(1 - \frac{n-1}{n}\right) \text{④}$$

$$a_{n+1} = 1 + 1 \text{———} \left(1 - \frac{n-1}{n+1}\right) \text{⑤}$$

$n < n+1$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$\frac{k}{n} > \frac{k}{n+1}$$

$$-\frac{k}{n} < -\frac{k}{n+1}$$

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$1-\frac{k}{n} < 1-\frac{k}{n+1}$.

$a_n < a_{n+1}$

a_n is cgt $\quad 2 \le \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le 3$

Que Prove that seq^n $\langle a_n \rangle$ defined by $a_1 = 1$, $a_n = \sqrt{2+a_{n-1}}$ Cgt the positive root of eq^n $x^2-x-2=0$.

A $\quad a_1 = 1$

$a_n = \sqrt{2+a_{n-1}}$

$a_2 = \sqrt{2+a_1} = \sqrt{2+1} = \sqrt{3} > a_1$

$a_{n+1} \ge a_n$

$a_m > a_{m-1}$

$2+a_m > 2+a_{m-1}$

$\sqrt{2+a_m} > \sqrt{2+a_{m-1}}$

$a_{m+1} \ge a_m$

$a_{n+1} \ge a_n$

$a_1 = 1 < 2$

$a_m < 2$

$2+a_m < 4, \ \sqrt{2+a_m} < 2$

$a_{m+1} < 2, \ a_m < 2$

a_n is Bounded above seq^n

a_n is Cgt

$\lim_{n\to\infty} a_n = \alpha$

$a_n \ge 0$

$a_{n+1} = \sqrt{2+a_n}$

$a_{n+1}^2 = 2+a_n$

$\lim_{n\to\infty} a_{n+1}^2 = 2+\lim_{n\to\infty} a_n$

$\alpha^2 = 2+\alpha$.

$\alpha^2-\alpha-2=0$

α is root of Eq^n $x^2-x-2=0$.

a_n is cgs to positive root of Eq^n $x^2-x-2=0$.

Limit point :- A real number l is said to be limit point or a cluster point of a seq^n if every neighbourhood of l contains an infinite number of terms of given seq^n.

l is a limit point of seq^n $\langle a_n \rangle$ if for given $\epsilon>0$ $a_n \in (l-\epsilon, l+\epsilon)$ for infinitely many value of n.

Bolzano weierstrass Th^m :-

Every Bounded seq^n has a limit point.

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An. Let a_n be Bounded Seq^n.
The range set S of the seq^n is Bounded
$S = \{a_n : n \in N\}$

I. When set S is finite.
$\exists$ atleast one member of S say l
$a_n = l$ for only many value of n
for each $\epsilon > 0$ interval $(l-\epsilon, l+\epsilon)$
l is a limit point of seq^n $\langle a_n \rangle$

II. When set S is infinite
the set S is an infinite Bounded set
By Bolzano weierstrass Th^m. for set
the set S has a limit l
The nbd of l contain only many
Element of set S.
nbd of l contains only many
term of seq^n $\langle a_n \rangle$
l is a limit point of seq^n $\langle a_n \rangle$

Cauchy's Seq^n.
A sequence $\langle a_n \rangle$ is said to be
a Cauchy's seq^n if for given $\epsilon > 0$
$\exists$ a positive integer m $|a_m - a_n| < \epsilon$

Ques Prove that Every Cauchy's seq^n
is Bounded. Is the converse true If
not show Ex

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Sol^n Let $\epsilon > 0$ be a given number and
a_n be any seq^n which is
Cauchy's seq^n.
$\exists$ a positive integer m $|a_n - a_m| < \epsilon$
$-\epsilon < a_n - a_m < \epsilon$
$a_m - \epsilon < a_n < a_m + \epsilon$
$k_1 = \min(a_1, \ldots, a_m - \epsilon)$
$k_2 = \max(a_1, \ldots, a_m + \epsilon)$
$k_1 \leq a_n \leq k_2$
a_n is Bounded

Converse of Th^m is not true
the seq^n $\langle a_n \rangle$
$\langle a_n \rangle = \langle -1, 1, -1, \ldots \rangle$
$-1 \leq a_n \leq 1 \ \forall n$,
a_n is Bounded seq^n But not a
Cauchy's seq^n as proved.

Cauchy's general principle of Convergence
Th^m A seq^n converge if and only if it
is a Cauchy's sequence

Proof I a_n is converges to l
a_n is Cauchy's seq^n
a_n converges to l.
$\epsilon > 0$ $\exists$ a positive integer m
$|a_n - l| < \epsilon$
$|a_m - l| < \frac{\epsilon}{2}$

$|a_n - a_m| = |a_n - l + l - a_m| \le |a_n - l| + |l - a_m|$
$= |a_n - l| + |a_m - l|$
$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$|a_n - a_m| < \epsilon$.

II Converse
a_n is Cauchy's Seq^n.
a_n is Convergent
a_n is a Cauchy's Seq^n
$\exists$ a positive integer m.
$|a_n - a_m| < \frac{\epsilon}{3}$.

Every Cauchy's Seq^n is a Bounded seq^n
$|a_n - l| < \frac{\epsilon}{3}$.

$|a_k - a_m| < \frac{\epsilon}{3}$

$|a_n - l| = |a_n - a_m + a_m - a_k + a_k - l|$
$\le |a_n - a_m| + |a_m - a_k| + |a_k - l|$
$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$

$|a_n - l| < \epsilon$ $\quad n \ge m$

a_n Converges to l.

Ques Show that the Seq^n $\langle a_n \rangle$ defined by $a_n = 1 + \frac{1}{3} \text{———} \frac{1}{2n+1}$ doesn't Converge

Proof Let $\epsilon > 0$

$a_m = 1 + \frac{1}{3} \text{———} \frac{1}{2m+1}$

$a_{2m} = 1 + \frac{1}{3} \text{———} \frac{1}{2m+2m+1}$

$|a_{2m} - a_m| = \left|\frac{1}{2m+1} \text{———} \frac{1}{2m+2m+1}\right|$

$= \frac{1}{2m+1} \text{———} \frac{1}{2m+2m+1}$.

$> \frac{1}{2m+2m} + \text{———} \frac{1}{2m+2m} = \frac{m}{4m} = \frac{1}{4}$.

$|a_{2m} - a_m| > \frac{1}{4}$.

$\epsilon = \frac{1}{4}$ $\quad |a_{2m} - a_m| \ge \epsilon$

Subsequence $\to$ Let a_n be given Seq^n If n_k is a Strictly increasing Seq^n of natural number such that $n_1 < n_2$ ——— then $\langle a_{n_k} \rangle$ is called Subseq^n.

Th^m If a Seq^n $\langle a_n \rangle$ Converge to l, then Every Subseq^n of $\langle a_n \rangle$ also Converge to l. Is the Converse true If not Show by Ex.

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Proof a_n Converge to l
for given $\epsilon > 0$
$|a_n - l| < \epsilon$
let a_{m_k} be any Subseq^n of sq^n a_n
$|a_{m_k} - l| < \epsilon$
a_{m_k} Converges to l

Converse is not true
$\langle a_n \rangle = (-1)^n = \langle -1, 1, -1, 1, -1 \ldots \rangle$
the seq^n $\langle a_n \rangle$ is not Cgt But its Subseq^n $\langle 1, 1, 1, \ldots, 1 \rangle$ is Cgt

Subsequential limit →
let $\langle a_n \rangle$ be a Sequence, A real number l is Subsequential limit of the Sequence $\langle a_n \rangle$ if ∃ a Subseq^n of $\langle a_n \rangle$ Converging to l

Th^m A real number l is a limit point of seq^n $\langle a_n \rangle$ if and only if ∃ a Subseq^n of a_n Converging to l

Proof let l be a limit point of sq^n $\langle a_n \rangle$
$|a_k - l| < \epsilon$
$|a_{m_1} - l| < 1$
$\epsilon = \frac{1}{2}$ $m > n_1$
$|a_{m_2} - l| < \frac{1}{2}$

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$\epsilon = \frac{1}{3}$ $m = n_2$
$|a_{m_3} - l| < \frac{1}{3}$.
$|a_{m_k} - l| < \frac{1}{k}$
$\epsilon = \frac{1}{k+1}$
$|a_{m_{k+1}} - l| < \frac{1}{k+1}$.

a_{m_k} Converges to l.
$|a_{m_k} - l| < \frac{1}{k} \leq \frac{1}{k_0} < \epsilon$.
$\langle a_{m_k} \rangle$ Converges to l.

Condition is Sufficient.
$\langle a_{m_k} \rangle$ is Subseq^n of a_n $a_{m_k} \to l$
$a_n \to l$
let $\epsilon > 0$ be any given number
$a_{m_k} \to l$
$a_{m_k} \in (l-\epsilon, l+\epsilon)$
a_{m_k} is a Subseq^n of $\langle a_n \rangle$
any nbd $(l-\epsilon, l+\epsilon)$ Contains Infinitely many member of $\langle a_n \rangle$
l is the limit point of $\langle a_k \rangle$.

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Chapter → 3

Infinite Series.

If a_n is a sequence of real numbers then, the expression of the form
$a_1 + a_2 + a_3 \ldots + a_n + \ldots$

Convergent and divergent of an infinite series +
The infinite series $\sum_{n=1}^{\infty} a_n = a_1 \ldots + a_n + \ldots$ $S_n = a_1 + \ldots a_n$.
$S_1 = a_1$, $S_2 = a_1 + a_2 \ldots$

Partial Sum of Series → The seqn S_n

Oscillating + if the seqn of partial sums of $\sum_{n=1}^{\infty} a_n$ oscillates finitely or infinitely respectively

Ques. Show that the series $\frac{1}{3} + \frac{1}{3^2} + \ldots$ converges

Ans. $S_n = \frac{1}{3} + \frac{1}{3^2} + \ldots \frac{1}{3^n}$

$$= \frac{\frac{1}{3}\left(1 - \frac{1}{3^n}\right)}{1 - \frac{1}{3}} = \frac{\frac{1}{3}\left(1 - \frac{1}{3^n}\right)}{\frac{2}{3}} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$$

$$\lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{3^n}\right) = \frac{1}{2}$$ which is a finite quantity

S_n converges to $\frac{1}{2}$

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Ques. Show that the series $1^2 + 2^2 \ldots n^2 + \ldots$ diverges to $+\infty$.

Ans. The given series is $1^2 + 2^2 \ldots n^2 + \ldots$

$S_n = 1^2 + 2^2 \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

$\lim_{n\to\infty} S_n = \infty$.

S_n diverges to $+\infty$.
The given series diverges to $+\infty$

Ques. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ converges to $\frac{1}{4}$.

Ans. Let $a_n = \frac{1}{n(n+1)(n+2)}$

$$= \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$= \frac{1}{2n} - \frac{1}{2(n+1)} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$a_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right).$$

Putt $n = 1, 2, 3 \ldots n$

$$a_1 = \frac{1}{2}\left(1 - \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right).$$

$$a_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right).$$

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$$a_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$a_1 + a_2 \text{ ——— } t_n = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2}\right)$

If S_n denotes the partial sum of first n terms of series

$$S_n = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$\lim n \to \infty$

$$\lim_{n\to\infty} S_n = \frac{1}{2}(1-0) - \frac{1}{2}\left(\frac{1}{2} - 0\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

the seq^n S_n converges to $\frac{1}{4}$,

infinite series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

converges to $\frac{1}{4}$.

Th^m The convergence, divergence or oscillation behaviour of a series remains unchanged by addition omissions or alteration of first finite number of terms of the series.

Proof Let $\sum_{n=1}^{\infty} a_n = a_1 \text{ ——— } a_n$ be given series st S_n denote the partial sum of first n terms

$$S_n = a_1 \text{ ——— } a_n$$

a new series $\sum_{n=1}^{\infty} b_n$ obtained after deleting first m terms of $\sum_{n=1}^{\infty} a_n$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 \text{ ——— } b_n + \text{ ——— }$$

$b_1 = a_{m+1}$ $b_2 = a_{m+2}$ ——— $b_n = a_{m+n}$

S_n' denote partial sum of first n term of $\sum_{n=1}^{\infty} b_n$

$$S_n' = b_1 + b_2 \text{ ——— } b_n$$
$$= a_{m+1} \text{ ——— } a_{m+n}$$
$$= (a_1 \text{ ——— } a_{m+n}) - (a_1 \text{ ——— } a_m)$$
$$= S_{m+n} - S_m$$

Case I When $\sum_{n=1}^{\infty} a_n$ is convergent.

S_n and S_{m+n} both are cgt series

S_n and S_n' both converge to finite limit and $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converget seq^n

Case II When $\sum_{n=1}^{\infty} a_n$ is divergent.

$\lim_{n\to\infty} S_n = \pm\infty$ and $\lim_{n\to\infty} S_{m+n} = \pm\infty$.

$\lim_{n\to\infty} S_n = \pm\infty$ and $\lim_{n\to\infty} S_n' = \pm\infty$.

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both are divergent se

Case III When $\sum_{n=1}^{\infty} a_n$ is oscillatry

S_n and S_{m+n} are oscillatry seq^n a

Seq^n S_n' is also oscillatry

the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are oscillatry

Thm If $\sum_{n=1}^{\infty} a_n$ converges to a and $\sum_{n=1}^{\infty} b_n$ diverges to ∞ (or $-\infty$) then $\sum_{n=1}^{\infty} (a_n+b_n)$ diverges to $+\infty$.

Proof Let S_n, S_n' and S_n'' denote the nth partial sums of the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} (a_n+b_n)$

$$S_n'' = S_n + S_n'$$

$$\lim_{n\to\infty} S_n'' = \lim_{n\to\infty} S_n' + \lim_{n\to\infty} S_n = a+\infty = \infty$$

The series $\sum_{n=1}^{\infty} (a_n+b_n)$ diverges

Cauchy's general principle of convergence

A necessary and sufficient condition for a series $\sum_{n=1}^{\infty} a_n$ to converge is that to each $\epsilon>0$ $\exists$ a positive integer m s.t

$|a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon \;\forall\; n \geq m$

Proof Let S_n denote the partial sum of first n terms of series $\sum_{n=1}^{\infty} a_n$

$S_n = a_1 + \dots + a_n$.

$\sum a_n$ converges if and only if the seq^n $\langle S_n \rangle$ converges iff by Cauchy general principle of convergence of seq^n

$|S_n - S_m| < \epsilon$

$|a_{m+1} + \dots + a_n| < \epsilon$

Ques. Show that the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ is not cgt.

Ans The given series is $\sum_{n=1}^{\infty} \frac{n}{2n+1} = \sum_{n=1}^{\infty} a_n$

$$a_n = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}.$$

$$\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$$

$\sum_{n=1}^{\infty} a_n$ is not convergent.

Convergence or divergence of geometric

(1) The geometrical series $a + ar + \dots \infty$.

Ans $S_n = a + ar + \dots + ar^{n-1}$

$= a(1-r^n)/(1-r)$

(1) Converges if $|r|<1$

Ans If $|r|<1$ then $r^n \to 0$ as $n\to\infty$.

$\lim_{n\to\infty} S_n = \frac{a}{1-r}$ which is unique and finite.

(2) Diverges if $r \geq 1$

Ans if $r>1$ then $r^n \to \infty$ as $n\to\infty$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} = \infty.$$

The given series is divergent.

If $r=1$, then $S_n = a + \dots$ to n terms $= na$

$$\lim_{n\to\infty} S_n = \infty$$

The series is divergent

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(3) oscillate finitely if $r = -1$.

A If $r = -1$, then
$S_n = a - a + a - a + \text{---}$ n terms
$= \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Seq^n $\langle S_n \rangle$ oscillate finitely b/w 0 and a
and series $\sum_{n=1}^{\infty} a_n$ oscillate finitely.

Series of positive term:-
A series $\sum_{n=1}^{\infty} a_n$ for which $a_n \geq 0 \ \forall n$

Th^m A positive term series either converges or diverges to ∞.

Proof Let $\sum_{n=1}^{\infty} a_n$ be series of positive term st S_n denote partial sum of it's first n terms.
$S_n = a_1 \text{———} a_n$
$S_{n+1} = a_1 \text{———} a_{n+1}$.
$S_n - S_{n+1} = a_n \geq 0$,
$S_n > S_{n+1}$.
S_n is monotonically increasing Seq^n.

General test for the convergence of positive term series.

Th^m A necessary and sufficient condition for the convergence of a positive term series $\sum_{n=1}^{\infty} a_n$ is that

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the Seq^n S_n of the partial sum of the series defined by $S_n = a_1 \text{—} a_n$ is bounded

Proof $S_n = a_1 + \text{———} a_n$.
$S_{n+1} = a_1 \text{———} a_{n+1}$.
$S_n - S_{n+1} = a_n > 0$.
$S_n > S_{n+1}$.
S_n is monotonically increasing
Condition is necessary.
Seq^n S_n is bounded above
Every monotonically increasing seq^n which is bounded above is cgt!.
S_n is cgt and series $\sum a_n$ is cgt.

Condition is sufficient.
Let the series $\sum_{n=1} a_n$ be cgt series
Seq^n S_n is cgt. But every cgt. Seq^n is bounded. S_n is bounded and bounded above.

Comparison Test $\rightarrow$

Th^m If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series of positive term st $\lim_{n\to\infty} a_n/b_n = 1$ then both the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1} b_n$ converges to diverge together.

Proof the series a_n and b_n are of positive terms. $a_n/b_n \geq 0 \ \forall n$
$\lim_{n\to\infty} \frac{a_n}{b_n} \geq 0$

Choose $\epsilon > 0$ $l - \epsilon > 0$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = l$.

$\left|\frac{a_n}{b_n} - l\right| < \epsilon$

$l - \epsilon < \frac{a_n}{b_n} < l + \epsilon$.

$(l-\epsilon) b_n < a_n < (l+\epsilon) b_n$

$k b_n < a_n < K b_n$.

$\sum a_n$ and $\sum b_n$ Converge together

(1) $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n = \infty$.

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ donot Converge or diverge together.

(2) $a_n = \frac{1}{n^2}$ $b_n = \frac{1}{n}$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$.

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1} b_n$ donot Converge

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Hyper Harmonic Series or p-series test.

Th^m The Series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} \text{———} \frac{1}{n^p} + \text{———}$

(1) Converges if $p > 1$.

(2) diverges if $p \leq 1$.

Where p any number, not necessarily an integer.

Proof Case I If $p > 1$

$\frac{1}{1^p} = 1$

$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$

$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{4^{p-1}} < \frac{1}{(2^{p-1})^2}$

$\frac{1}{(2^n)^p} + \text{———} \frac{1}{(2^{n+1}-1)^p} < \frac{1}{(2^{p-1})^n}$

Adding inequalities

$S_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \text{———} \frac{1}{(2^{p-1})^n}$

RHS is a geometrical series $\frac{1}{2^{p-1}}$

$S_{2^{n+1}-1} < \frac{1\left(1 - \frac{1}{(2^{p-1})^{n+1}}\right)}{1 - \frac{1}{2^{p-1}}}$

$S_{2^{n+1}-1} < \frac{1}{1 - \frac{1}{2^{p-1}}}$.

$S_{2^{n+1}-1} < k$ $k = \frac{1}{1 - \frac{1}{2^{p-1}}}$

$S_n < S_{2^{n+1}-1}$, $S_n < k$, S_n is Bounded

Case II If $p = 1$

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} \text{———} \frac{1}{n}$.

$1 + \frac{1}{2} > \frac{1}{2}$

$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

There we $1+\frac{1}{2} > \frac{1}{2}$

$\frac{1}{3}+\frac{1}{4} > \frac{1}{2}$ ——

$\frac{1}{2^{n-1}+1}$ — $\frac{1}{2^n} > \frac{1}{2}$

adding alone inequality

$S_{2^n} > \frac{1}{2}+\frac{1}{2}$ —— n terms

$S_{2^n} > \frac{n}{2}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent to $+\infty$

Corollary $p<1$

$n^p < n$ or $\frac{1}{n^p} > \frac{1}{n}$

$\sum \frac{1}{n}$ is divergent

By Comparison Test $\sum \frac{1}{n^p}$ is divergent

Ques Show that the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^{1/n}$ is divergent

Ans let $y = (n)^{1/n}$

$\log y = \log n^{1/n} = \frac{1}{n}\log n = \frac{\log n}{n}$

$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{\log n}{n}$

$= \lim_{n\to\infty} \frac{1}{n}/1 = 0$

$\lim_{n\to\infty} y = e^0 = 1 \neq 0$

$\lim_{n\to\infty} (n)^{1/n} = 1$

let $\sum a_n$ be given series

$a_n = \left(\frac{1}{n^2}\right)^{1/n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(n^{1/n})^2} = 1$

$\sum_{n=1}^{\infty} a_n$ is a series of positive terms

$\lim_{n\to\infty} a_n \neq 0$

the series $\sum_{n=1}^{\infty} a_n$ is divergent

Ques. Test the Convergence of Series

$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \text{—} \text{—}$

Sol^n let $\sum_{n=1} a_n$ be given Series

$a_n =$ nth term $1\,3\,5\ldots/(n-1\,2\,3\ldots)(n+1\,2\,3\,4\ldots)$

$= 2n-1 / n(n+1)(n+2) = (2-\frac{1}{n}) / n^2(1+\frac{1}{n})(1+\frac{2}{n})$

$b_n = \frac{1}{n^2}$

$\frac{a_n}{b_n} = (2-\frac{1}{n}) / (1+\frac{1}{n})(1+\frac{2}{n})$

$\lim_{n\to\infty} a_n/b_n = 2$ which is non zero

By Comparison Test the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ behave alike

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is Cgt by p test

$\sum_{n=1}^{\infty} a_n$ is also Convergent.

Ques Test the Convergence of following Series

(i) $\sum_{n=1}^{\infty} \sqrt{n^4+1} - \sqrt{n^4-1}$

$= \sqrt{n^4+1} - \sqrt{n^4-1} \left(\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right)$

$\frac{n^4+1-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$

$\log(\log n) \geq 2$

$\log n(\log(\log n)) > 2 \log n$

$\log(\log n)^{\log n} > \log n^2$

$(\log n)^{\log n} > n^2$

$\frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$

$\sum_{n=2}^{\infty} 1/n^2$ is Converget

By Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$ is also Converget

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Chapter – 4.

Infinite Series (Continued)

D'Alembert's ratio test →

If $\sum_{n=1}^{\infty} a_n$ is a Series of positive term. St $\lim_{n\to\infty} a_n/a_{n+1} = l$, then.

1. for $l>1$, the Series is Cgt
2. for $l<1$, the series is diverget
3. for $l=\infty$, the Series is Cgt
4. for $l=1$, non Conclusion, the test fails

Proof :- $\lim_{n\to\infty} a_n/a_{n+1} = l$.

for Each $\epsilon>0$, a positive integer m

$$\left|\frac{a_n}{a_{n+1}} - l\right| < \epsilon$$

$$l-\epsilon < a_n/a_{n+1} < l+\epsilon$$

Case I If $l>1$

Chose $\epsilon>0$ st $l-\epsilon>1$

$a_n/a_{n+1} > l-\epsilon$

$a_n/a_{n+1} > R$ $\quad R = l-\epsilon > 1$.

$n = m, m+1 \text{ —— } n-1$

$a_m/a_{m+1} > R$

$a_{m+1}/a_{m+2} > R$

$a_{n-1}/a_n \geq R$

multiply the above $(n-m)$ inequality

$a_m/a_n > R^{n-m}$

$a_n/a_m < 1/R^{n-m}$

$$a_n < \frac{a_m}{R^{n-m}}$$

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$a_n < \frac{(a_m)\, R^m}{R^n}$

Taking $(a_m)\, R^m = K$

$a_n < K\,(1/R)^n$

$\sum_{n=1}^{\infty} \frac{1}{R^n} = \frac{1}{R} + \frac{1}{R^2} + \text{———}$

$\frac{1}{R} < 1$

Case II If $l < 1$

$l + \epsilon < 1$

$a_m / a_{n+1} < l + \epsilon$

$a_m / a_{n+1} < r$

$n = m, m+1$ ——— $n-1$

$a_m / a_{m+1} < r$

$a_{m+1} / a_{m+2} < r$

$a_{n-1} / a_n < r$

multiplying above $(n-m)$

$a_m / a_n < r^{n-m}$

$a_m / a_m > 1/r^{n-m}$

$a_n > a_m / r^{n-m}$

$a_n > a_m \cdot r^m / r^n$

$a_m \cdot r^n = K$

$a_m > K' \cdot \frac{1}{r^n}$

$\sum_{n=1}^{\infty} \frac{1}{r^n} = \frac{1}{r} + \frac{1}{r^2} + \text{———}$

$\frac{1}{r} > 1$

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Case III If $l = \infty$ $\lim_{n\to\infty} \frac{a_m}{a_{n+1}} = \infty$

$a_m / a_{n+1} > 2$

$n = m, m+1$ ——— $(n-1)$

$a_m / a_{m+1} > 2$

$a_{n-1} / a_n > 2$

multiply above $(n-m)$ in Equalities

$a_m / a_n > 2^{n-m}$

$a_n / a_m < 1/2^{n-m}$

$a_n < 2^m \cdot a_m / 2^n$

$2^m \cdot a_n = K''$

$a_n < K''\,(1/2^n)$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2}$ ———

$\sum a_n$ is also Cgt

IV If $l = 1$ $\lim a_m / a_{n+1} = 1$

$\sum_{n=1}^{\infty} a_m$ may Cgt or diverges series

$a_m = 1/n$, $a_m / a_{m+1} = n+1/n = 1 + 1/n$

$a_m = \frac{1}{n^2}$ $\frac{a_m}{a_{m+1}} = \frac{(n+1)^2}{n^2} = \left(1+\frac{1}{n}\right)^2$

$\lim_{n\to\infty} \frac{a_m}{a_{m+1}} = 1$

$a_m = \frac{1}{n}$ $\sum a_m$ So dvg $a_m = \frac{1}{n^2}$ Series dvg

$\lim_{n\to\infty} \frac{a_m}{a_{m+1}} = 1$ Cgt dvgt of Series

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Ques Examine the Convergence or divergence of the series $(1/3)^2 + (1\cdot 2/3\cdot 5)^2 + \ldots$

A The given series is $\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \ldots$

$$a_n = \left[\frac{1\cdot 2\cdot 3 \ldots n}{3\cdot 5\cdot 7 \ldots (2n+1)}\right]^2$$

$$a_{n+1} = \left[\frac{1\cdot 2 \ldots n(n+1)}{3\cdot 5 \ldots (2n+3)}\right]^2$$

$$\frac{a_n}{a_{n+1}} = \left(\frac{2n+3}{n+1}\right)^2 = \left[\frac{2+3/n}{1+1/n}\right]^2$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = (2)^2 = 4 > 1$$

By D'Alembert's Ratio Test.

Ques Test the following series for Converge

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \ldots \quad x > 0$$

Sol^n Neglecting the first term of the series

$a_n = x^{2n}/2n \quad a_{n+1} = x^{2n+2}/2n+2$

$$\frac{a_n}{a_{n+1}} = \frac{x^{2n}}{2n}\left(\frac{2n+2}{x^{2n+2}}\right) = \frac{2n+2}{2n}\cdot\frac{1}{x^2}$$

$$\left(1+\frac{1}{n}\right)\frac{1}{x^2} =$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x^2}$$

By D'Alembert Ratio test.

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$\sum_{n=1}^{\infty} a_n$ Converges if $\frac{1}{x^2}$ and

$\sum_{n=1} a_n$ is diverges if $x \leq 1$

Ratio test fails at $x = 1$

$a_n = 1/2n \qquad b_n = 1/n$

$\lim_{n\to\infty} a_n/b_n = 1/2$ which is nonzero.

By Comparison Test $\sum a_n$ and $\sum b_n$.

$\sum_{n=1}^{\infty} b_n = \sum_{n=1} \frac{1}{n}$ is divergent,

$\sum_{n=1}^{\infty} a_n$ is also divergent.

The given series is cg^t if $x<1$ and dv if $x \geq 1$

Cauchy's Root Test :-

(a) If $\sum_{n=1}^{\infty} a_n$ is a positive terms series and $\lim_{n\to\infty} (a_n)^{1/n} = l$, then (i) the series only if $l<1$ and (ii) the series is diverges if $l>1$

(b) If $\lim_{n\to\infty} (a_n)^{1/n} = \infty$, then the series $\sum_{n=1} a_n$ is divergent.

A(a) $\lim_{n\to\infty} (a_n)^{1/n} = l$

for given $\epsilon > 0$, $\exists$ a positive integer m.

$|(a_n)^{1/n} - l| < \epsilon$.

$-\epsilon < a_n^{1/n} - l < \epsilon, \quad l - \epsilon < (a_n)^{1/n} < l+\epsilon$

Case 1 If $l<1$, $\quad l < l+\epsilon < 1$.

$(a_n)^{1/n} < l+\epsilon < 1$

put $l+\epsilon = r$

$(a_n)^{1/n} < r$ where $0<r<1$

$\sum_{n=1}^{\infty} r^n$ being a geometric series with common ratio $r<1$

$\sum_{n=1}^{\infty} a_n$ is cgt.

Case II If $l > 1$, $l - \epsilon > 1$

$l - \epsilon < (a_n)^{1/n}$

$(a_n)^{1/n} > (l - \epsilon)$.

put $l - \epsilon = R$.

$(a_n)^{1/n} > R$

$a_n > R^n$.

$R > 1$ so $\sum_{n=1}^{\infty} R^n$ being a geometric series

$\sum_{n=1}^{\infty} a_n$ is divergent.

(2) $\lim_{n\to\infty} (a_n)^{1/n} = \infty$.

$(a_n)^{1/n} \geq 3$

$a_n > (3)^n$.

$\sum_{n=1}^{\infty} 3^n$ being a geometric series with common ratio $= 3 > 1$ is div.

By Comparison Test $\sum_{n=1}^{\infty} a_n$ is divergent

Ques Discuss the convergence of series

$\frac{1}{2} + \left(\frac{2}{3}\right)x + \_\_\_\_$

Sol^n The given series is $\frac{1}{2} + \left(\frac{2}{3}\right)x + \_\_\_\_$

Leave first term and remaining series be denoted by $\sum a_n$

$a_n = \left(\frac{n+1}{n+2}\right)^n x^n$.

$(a_n)^{1/n} = \left(\frac{n+1}{n+2}\right)x = \left[\frac{1+1/n}{1+2/n}\right]x$

$\lim_{n\to\infty} (a_n)^{1/n} = x$.

By Cauchy's root Test, the given series is cgt if $x < 1$ and div. if $x > 1$. Test fails $x = 1$

at $x = 1$ $a_n = \left(\frac{n+1}{n+2}\right)^n = \frac{(1+1/n)^n}{(1+2/n)^{\frac{n}{2}\cdot 2}}$.

$\lim_{n\to\infty} a_n = \frac{e}{e^2} = \frac{1}{e} \neq 0$.

given positive term series is divergent when $x = 1$.

the series is cgt when $x < 1$ and divergent when $x \geq 1$.

Raabe's Test → If $\sum_{n=1}^{\infty} a_n$ is a series of positive term s.t. $\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = l$ then the series is

(1) Convergent if $l > 1$ (2) divergent if $l < 1$

Proof Case I If $l > 1$.

a number $p >$ s.t. $l > p > 1$.

let $b_n = \frac{1}{n^p}$ $\therefore \sum_{n=1}^{\infty} b_n$ is cgt.

$\sum_{n=1}^{\infty} a_n$ will be cgt if $\frac{a_n}{a_{n+1}} > \frac{b_n}{b_{n+1}}$ $\forall n$

$\frac{a_n}{a_{n+1}} > \frac{(n+1)^p}{n^p} \quad \forall n$

$\frac{a_n}{a_{n+1}} > \left(1+\frac{1}{n}\right)^p$

$\frac{a_n}{a_{n+1}} > 1+\frac{p}{n}+\frac{p(p-1)}{2n^2} ——$

$\frac{a_n}{a_{n+1}} - 1 > \frac{p}{n}+\frac{p(p-1)}{2n^2}+ ——$

$n\left(\frac{a_n}{a_{n+1}}-1\right) > p+\frac{p(p-1)}{2n}+ ——$

$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) \geq \lim_{n\to\infty}\left(p+\frac{p(p-1)}{2n}+ ——\right)$

$l > p > 1$ which is true.

$\sum_{n=1}^{\infty} a_n$ is cgt.

Case II if $l<1$

Choose a real number p st $l<p<1$

$b_n = \frac{1}{n^p}$ then $\sum_{n=1}^{\infty} b_n$ diverges

$\sum_{n=1}^{\infty} a_n$ will be divergent if $\frac{a_n}{a_{n+1}} < \frac{b_n}{b_{n+1}}$

$\frac{a_n}{a_{n+1}} < \left(\frac{n+1}{n}\right)^p$

$\frac{a_n}{a_{n+1}} < \left(1+\frac{1}{n}\right)^p$

$\frac{a_n}{a_{n+1}} < 1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+ ——$

$\frac{a_n}{a_{n+1}} - 1 < \frac{p}{n}+\frac{p(p-1)}{2n^2}+ ——$

$n\left(\frac{a_n}{a_{n+1}}-1\right) < p+\frac{p(p-1)}{2n}+ ——$

If $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) < \lim_{n\to\infty}\left(p+\frac{p(p-1)}{2n}+ ——\right)$

$\sum_{n=1}^{\infty} a_n$ is divergent. $l<p<1$

Logarithmic Test :-

If the series $\sum_{n=1}^{\infty} a_n$ is a positive term series st $\lim_{n\to\infty} n \log a_n/a_{n+1} = l$ then the series is cgt if $l>1$ and divergent if $l<1$.

Proof Case I if $l>1$

a real number p $l \geq p > 1$

If $b_n = \frac{1}{n^p}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is cgt.

$\sum_{n=1}^{\infty} a_n$ will also be cgt if.

$a_n/a_{n+1} \geq b_n/b_{n+1}$

$a_n/a_{n+1} > (n+1/n)^p$

$a_n/a_{n+1} > (1+1/n)^p$

$\log(a_n/a_{n+1}) > \log(1+1/n)^p$

$\log(a_n/a_{n+1}) > p\log(1+1/n)$

$n\log(a_n/a_{n+1}) > np\log(1+1/n)$

$n\log(a_n/a_{n+1}) > p\left(1-\frac{1}{2n}+\frac{1}{3n^2} ——\right)$

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$\lim_{n\to\infty} n \log\left(\frac{a_n}{a_{n+1}}\right) > p$

$l > p > 1$ which is true

The Series $\sum_{n=1}^{\infty} a_n$ converges if $p > 1$

Case II If $l < 1$

If $b_n = \frac{1}{n^p}$ then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

the Series $\sum_{n=1}^{\infty} a_n$ will also diverge

$\frac{a_n}{a_{n+1}} < \frac{b_n}{b_{n+1}}$

$\frac{a_n}{a_{n+1}} < \left(\frac{n+1}{n}\right)^p$

$\log a_n/a_{n+1} < p \log\left(1+\frac{1}{n}\right)$

$n \log \frac{a_n}{a_{n+1}} < np \log\left(1+\frac{1}{n}\right)$

$n \log \frac{a_n}{a_{n+1}} < pn\left(\frac{1}{n} - \frac{1}{2n^2} - \right)$

$n \log \frac{a_n}{a_{n+1}} < p\left(1 + \frac{1}{2n} + \frac{1}{3n^2} - \right)$

$\lim_{n\to\infty} n \log \frac{a_n}{a_{n+1}} < \lim_{n\to\infty} p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \right)$

$l < p < 1$

the Series $\sum_{n=1}^{\infty} a_n$ diverges if $l < 1$

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Ques. Test the Convergence of the Series
$x^2 + \frac{2^2}{3\cdot 4} x^4 + \text{——} \quad (x > 0)$

Sol^n Let the given Series be denoted by $\sum_{n=1}^{\infty} a_n$. $a_n = \frac{2^2 \text{——} (2n)^2}{3\cdot 4\cdot 5\cdot 6 \text{——} (2n+2)} x^{2n+2}$

$a_{n+1} = \frac{2^2 \cdot \text{——} (2n+2)^2}{3\cdot 4\cdot 5 \text{——} (2n+3)(2n+4)} x^{2n+4}$

$\frac{a_n}{a_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2} \cdot \frac{1}{x^2}$

$\frac{4n^2+14n+12}{4n^2+8n+4} \cdot \frac{1}{x^2} = \frac{4+\frac{14}{n}+\frac{12}{n^2}}{4+\frac{8}{n}+\frac{4}{n^2}} = \frac{1}{x^2}$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x^2}$

By D'Alembert's ratio test $\sum_{n=1}^{\infty} a_n$ is Cgt if $1/x^2 > 1$ $x^2 < 1$ $\sum_{n=1}^{\infty} a_n$ is divergent if $1/x^2 < 1$ $x^2 > 1$

At $x^2 = 1$ $\lim_{n\to\infty} a_n/a_{n+1} = 1$

$\frac{a_n}{a_{n+1}} = \frac{4n^2+14n+12}{4n^2+8n+4}$

$\frac{a_n}{a_{n+1}} - 1 = \frac{4n^2+14n+12}{4n^2+8n+4} - 1$

$= \frac{4n^2+14n+12-4n^2-8n-4}{4n^2+8n+4}$

$= 6n+8 / 4n^2+8n+4 = \frac{\left(6+\frac{8}{n}\right) n}{n\left[4+\frac{8}{n}+\frac{4}{n^2}\right]}$

$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{6+8/n}{4+8/n+4/n^2}$

$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{6}{4} > 1$

By Raabe's Test $\sum_{n=1}^{\infty} a_n$ is cgt

Ques Test the Convergence of the Series $(x>0)$

$1 + \frac{x}{2} + \frac{2!}{3^2} x^2 \ldots$

Ans Neglecting the first term

$a_n = \frac{n!}{(n+1)^n} x^n \quad a_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}} x^{n+1}$

$\frac{a_n}{a_{n+1}} = \frac{n!\,x^n}{(n+1)^n} \times \frac{(n+2)^{n+1}}{(n+1)!\,x^{n+1}}$

$\frac{(n+2)^{n+1}}{(n+1)^n (n+1)} \cdot \frac{1}{x} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{1}{x}$

$\frac{n^{n+1}\left(1+2/n\right)^{n+1}}{n^{n+1}\left(1+1/n\right)^{n+1}} \cdot \frac{1}{x}$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(1+2/n)^{n+1}}{(1+1/n)^{n+1}} = \frac{1}{x}$

$\lim_{n\to\infty} \frac{(1+2/n)(1+\frac{2}{n})^n}{(1+\frac{1}{n})(1+\frac{1}{n})^n} \cdot \frac{1}{x}$

$= \frac{(1)(e^2)}{(1)(e)} \cdot \frac{1}{x} = \frac{e}{x}$.

Case I When $\frac{e}{x} > 1 \Rightarrow \frac{x}{e} < 1 \quad x<e$

By D'Alembert's ratio Test

$\sum_{n=1}^{\infty} a_n$ Converges if $x<e$

Case II when $\frac{e}{x} < 1 \quad \frac{x}{e} > 1 \quad x>e$

By D'Alembert's Ratio Test $\sum_{n=1}^{\infty} a_n$ diverges if $x>e$

Case III $\frac{e}{x} = 1 \quad x=e$

D'alembert Ratio Test

$\frac{a_n}{a_{n+1}} = \frac{(1+2/n)^{n+1}}{(1+1/n)^{n+1}} \cdot \frac{1}{x} = \frac{(1+2/n)^{n+1}}{(1+1/n)^{n+1}} \cdot \frac{1}{e}$

$\log \frac{a_n}{a_{n+1}} = \log \frac{(1+2/n)^{n+1}}{(1+1/n)^{n+1}} \cdot \frac{1}{e}$

$\log\left(1+2/n\right)^{n+1} - \log\left(1+1/n\right)^{n+1} - \log e$.

$n+1 \cdot \log\left(1+2/n\right) - (n+1)\log\left(1+\frac{1}{n}\right) - 1$

$(n+1)\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} \ldots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} \ldots\right) - 1$

$(n+1)\left(\frac{1}{n} - \frac{3}{2n^2}\right) - 1$

$= \left[1 - \frac{3}{2n} + \ldots\right] - 1$

$= -\frac{1}{2n} - \frac{3}{2n^2} + \ldots$

$n \log \frac{a_n}{a_{n+1}} = -\frac{1}{2} - \frac{3}{2n} + \ldots$

$n \log \frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\left(-\frac{1}{2} - \frac{3}{2n} - \right) = \frac{1}{2} < 1$

arithmic test $\sum_{n=1}^{\infty} a_n$ is dagt.

Series Converges if $x < e$ and diverges if $x \geq e$

Ques Convergence of the series $x + \frac{2^2 x^2}{2!} + \ldots$

A $a_n = \frac{n^n x^n}{n!}$ $a_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$\frac{a_n}{a_{n+1}} = \frac{n^n x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}}$

$= \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)!}{n!} \cdot \frac{x^n}{x^{n+1}}$

$= \frac{n^n}{(n+1)^n (n+1)} \cdot \frac{(n+1)\, n!}{n!} \cdot \frac{1}{x}$

$\frac{n^n}{(n+1)^n} \cdot \frac{1}{x} = \frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x}$

$\frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x} = \frac{1}{ex}$

Case I $\frac{1}{ex} > 1$ $ex < 1$ $x < \frac{1}{e}$

By D'Alembert's Ratio Test

$\sum_{n=1}^{\infty} a_n$ Cgs if $x < \frac{1}{e}$

Case II when $\frac{1}{ex} < 1$ $ex > 1$ $x > \frac{1}{e}$

By D'Alembert's ratio test

$\sum_{n=1}^{\infty} a_n$ diverges if $x > \frac{1}{e}$.

III when $\frac{1}{ex} = 1$ $x = \frac{1}{e}$

D'Alembert's ratio test fails

$\frac{a_n}{a_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$\log \frac{a_n}{a_{n+1}} = \log \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$\log e - \log\left(1+\frac{1}{n}\right)^n = \log e - n \log\left(1+\frac{1}{n}\right)$

$1 - n\left(\frac{1}{n} - \frac{1}{2n^2} - \right] = 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} -$

$= \frac{1}{n}\left(\frac{1}{2} - \frac{1}{3n} + - \right)$

$n \log \frac{a_n}{a_{n+1}} = \frac{1}{2} - \frac{1}{3n} + -$

$\lim_{n\to\infty} n \log \frac{a_n}{a_{n+1}} = \frac{1}{2} < 1$

By logarithmic test $\sum_{n=1}^{\infty} a_n$ is dvgt

Series Cgt if $x < \frac{1}{e}$ and dvgt if $x \geq \frac{1}{e}$

Date :

De Morgan's and Bertrand's Test

Th^m If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms such that

$$\lim_{n\to\infty} \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right] \log n = l,$$ the

the series is

① Convergent if $l > 1$

② divergent if $l < 1$.

Proof Case I when $l > 1$.

choose $p > 1$ st $l > p > 1$.

$b_n = 1/n(\log n)^p$

$\sum_{n=2}^{\infty} b_n$ is cgt series

$\sum_{n=1}^{\infty} a_n$ is cgt if $a_n/a_{n+1} > b_n/b_{n+1}$

$a_n/a_{n+1} > (n+1)\log(n+1)^p / n(\log n)^p$

$(1+1/n)\left[\frac{\log n(1+1/n)}{\log n}\right]^p =$

$= (1+1/n)\left[\frac{\log n + \log(1+1/n)}{\log n}\right]^p$

$(1+1/n)\left[\frac{\log n + 1/n - 1/2n^2 - \cdots}{\log n}\right]^p$

$(1+1/n)\left[1 + p\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n}\right) + \frac{p(p-1)}{2n^2(\log n)^2} - \cdots\right] -$

$\frac{a_n}{a_{n+1}} \geq 1 + 1/n + \frac{p}{n\log n} + \frac{p(p-1)}{2n^2(\log n)^2} - \cdots$

$n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1 + \frac{p}{\log n} + \frac{p(p-1)}{2n(\log n)^2} - \cdots$

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$n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 > \frac{p}{\log n} + \frac{p(p-1)}{2n(\log n)^2} - \cdots$

$\log n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) > p + \frac{p(p-1)}{2n\log n} - \cdots$

$\lim_{n\to\infty} \log n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right] > p$

$l > p$ which is true

$\sum_{n=1}^{\infty} a_n$ is cgt if $l > 1$

Case II when $l < 1$

$\sum_{n=1}^{\infty} a_n$ is divergent if $l < 1$

Ques. Test the convergence of series

$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \cdots \quad (x > 0)$$

Sol^n Neglecting the first term.

$a_n = \frac{1^2 \cdot 3^2 \cdots (2n+1)^2}{2^2 \cdot 4^2 \cdots (2n+2)^2}$

$a_{n+1} = \frac{1^2 \cdot 3^2 \cdots (2n+3)^2}{2^2 \cdot 4^2 \cdots (2n+4)^2}$

$\frac{a_n}{a_{n+1}} = \frac{(2n+4)^2}{(2n+3)^2} \cdot \frac{1}{x}$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}$.

Case I When $\frac{1}{x} > 1 \Rightarrow x < 1$

$\sum_{n=1}^{\infty} a_n$ is cgt by Ratio Test.

Case II when $\frac{1}{x} < 1 \Rightarrow x > 1$

$\sum_{n=1}^{\infty} a_n$ is divgt by Ratio Test.

Case III when $\frac{1}{x} = 1 \Rightarrow x = 1$

Ratio Test is not applicable,

$$\frac{a_n}{a_{n+1}} = \frac{(2n+4)^2}{(2n+3)^2} \cdot \frac{1}{x} = \frac{(2n+4)^2}{(2n+3)^2}$$

$$= \frac{4n^2+16n+16}{4n^2+12n+9}$$

$$\frac{a_n}{a_{n+1}} - 1 = \frac{4n^2+16n+16}{4n^2+12n+9} - 1$$

$$= \frac{4n^2+16n+16-4n^2-12n-9}{4n^2+12n+9} =$$

$$\frac{4n+7}{4n^2+12n+9}$$

$$\frac{a_n}{a_{n+1}} - 1 = \frac{n\left(4+7/n\right)}{n^2\left(4+\frac{12}{n}+\frac{9}{n^2}\right)}$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{4+7/n}{4+\frac{12}{n}+\frac{9}{n^2}}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{4}{4} = 1$$

Raabe's Test is not applicable

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{4+7/n}{4+\frac{12}{n}+\frac{9}{n^2}}$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 = \frac{4+7/n-4-\frac{12}{n}-\frac{9}{n^2}}{4+\frac{12}{n}+\frac{9}{n^2}}$$

$$= \frac{-\frac{5}{n}-\frac{9}{n^2}}{4+\frac{12}{n}+\frac{9}{n^2}}$$

multiply by n.

$$\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right]\log n = \frac{-\frac{5}{n}-\frac{9}{n^2}}{4+\frac{12}{n}+\frac{9}{n^2}}\log n$$

$$= \frac{-5-\frac{9}{n}}{4+\frac{12}{n}+\frac{9}{n^2}} \cdot \frac{\log n}{n}$$

Take $\lim n\to\infty$.

$$\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right]\log n$$

$$\lim_{n\to\infty}\left[\frac{-5-\frac{9}{n}}{4+\frac{12}{n}+\frac{9}{n^2}}\right] \cdot \lim_{n\to\infty}\frac{\log n}{n}$$

$$\left(\frac{-5}{4}\right)\cdot 0 = 0 < 1$$

By DeMorgan's Test the Series $\sum_{n=1}^{\infty} a_n$ is divergent.

Series is Cgt if $x<1$ and divgt

Date :

Gauss Test.

Thm: If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms of $a_n/a_{n+1} = 1 + \lambda/n + \gamma_n/n^{1+\delta}$ where $\delta > 0$ and γ_n is bounded seq then
(1) cgt if $\lambda > 1$ (2) diverges if $\lambda \leq 1$.

Proof: it is given that $\frac{a_n}{a_{n+1}} = 1 + \frac{\lambda}{n} + \frac{\gamma_n}{n^{1+\delta}}$

γ_n is bounded

$|\gamma_n| \leq K$

$\lim_{n\to\infty} \frac{\gamma_n}{n^{1+\delta}} = 0$

$\frac{a_n}{a_{n+1}} - 1 = \frac{\lambda}{n} + \frac{\gamma_n}{n^{1+\delta}}$

$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lambda + \frac{\gamma_n}{n^{\delta}}$

$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lambda$

By Raabe's Test $\sum a_n$ is cgt if $\lambda > 1$

If $\lambda = 1$ Raabe's Test is not applicable

$n\left(\frac{a_n}{a_{n+1}} - 1\right) = 1 + \frac{\gamma_n}{n^{\delta}}$

$n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 = \frac{\gamma_n}{n^{\delta}}$

Multiply (2) by $\log n$

$\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right)\log n = \frac{\gamma_n \cdot \log n}{n^{\delta}}$

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$\lim_{n\to\infty}\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right)\log n = \lim_{n\to\infty} \gamma_n \cdot \frac{\log n}{n^{\delta}} \leq K \lim_{n\to\infty} \frac{1/n}{\delta n^{\delta-1}}$

$\frac{K}{\delta} \cdot \lim_{n\to\infty} \frac{1}{n^{\delta}} = 0 < 1$

By De Morgan's and Bertrand's Test $\sum_{n=1}^{\infty} a_n$ is diverges.

Ques. Test for Convergence, the Series
$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1\cdot 3}{2\cdot 4}\right)^p$ ——

Soln: Neglecting first term

$a_n = \left[\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}\right]^p$

$a_{n+1} = \left[\frac{1\cdot 3 \cdots (2n+1)}{2\cdot 4 \cdots (2n+2)}\right]^p$

$\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \left[\frac{2n(1+1/n)}{2n(1+1/2n)}\right]^p$

$= \left(\frac{1+1/n}{1+1/2n}\right)^p$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\left(\frac{1+1/n}{1+1/2n}\right)^p = 1$

D'Alembert's Ratio test is not applicable

$$\frac{a_n}{a_{n+1}} = \left(1+\frac{1}{n}\right)^{p}\left(1+\frac{1}{2n}\right)^{-p}$$

$$= \left(1+\frac{p}{n} - \right)\left(1-\frac{p}{2n} - \right)$$

$$= \left(1+\frac{p}{n} - \right)\left(1-\frac{p}{2n} - \right)$$

$$\frac{a_n}{a_{n+1}} = 1+\frac{p}{n} - \frac{p}{2n} + O\left(\frac{1}{n^{1+\delta}}\right)$$

$O\left(\frac{1}{n^{1+\delta}}\right)$ stand for terms of higher power of $\frac{1}{n}$

$$\frac{a_n}{a_{n+1}} = 1+\frac{p/2}{n} + O\left(\frac{1}{n^{1+\delta}}\right)$$

By gauss test the series is cgt if $\frac{p}{2}>1$ $p>2$ and div. if $\frac{p}{2}\le 1$

Ques. If x, α, β, γ are all positive, discuss convergence of the hypergeometric series

$$1+\frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\,\gamma(\gamma+1)} - -$$

Sol^n All term in given series are positive neglecting first term we have

$$a_n = \frac{\alpha(\alpha+1) - (\alpha+n-1)\beta - (\beta+n-1)}{1\cdot 2 - \gamma(\gamma+1) - (\gamma+n-1)}\,x^n$$

$$a_{n+1} = \frac{\alpha(\alpha+1) - (\alpha+n)\cdot + - (\beta+n)}{1\cdot 2 - (n+1)\gamma - \gamma+n}\,x^{n+1}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(\frac{\gamma}{n}+1\right)}{\left(\frac{\alpha}{n}+1\right)\left(\frac{\beta}{n}+1\right)}\cdot\frac{1}{x} = \frac{1}{x}$$

By Ratio Test $\sum^{\infty} a_n$ is cgt if $\frac{1}{x}>1$ divergent if $\frac{1}{x}<1$.

at $x=1$ Ratio test is not applicable

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)} \quad \frac{n^2(1+1/n)(1+\gamma/n)}{n^2(1+\alpha/n)(1+\beta/n)}$$

$$= (1+1/n)(1+\gamma/n)(1+\alpha/n)^{-1}(1+\beta/n)^{-1}.$$

$$= (1+1/n)(1+\gamma/n)\left(1-\frac{\alpha}{n} - \right)\left(1-\frac{\beta}{n} - \right)$$

$$\frac{a_n}{a_{n+1}} = \left(1+\frac{1}{n} - \right)\left(1-\frac{\alpha}{n} - \right)$$

$O\left(\frac{1}{n^{1+\delta}}\right)$ stand for term containing higher power of $1/n$.

$$\frac{a_n}{a_{n+1}} = 1+\frac{1}{n} - O\left(\frac{1}{n^{1+\delta}}\right)$$

$$= 1+\frac{1+\gamma-\alpha-\beta}{n} + O\left(\frac{1}{n^{1+\delta}}\right)$$

By gauss test $\sum_{n=1}^{\infty} a_n$ converges if

$1+\gamma-\alpha-\beta>1$ $\gamma>\alpha+\beta$. and

diverges if $1+\gamma-\alpha-\beta\le 1$ $\gamma\le\alpha+\beta$

cgt $\gamma>\alpha+\beta$ div. $\gamma\le\alpha+\beta$.

Cauchy's Condensation Test :-

If $a_n \geq a_{n+1} \geq 0$ for each n, then series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges or diverges together.

Proof Let S_n and σ_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n}$.

$S_n = a_1 + a_2 \text{ —— } a_n.$

$\sigma_n = 2a_2 \text{ —— } 2^n a_{2^n}.$

Case I Let the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ be convergent. The seq^n σ_n is cgt and bounded

$\sigma_n \leq M$

$S_n = a_1 + a_2 \text{ —— } + a_n.$

$= a_1 + a_2 \text{ - - - - } a_{2^{k+1}}$

$= a_1 + (a_2 + a_3) \text{ —— } a_{2^{k+1}}$

$\leq a_1 + \text{ —— } a_{2^{k+1}}$

$= a_1 + 2a_2 + \text{ —— } a_{2^{k+1}}$

$= a_1 + (2a_2 \text{ —— } 2^k a_{2^k}) + a_{2^{k+1}}$

$= a_1 + \sigma_k + a_{2^{k+1}}$

$\leq a_1 + \sigma_k + a_1$

$S_n \leq 2a_1 + \sigma_k$

$\leq 2a_1 + M$

The seq^n S_n is bounded above and convergent.

The series $\sum_{n=1}^{\infty} a_n$ is convergent

Case II Let the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ be dvgt.

The seq^n σ_n is divergent to ∞

The seq^n σ_n is unbounded above

$S_{2^n} = a_1 \text{ —— } a_{2^n}.$

$= (a_1 + a_2) \text{ —— } (a_{2^{n-1}} + a_{2^n})$

$\geq (a_1 + a_2 \text{ - - —— } (a_{2^n} + \text{—} a_{2^n})$

$= (a_1 \text{ —— } 2^{n-1} a_{2^n}.$

$a_1 + \frac{1}{2}(2a_2 \text{ —— } 2^n a_{2^n})$

$a_1 + \frac{1}{2}\sigma_n$

But S_{2^n} is subseq^n of seq^n S_n

Seq^n S_n is divergent to ∞

the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Ques Using Cauchy's Condensation test, discuss convergence of the series $\sum_{n=2}^{\infty} 1/n(\log n)^p$

Ans Let $a_n = 1/n(\log n)^p$.

Case (i) when $p \leq 0$

$\frac{1}{n(\log n)^p} \geq \frac{1}{n} \quad \forall\, n \geq 2$

As $\sum_{n=2}^{\infty} 1/n$ is divergent, so by comparison test $\sum_{n=2}^{\infty} 1/n(\log n)^p$ is divergent.

Case (ii) when $p > 0$

$n(\log n)^p >$ is an increasing seq^n

a_n is a decreasing seq^n

$a_n > a_{n+1} > 0 \quad \forall\, n \geq 2$

By Cauchy's Condensation test

the series $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} 2^n a_{2^n}$ Converge and diverge together

$\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot 1/2^n (\log 2^n)^p$

$= \sum_{n=2}^{\infty} 1/(n \log 2)^p = 1/(\log 2)^p \cdot \sum_{n=2}^{\infty} 1/n^p$

$\sum_{n=2}^{\infty} 1/n^p$ is cgt if $p > 1$ and div if $p \le 1$

$\sum_{n=2}^{\infty} 2^n a_{2^n}$ is cgt if $p > 1$ and div. if $p \le 1$

$\sum_{n=2}^{\infty} a_n$ is cgt if $p > 1$ and divgt if $p \le 1$

Chapter 8 Alternating Series

A.S → A Series in which the term are alternatively positive and negative

Leibnitz's Test :- The alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 \text{———} (a_n > 0)$ cgt

(1) $a_{n+1} \le a_n$ (2) $\lim_{n \to \infty} a_n = 0$

Proof :- $S_{2n} = a_1 - a_2 \text{———} a_{2n-1} - a_{2n}$

$= a_1 - \text{———} a_{2n})$

$a_n \ge a_{n+1}$ $\forall n$ and $a_{2n} > 0$

Each term $S_{2n} \le a_1$

Seq^n S_{2n} is Bounded above

$S_{2n+2} - S_{2n} = a_{2n+1} - a_{2n+2} \ge 0$

$S_{2n+2} \ge S_{2n}$ $\forall n$

S_{2n} is a monotonically increasing Seq^n

Seq^n $\langle S_{2n} \rangle$ is monotonically increasing and Bounded and Convergent

$\lim_{n \to \infty} S_{2n} = S.$

$S_{2n+1} = S_{2n} + a_{2n+1}.$

$\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} a_{2n+1} = S + 0 = S$

the Seq^n $\langle S_{2n} \rangle$ and $\langle S_{2n+1} \rangle$ Converges to the Same limit S,

$|S_{2n} - S| < \epsilon$ $\forall$ $2n \ge m_1$

S_{2n+1} Converges to S, $\therefore$ Some $\epsilon > 0$

$|S_{2n+1} - S| < \epsilon$

Let $m = \max(m_1, m_2)$

$|S_{2n} - S| < \epsilon$ $|S_{2n+1} - S| < \epsilon$

$|S_n - S| < \epsilon$

Seq^n S_n Converge to S

the given Series is cgt.

Absolute Convergence → A Series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely Convergent if the series $\sum_{n=1}^{\infty} |a_n|$ Converges.

Conditional Convergence →
A Series $\sum_{n=1}^{\infty} a_n$ is said to be Condition Convergent.

(1) $\sum_{n=1}^{\infty} a_n$ is Cgt

(2) $\sum_{n=1}^{\infty} a_n$ is not absolutely Converge

Date :

Thm Show that every absolutely cgt series is convergent. Is the converse true? If not, show by an example.

Proof Let $\sum_{n=1} a_n$ be an absolutely cgt series.

for $\sum_{n=1}^{\infty} |a_n|$ is cgt

for given $\epsilon > 0$ by Cauchy's general principle of convergence there exist a positive integer m

$||a_{m+1}| + \cdots |a_n|| < \epsilon$

$|a_{m+1}| \cdots + |a_n| < \epsilon$.

By tringle inequality

$|a_{m+1} + \cdots a_n| \le |a_{m+1}| \cdots + |a_m| < \epsilon$

By Cauchy general principle of convergence it series $\sum_{n=1}^{\infty} a_n$ cgt

Every absolutely cgt series is cgt.

Converese

The series $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} \cdots$

$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$.

$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1} a_n$ is cgt by Leibnitz Test

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Thm If $\sum_{n=1}^{\infty} a_n$ is an absolutely cgt series then the series of its positive terms and the series of its negative terms are both convergent

Proof Let $S_n = a_1 \cdots a_n$

$\sigma_n = |a_1| \cdots |a_n|$

Let P_n and $-Q_n$ denote the sum of positive and negative term in S_n

$S_n = P_n - Q_n \qquad \sigma_n = P_n + Q_n$

$P_n = \frac{1}{2}(\sigma_n + S_n) \quad Q_n = \frac{1}{2}(\sigma_n - S_n)$

The series $\sum_{n=1}^{\infty} a_n$ is absolutely cgt $\therefore$ both S_n and σ_n tend to finite limit as $n \to \infty$

Let $\lim_{n\to\infty} S_n = l \quad \lim_{n\to\infty} \sigma_n = l'$

$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{1}{2}(\sigma_n + S_n) = \frac{1}{2}(l' + l)$

$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} \frac{1}{2}(\sigma_n - S_n) = \frac{1}{2}(l' - l)$

Ques Test for convergence and absolute convergence of the series $\sum_{n=1} \frac{(-1)^n \sin n x}{n^3}$

A The given series is

$\sum_{n=1}^{\infty} \frac{(-1)^n \sin n x}{n^3} = \sum_{n=1} a_n$

$$a_n = \frac{(-1)^n \sin nx}{n^3}$$

$$|a_n| = \left|\frac{(-1)^n \sin nx}{n^3}\right| \leq \frac{1}{n^3}$$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is Cgt by p test

By Comparison Test

$\sum_{n=1}^{\infty} |a_n|$ is also Cgt

Ques Discuss the Convergence and absolute Convergence of the Series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sqrt{n+1} - \sqrt{n-1}\right)$$

A Let $a_n = \frac{(-1)^{n-1}}{n}\left(\sqrt{n+1} - \sqrt{n-1}\right)$

$$|a_n| = \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$$

$$= \frac{\sqrt{n+1} - \sqrt{n-1}}{n} \times \frac{\sqrt{n+1} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n-1}}$$

$$\frac{(n+1) - (n-1)}{n\left(\sqrt{n+1} + \sqrt{n-1}\right)} = \frac{2}{n\left(\sqrt{n+1} + \sqrt{n-1}\right)}$$

$$= \frac{2}{n\sqrt{n}\left[\sqrt{1+1/n} + \sqrt{1-1/n}\right]} = \frac{2}{n^{3/2}\left(\sqrt{1+1/n} + \sqrt{1-1/n}\right)}$$

$$b_n = \frac{1}{n^{3/2}}$$

$$|a_n| / b_n = 2 / \left(\sqrt{1+1/n} + \sqrt{1-1/n}\right)$$

By Comparison Test $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} b_n$ Converge or diverge together But $\sum_{n=1}^{\infty} b_n$ is Cgt by p test

$\sum_{n=1}^{\infty} |a_n|$ is also Convergent

Arbitrary Series.

Abel's Lemma – If $\sum_{n=1}^{\infty} a_n$ is a Series of real number s.t. $m \leq a_1 + \ldots + a_n \leq M$ and b_n is a non increasing seq^n of non negative real numbers

$$m b_1 \leq \sum_{k=1}^{n} a_k b_k \leq M b_1$$

Proof Let S_n denote the seq^n of partial Sum

$S_1 = a_1 \quad S_2 = a_1 + a_2 \quad S_3 = a_1 + a_2 + a_3$ ———

$S_{n-1} = a_1 + a_2 + a_3 + \text{———} + a_{n-1}$

$S_n = a_1 + \text{———} a_n$

$a_1 = S_1, \; a_2 = S_2 - S_1, \; a_3 = S_3 - S_2$ ——— $a_n = S_n - S_{n-1}$

$m \leq S_n \leq M$.

$\sum_{k=1}^{n} a_k b_k = a_1 b_1$ ——— $a_{n-1} b_{n-1} + a_n b_n$

$= S_1 b_1 +$ ——— $(S_n - S_{n-1}) b_n$.

$= S_1 (b_1 - b_2)$ ——— $S_n b_n$ — (2)

from (1) and (2)

$\sum_{k=1}^{n} a_k b_k \leq M (b_1 - b_2)$ ——— $M b_n$

$= M (b_1 - b_2$ ——— $b_n)$

$\sum_{k=1}^{n} a_k b_k \leq M b_1$.

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$$\sum_{k=1}^{n} a_k b_k \leq M(b_1-b_2) \text{———} Mb_n$$
$$= m(b_1-b_2) \text{———} b_n$$
$$\sum_{k=1}^{n} a_k b_k \geq m b_1$$

from ① and ②
$$m b_1 \leq \sum_{k=1}^{n} a_k b_k \leq M(b_1-b_2$$
$$\text{———} Mb_n$$
$$= M(b_1-b_2) \text{———} b_n)$$

$$\sum_{k=1}^{n} a_k b_k \leq M b_1$$

$$\sum_{k=1}^{n} a_k b_k \geq m b_1$$

$$m b_1 \leq \sum_{k=1}^{n} a_k b_k \leq M b_1$$

Abel's Test.
If $\sum_{n=1}^{\infty} a_n$ is Cgt and Seq^n b_n is monotonic and Bounded, then $\sum_{n=1}^{\infty} a_n b_n$ is Cgt

Proof The Seq^n b_n is monotonic and Bounded

$$b = \begin{cases} \text{glb of } b_n \text{ if } b_n \text{ is decreasing} \\ \text{lub of } b_n \text{ if } b_n \text{ is increasing} \end{cases}$$

$$u_n = \begin{cases} b - b_n \text{ if } b_n \text{ is increasing} \\ b_n - b \text{ if } b_n \text{ is decreasing} \end{cases}$$

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$u_n \geq 0 \ \forall n \quad \lim_{n\to\infty} u_n = 0$

$$u_n - u_{n+1} = \begin{cases} (b-b_n) - (b-b_{n+1}) \text{ if } b_n \text{ incr} \\ (b_n-b) - (b_{n+1}-b) \text{ if } b_n \text{ decr} \end{cases}$$

$u_n - u_{n+1} \geq 0$.

$$b_n = \begin{cases} b - u_n \text{ if } b_n \text{ is increasing} \\ b + u_n \text{ if } b_n \text{ is decreasing.} \end{cases}$$

$$a_n b_n = \begin{cases} b a_n - a_n u_n \\ b a_n + a_n u_n \end{cases}$$

$$\sum_{n=1}^{\infty} a_n b_n = \begin{cases} b \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n u_n \\ b \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} a_n u_n \end{cases}$$

$\sum_{n=1}^{\infty} a_n$ is Cgt

$|a_{n+1} \text{———} a_{n+p}| < \epsilon/u_1$

By Abel's lemma

$|a_{n+1}u_{n+1} \text{———} a_{n+p}u_{n+p}| < \epsilon/u_1$

$|a_{n+1}u_{n+1} \text{———} a_{n+p}u_{n+p}| < \frac{\epsilon}{(u_1)}(u_1)$

$|a_{n+1}u_{n+1} \text{———} a_{n+p}u_{n+p}| < \epsilon$

$\sum_{n=1}^{\infty} a_n u_n$ is Cgt

Dirichlet Test →

If $\sum_{n=1}^{\infty} a_n$ is series of real number

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sequence S_n of partial sum Bounded
and if b_n is a non negative monotonically
decreasing seq^n tending to zero
then the series $\sum_{n=1}^{\infty} a_n b_n$ Converges.

Proof $S_n = a_1 + a_2 \text{——} a_n$.
seq^n S_n of partial sum is Bounded
a positive real number K s.t
$|S_n| \le K$

$S_n = a_1 + \text{——} a_n$.

$S_{n+p} = a_1 \text{——} a_n + a_{n+1} \text{——} a_{n+p}$

$S_{n+p} - S_n = a_{n+1} \text{——} a_{n+p}$

$|a_{n+1} \text{——} a_{n+p}| = |S_{n+p} - S_n|$
$\le |S_{n+p}| + |S_n|$
$\le K + K = 2K$

$\lim_{n\to\infty} b_n = 0$

for given $\epsilon > 0$,
$|b_n - 0| < \frac{\epsilon}{2K}$

$b_n < \frac{\epsilon}{2K}$

Apply Abel's lemma
$|a_{n+1} b_{n+1} \text{——} a_{n+p} b_{n+p}| \le 2K\, b_{n+1}$

$< 2K \frac{\epsilon}{2K} = \epsilon$

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Ques Show that series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1+\frac{1}{n}\right)^{-n}$
is Convergent

Sol The given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1+\frac{1}{n}\right)^{-n}$
$= \sum_{n=1}^{\infty} a_n b_n$

$a_n = \frac{(-1)^{n-1}}{n}$ $b_n = \left(1+1/n\right)^{-n}$

To show $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$
is Cgt

$t_n = \frac{1}{n}$ $\lim_{n\to\infty} t_n = 0$.

$t_{n+1} = \frac{1}{n+1}$

$n < n+1$

$\frac{1}{n} > \frac{1}{n+1}$

$t_n > t_{n+1}$

By Leibnitz Test $\sum_{n=1}^{\infty} (-1)^{n-1} t_n$
is Cgt

$\sum_{n=1}^{\infty} a_n$ is Cgt

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$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n} + \cdots \frac{1}{n^n}$

$= 1+1+ \cdots \frac{1}{n^n}$

$\left(1+\frac{1}{n+1}\right)^{n+1} = 1+1 \cdots \frac{1}{(n+1)^{n+1}}$

from (2) and (3),

$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$

$\left(1+\frac{1}{n}\right)^{-n} > \left(1+\frac{1}{n+1}\right)^{-(n+1)}$

$b_n > b_{n+1}$.

Seq^n b_n is a decreasing Seq^n

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}$

$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$

$b_1 = \frac{1}{2}$ so $\frac{1}{e} \leq b_n \leq \frac{1}{2}$

Seq^n b_n is monotonic and Bounded.

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$\sum_{n=1}^{\infty} a_n b_n$ is cgt

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(1+\frac{1}{n}\right)^{-n}$ is cgt

Ques Discuss Convergence of the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}\cos n\theta$

A The given series is $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}\cos n\theta$

$= \sum_{n=1}^{\infty} a_n b_n$

$a_n = \cos n\theta \quad b_n = \frac{(n!)^2}{(2n)!}$

$S_n = a_1 + \cdots a_n$

$= \cos\theta + \cos 2\theta + \cdots \cos n\theta$

$= \frac{\cos\left(\frac{\theta+n\theta}{2}\right)\sin\frac{n\theta}{2}}{\sin\theta/2}$

$|S_n| = \frac{\left|\cos\left(\frac{\theta+n\theta}{2}\right)\right|\left|\sin\frac{n\theta}{2}\right|}{|\sin\theta/2|}$

$|S_n| \leq \frac{1}{|\sin\theta/2|}$

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Case I when $\left|\sin \theta/2\right| \neq 0$

$\sum_{n=1}^{\infty} a_n$ has Bounded partial sum

$b_n = \frac{(n!)^2}{(2n)!}$

$b_{n+1} = \frac{((n+1)!)^2}{(2n+2)!}$

$\frac{b_n}{b_{n+1}} = \frac{(n!)^2}{(2n)!} \times \frac{(2n+2)!}{(n+1)!^2}$

$= \frac{(n!)^2}{2n!} \times \frac{(2n+2)(2n+1)(2n)!}{((n+1)n!)^2}$

$= \frac{(n!)^2}{2n!} \times \frac{(2n+2)(2n+1)(2n)!}{(n+1)n!}$

$\frac{(n!)^2}{2n!} \times \frac{(2n+2)(2n+1)(2n!)}{(n+1)^2 (n)^2}$

$= \frac{(2n+2)(2n+1)}{(n+1)^2}$

$= \frac{4n^2+6n+2}{n^2+2n+1} = \frac{n^2\left(4+\frac{6}{n}+\frac{2}{n^2}\right)}{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}$

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$\frac{4+\frac{6}{n}+\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}}$

$\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = 4 > 1$

By ratio test $\sum_{n=1}^{\infty} b_n$ Converges.

$\lim_{n\to\infty} b_n = 0$

$\frac{b_n}{b_{n+1}} > 1 \ \forall n.$

$\lim_{n\to\infty} b_n$ decrease to zero.

Dirichlet's Test $\sum_{n=1}^{\infty} a_n b_n$ Converges

$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \cos n\theta$ Converges.

Case II when $\theta = 2k\pi$ k for any integer

$\cos n\theta = \cos n(2k\pi) = 1$

$= \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \cos n\theta$

$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} = \sum_{n=1}^{\infty} b_n.$

series is cgt.

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INSERTION AND REMOVAL OF PARENTHESIS →

Consider the series $\sum_{n=1}^{\infty} (-1)^{n-1} = 1-1+1-1+1-1+\ldots$

this is an oscillating series oscillating between 0 and 1. If Parenthesis are inserted then the new series is $(1-1)+(1-1)+(1-1)+(1-1)+ \ldots$

Removal of Parenthesis :-

If $\sum_{n=1}^{\infty} b_n$ be a series with parenthesis and $\sum_{n=1}^{\infty} a_n$ is the series obtained from it by removing parenthesis, then it is not necessary that $\sum_{n=1}^{\infty} a_n$ is Convergent $\sum_{n=1}^{\infty} b_n$ is Convergent.

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Que :- Can be brackets be removed from series :-

$$\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}+\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+ \ldots$$

—— without affecting its convergence?

Soln :-

$$\therefore \sum_{n=1}^{\infty} a_n = \left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+ \ldots + \left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right) + \ldots$$

$$\therefore a_n = \frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}$$

$$a_{n+1} = \frac{1}{4n+1}+\frac{1}{4n+3}-\frac{1}{2n+2}$$

$$< \frac{1}{4n} + \frac{1}{4n} - \frac{1}{2(n+1)}$$

$$= \frac{2}{4n} - \frac{1}{2(n+1)} = \frac{1}{2n} - \frac{1}{2(n+1)}$$

$$= \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2}\left(\frac{\cancel{n}+1-\cancel{n}}{n(n+1)}\right)$$

$$= \frac{1}{2n(n+1)}$$

$$= \frac{1}{2n^2\left(1+\frac{1}{n}\right)} < \frac{1}{2n^2}$$

$\therefore a_{n+1} < \frac{1}{2n^2}$ for all n

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent ~~by~~

In a_{n+1}, sum of absolute values of terms.

$$= \left|\frac{1}{4n+1}\right| + \left|\frac{1}{4n+3}\right| + \left|\frac{1}{2n+2}\right|$$

$$= \frac{1}{4n+1} + \frac{1}{4n+3} + \frac{1}{2n+2},$$

which tends to zero as $n \to \infty$

<u>Que</u>:- Investigate what rearrangement of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$ will reduce its sum to zero.

<u>Sol</u>

Sol^n Clearly the given series is conditionally convergent with sum $\log 2$.

Now re-arrange the terms of the series by taking m positive terms followed by n negative terms ~~to~~ so that

$$\lim_{n\to\infty} \frac{m}{n} = k \text{ (say)}$$

Sum of the resulting series =

$$= S + \frac{1}{2} \lambda \log k$$

$$= \log 2 + \frac{1}{2} \lambda \log k$$

$$= \log 2 + \frac{1}{2} \log k$$

For sum of the resulting series to be equal to zero, we have

$$0 = \log 2 + \frac{1}{2} \log k$$

$$\log k = -2 \log 2 = \log \frac{1}{4}$$

$$k = \frac{1}{4} \Rightarrow \frac{m}{n} = \frac{1}{4}$$

$\therefore$ thus, to get the sum zero, one positive term should be followed by four negative term hence the arranged series a of the type

$$1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} \ldots$$

Multiplication of series →

If $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ and $\sum_{n=1}^{\infty} b_n = b_1 + b_2 \ldots$

series that want to consider how the product

$\left[\sum_{n=1}^{\infty} a_n\right] \left[\sum_{n=1}^{\infty} b_n\right]$ can be allowed worked out. there are different ways to find out this product but here we shall consider only two ways of arranging the term so as to form a simple series.

Ques- Show that the cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ with itself is divergent

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Solu- Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$ for all $n \in N$

By Leibnitz's test of alternating series both series are convergent but these are not convergent absolutely $\left[\because \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0 \text{ and } a_n > a_{n+1}\right]$

Let $\sum_{n=1}^{\infty} c_n$ denote Cauchy product of two series. then

$c_n = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} \cdots + a_n b_1$

$= \frac{(-1)^1}{\sqrt{2}\sqrt{n+1}} + \frac{(-1)^{n+1}}{\sqrt{3}}$

$= \frac{(-1)^2}{\sqrt{2}} \frac{(-1)^n}{\sqrt{n+1}} + \frac{(-1)^2}{\sqrt{3}} \frac{(-1)^{n-1}}{\sqrt{n}} +$

$\frac{(-1)^3}{\sqrt{4}} \cdot \frac{(-1)^{n-2}}{\sqrt{n-1}} \cdots + \frac{(-1)^n}{\sqrt{n+1}} \frac{(-1)^1}{\sqrt{2}}$

$= \frac{(-1)^{n+1}}{\sqrt{2}\sqrt{n+1}} + \frac{(-1)^{n+1}}{\sqrt{3}\sqrt{n}} + \frac{(-1)^{n+1}}{\sqrt{4}\sqrt{n-1}} + \cdots + \frac{(-1)^{n+1}}{\sqrt{n+1}\sqrt{2}}$

$= (-1)^{n+1} \left\{ \frac{1}{\sqrt{2}\sqrt{n+1}} + \frac{1}{\sqrt{3}\sqrt{n}} + \frac{1}{\sqrt{4}\sqrt{n-1}} + \cdots + \frac{1}{\sqrt{n+1}\sqrt{2}} \right\}$

$\therefore |c_n| = \frac{1}{\sqrt{2}\sqrt{n+1}} + \frac{1}{\sqrt{3}\sqrt{n}} + \frac{1}{\sqrt{4}\sqrt{n-1}} + \cdots + \frac{1}{\sqrt{n+1}\sqrt{2}}$

$\geq \frac{1}{\sqrt{n+1}\sqrt{n+1}} + \frac{1}{\sqrt{n+1}\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{n+1}+\sqrt{n+1}}$

$|c_n| \geq \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$

$|c_n| \geq \frac{1}{2}$ for all n

$|c_n| \geq 1$ as $n \to \infty$

$\therefore |c_n|$ and hence c_n does not tend to 0 as $n \to \infty$.

Ques- Prove that the Cauchy product of the two series $3 + \sum_{n=1}^{\infty} 3^n$ and $-2 + \sum_{n=1}^{\infty} 2^n$ is absolutely convergent although both the given series are divergent

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A $\sum_{n=0}^{\infty} a_n = 3 + \sum_{n=1}^{\infty} 3^n \Rightarrow a_0 = 3$

and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 3^n$

$\sum_{n=0}^{\infty} b_n = -2 + \sum_{n=1}^{\infty} 2^n \Rightarrow b_0 = -2$

and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 2^n$

The Series $\sum_{n=1}^{\infty} 3^n$ and $\sum_{n=1}^{\infty} 2^n$ are geometric Series with Common ratio greater than one and Both are divergent. $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are divergent series

$\sum_{n=0}^{\infty} C_n$ be Cauchy product of given series.

$C_0 = a_0 b_0 = 3(-2) = -6$

$C_n = a_0 b_n + \_\_\_\_ a_n b_0$

$= 3(2^n) + \_\_\_\_ 3^n(-2)$

$= 2^n \left[3 + 3/2 + \_\_\_ \left(3/2\right)^{n-1} \right] - 2 \cdot 3^n$

$2^n \left(3 + 3/2 \left((3/2)^{n-1} - 1 \right) \right) - 2 \cdot 3^n.$

$3 \cdot 2^n \left(\frac{3}{2}\right)^{n-1} - 2 \cdot 3^n = 2 \cdot 3^n - 2 \cdot 3^n = 0.$

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$\sum_{n=0}^{\infty} |C_n| = |-6| + \sum_{n=1}^{\infty} |C_n|$

$= 6 + 0 \_\_\_\_ = 6.$

Cauchy's Th^m |— Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely Convergent Series s.t. $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then the Cauchy product series $\sum_{n=1}^{\infty} C_n$ is also Bounded absolutely Cgt and Converges to AB.

Proof The Series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely Cgt $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ are Cgt.

Let $\sum_{n=1}^{\infty} |a_n|$ Converges to A' and $\sum_{n=1}^{\infty} |b_n|$ Converges to B'.

If $\langle A_n' \rangle$ and $\langle B_n' \rangle$ denote the seq^n of partial sum of $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$.

$\lim_{n\to\infty} A_n' = A'$ and $\lim_{n\to\infty} B_n' = B'$

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Cauchy's Th^m

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely cgt series st $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$.

Then Cauchy product series $\sum_{n=1}^{\infty} c_n$ is also absolutely cgt and cgt to AB.

Proof The series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely cgt $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ is cgt,

Let $\sum_{n=1}^{\infty} |a_n|$ converges to A' and $\sum_{n=1}^{\infty} |b_n|$ converges to B'

If A_n' and $\langle B_n' \rangle$ denote the seq^n of partial sums of $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$

$\lim_{n\to\infty} A_n' = A'$ $\quad \lim_{n\to\infty} B_n' = B'$

$\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ are series of positive term converging to A', B'.

$\sum_{n=1}^{\infty} |a_n| \sum_{n=0}^{\infty} |b_n| = \sum_{n=1}^{\infty} d_n$

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$d_n = |a_1||b_n| + \text{———} + |a_n||b_1|$ — (1).

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is cgt,

$\sum_{n=1}^{\infty} c_n$ is cgt

$|c_n| = |a_1 b_n + \text{———} + a_n b_1|$

$\leq |a_1||b_n| + \text{———} + |a_n||b_1| = d_n$

$|c_n| \leq d_n$

$\sum_{n=1}^{\infty} d_n$ is cgt By Comparison Test

$\sum_{n=1}^{\infty} |c_n|$ converges.

$\sum_{n=1}^{\infty} c_n$ converges to AB.

Let S_n be seq^n of partial sum of $\sum_{n=1}^{\infty} c_n$ and S_n' be sequential of partial sums of $\sum_{n=1}^{\infty} d_n$

$S_n = c_1 + \text{———} + c_n$

$= a_1 b_1 + \text{———} + a_n b_1$

$= a_1 (b_1 + \text{———} + b_n) + \text{———} + a_n b_1$

$A_n B_n = a_1 (b_1 + \text{———} + b_n) + \text{———} + a_n (b_1 + \text{———} + b_n)$

Subtracting (2) from (3).

$S_n - A_n B_n = -a_2 b_n - \dots - a_n (b_2 - b_n)$

$|S_n - A_n B_n| \le |a_2| |b_n| - \dots - |a_n| |b_2 - b_n|$

$|S_n - A_n B_n| \le A_n' B_n' - S_n'$.

$\lim_{n\to\infty} A_n' = A'$ $\lim_{n\to\infty} B_n' = B'$

$\lim_{n\to\infty} S_n' = A'B'$.

$|S_n - A_n B_n| \to 0$

$|A_n B_n - AB| \to 0$

$|S_n - AB| = |S_n - A_n B_n + A_n B_n - AB|$

$\le |S_n - A_n B_n| + |A_n B_n - AB|$

$|S_n - AB| \to 0$

$\lim_{n\to\infty} S_n = AB$.

S_n Converges to AB,

$\sum_{n=1}^{\infty} c_n$ Converges to AB,

Abel's Thm:-

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two C-series
& $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ Converges to
A and B. If Their Cauchy product
$\sum_{n=1}^{\infty} c_n$ Converges that it Converges to AB

Proof Let $\langle A_n \rangle$ $\langle B_n \rangle$ and $\langle S_n \rangle$ denote Seq^n
of partial sum of $\sum_{n=1}^{\infty} a_n$ $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$.

$\lim_{n\to\infty} A_n = A$ $\lim_{n\to\infty} B_n = B$.

$S_n = c_1 - \dots - c_n$

$= a_1 b_1 - \dots - a_n b_1$

$= a_1 (b_1 - \dots - b_n) - \dots - a_n b_1$

$= a_1 B_n - \dots - a_n B_1$ — (2)

Changing n to $(n+1)$ $(n-2)$ in (2)

$S_{n+1} = a_1 B_{n+1} - \dots - a_{n+1} B_1$

$S_1 = a_1 B_1$

on adding There

$S_1 - \dots - S_n = a_1 B_n - \dots - (a_1 - a_n) B_1$

$= A_1 B_n - \dots - A_n B_1$

$\dfrac{S_1 - S_n}{n} = \dfrac{A_1 B_n - A_n B_1}{n}$ (3)

$\sum_{n=1}^{\infty} c_n$ Converges

$\lim_{n\to\infty} S_n = S$.

$\lim_{n\to\infty} \frac{S_1 + \dots + S_n}{n} = S$.

B_n Converges to B

$\lim_{n\to\infty} \frac{AB_n + \dots + A_nB_1}{n} = AB$

$\lim_{n\to\infty} \frac{S_1 + \dots + S_n}{n} = \lim_{n\to\infty} \frac{AB_n + \dots + A_nB_1}{n}$

$= AB$

$S = AB$.

$\sum_{n=1}^{\infty} C_n$ Converges to AB.

Ques. If for $|x|<1$ the Series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely Cgt to $A(x)$ then Show that $(1-x)^{-1} A(x) = \sum_{n=0}^{\infty} s_n x^n$, $s_n = a_0 + \dots + a_n$

$\sum b_n = 1 + x + \dots + x^n, \sum_{n=0}^{\infty} x^n$

$|x|<1$ Cgs to $\frac{1}{1-x} = (1-x)^{-1}$

The Series $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n x^n$ is

absolutely Cgt to Say $A(x)$

The Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ Converges absolutely

and $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = (1-x)^{-1} A(x)$ —— (1)

where $c_n = \sum_{k=0}^{n} a_{n-k} b_k = \sum_{k=0}^{n} x^{n-k} a_k x^k$

$= x^n \sum_{k=0}^{n} a_k = x^n (a_0 + a_1 + a_2 + \dots + a_n)$

$= x^n s_n$ —— (2)

where $s_n = a_0 + a_1 + \dots + a_n$

$\therefore$ from (1) and (2), we have

for $|x|<1$

$$(1-x)^{-1} A(x) = \sum_{n=0}^{\infty} x^n . s_n \quad \text{—— (3)}$$

where

$s_n = a_0 + a_1 + \dots + a_n$

Hence the result

Deduction. Put $a_n = 1$ for all $n \geq 0$.

then

$s_n = 1 + 1 + 1 + \dots + 1 (n+1)$ times.

$= n+1$

$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$(1-x)^{-1}$$

from (8) we get

$$(1-x)^{-1}(1-x)^{-1} = \sum_{n=0}^{\infty} x^n (n+1)$$

$$\sum_{n=0}^{\infty} (n+1)x^n = (1-x)^{-2}$$

Chapter → 7.

Infinite Products

Ques → Show that the infinite product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)$ --- is convergent

Sol^n Let $\prod_{n=1}^{\infty}(1+a_n) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)$ ——

where

$$1+a_n = 1-\frac{1}{(n+1)^2} = \frac{(n+1)^2-1}{(n+1)^2}$$

$$= \frac{n^2+2n+1-1}{(n+1)^2}$$

$$= \frac{n^2+2n}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2} = \frac{n}{n+1} \cdot \frac{n+2}{n+1}$$

$$P_n = (1+a_1)(1+a_2)(1+a_3) \ldots (1+a_n)$$

$$= \left(\frac{1}{2}\cdot\frac{3}{2}\right)\cdot\left(\frac{2}{3}\cdot\frac{4}{3}\right)\cdot\left(\frac{3}{4}\cdot\frac{5}{4}\right) \cdots \left(\frac{n}{n+1}\cdot\frac{n+2}{n+1}\right)$$

$$= \frac{1}{2}\cdot\frac{n+2}{n+1} = \frac{1}{2}\left(1+\frac{1}{n+1}\right)$$

$$\lim_{n\to\infty} P_n = \frac{1}{2}$$

Hence, the given infinite product

is convergent.

Alternative Method →

$P = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) \ldots \infty$

$\log P = \log\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) \ldots$ to ∞

$= \log\left(1-\frac{1}{2^2}\right) + \log\left(1-\frac{1}{3^2}\right) + \log\left(1-\frac{1}{4^2}\right) + \ldots$ to ∞

$= \sum_{n=2}^{\infty} \log\left(1-\frac{1}{n^2}\right)$

$\log P = \sum_{n=2}^{\infty} a_n$ ——— (1)

$a_n = \log\left(1-\frac{1}{n^2}\right)$

$= \log(1-x)$ where $x = \frac{1}{n^2}$

$= -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right]$

$= -\left[\frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \ldots\right]$

$= -\frac{1}{n^2}\left[1 + \frac{1}{2n^2} + \frac{1}{3n^4} + \ldots\right]$

Let $b_n = \frac{1}{n^2}$

$\frac{a_n}{b_n} = -\left[1 + \frac{1}{2n^2} + \frac{1}{3n^4} + \ldots\right]$

Thus $\lim_{n\to\infty} \frac{a_n}{b_n} = -1$ which is non-zero & finite.

∴ By Comparison test, $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} b_n$ converge or diverge together.

But $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent by P-test.

$\sum_{n=2}^{\infty} a_n$ is also convergent to a finite no. S (say)

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$\therefore$ from (1), log P is converges to S

Hence, P is convergent to e^S.

Ques – Show that the infinite product $\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}$ converges as $n\to\infty$

Sol^n Let $P = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\cdots$ to ∞

$$\log P = \log\left(\frac{1}{2}\cdot\frac{3}{2}\right) + \log\left(\frac{3}{4}\cdot\frac{5}{4}\right) + \log\left(\frac{5}{6}\cdot\frac{7}{6}\right) + \cdots + \log\left(\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right)$$

$$= \sum_{n=1}^{\infty}\log\left(\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right)$$

$$= \sum_{n=1}^{\infty}\log\left(\frac{4n^2-1}{4n^2}\right) = \sum_{n=1}^{\infty}\log\left(1-\frac{1}{4n^2}\right)$$

$$= \sum_{n=1}^{\infty}\log(1-x), \text{ where } x = \frac{1}{4n^2}$$

$$= \sum_{n=1}^{\infty}\left[-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\text{ to }\infty\right)\right]$$

$$= \sum_{n=1}^{\infty}\left[-\left(\frac{1}{4n^2}+\frac{1}{32n^4}+\frac{1}{192n^6}+\cdots\right)\right]$$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{n^2}\left(\frac{1}{4}+\frac{1}{32n^2}+\frac{1}{192n^4}+\cdots\right)\right)$$

or $\log P = \sum_{n=1}^{\infty} a_n$ —— (1)

where

$$a_n = -\frac{1}{n^2}\left[\frac{1}{4}+\frac{1}{32n^2}+\frac{1}{192n^4}+\cdots\right]$$

Let $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = -\left[\frac{1}{4}+\frac{1}{32n^2}+\frac{1}{192n^4}+\cdots\right]$$

thus $\lim_{n\to\infty}\frac{a_n}{b_n} = -\frac{1}{4}$, which is non-zero and finite.

∴ By comparison test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

But $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-test.

∴ $\sum_{n=1}^{\infty} a_n$ is also convergent to a finite no. S (say)

∴ from (1), $\log P$ is convergent to S.

∴ P is convergent to e^S.

Hence, the given infinite product is convergent.

Show that the following infinite product are divergent:

(1) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$

the given infinite product =

$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{\infty}(1+a_n)$

where,

$1+a_n = 1+\frac{1}{n} = \frac{n+1}{n}$

Let

$P_n = (1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n)$

$= \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\ldots\left(\frac{n+1}{n}\right)$

$= n+1$

$\lim_{n\to\infty} P_n = \infty$

Hence, the given infinite product is divergent.

Q→ Discuss the convergence of the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$.

Sol^n the given infinite product

$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$

$= \prod_{n=1}^{\infty}(1+a_n)$ where $a_n = \frac{1}{n^2} > 0$ for all n

$\therefore \sum_{n=1}^{\infty} a_n$ and $\prod_{n=1}^{\infty}(1+a_n)$ converge or diverge together.

But $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-test

$\therefore$ Infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is convergent.

Hence, the given infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ is convergent.

Q) Discuss the convergence of the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^\alpha}\right)$, $0<\alpha<1$.

Solⁿ the given infinite product $= \prod_{n=1}^{\infty}\left(1+\frac{1}{n^\alpha}\right)$

$= \prod_{n=1}^{\infty}(1+a_n)$ where $a_n = \frac{1}{n^\alpha}$, $0<\alpha<1$

Here $a_n \geq 0$ for all n.

$\therefore$ infinite product $\prod_{n=1}^{\infty}(1+a_n)$ and series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

But $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges to ∞ as $0<\alpha<1$

Hence, the given infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^\alpha}\right)$ is divergent.

Ques Discuss the convergence of infinite product $\prod_{n=2}^{\infty}\left(1-\frac{1}{\sqrt{n}}\right)$

Solⁿ The infinite product $= \prod_{n=2}^{\infty}\left(1-\frac{1}{\sqrt{n}}\right)$

$\prod_{n=2}^{\infty}(1-b_n)$ $b_n = \frac{1}{\sqrt{n}}$

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$\sum_{n=2}^{\infty} b_n$ and $\prod_{n=2}^{\infty}(1-b_n)$ converge.

$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent

by p test.

$\therefore$ Infinite product $\prod_{n=2}^{\infty}(1-b_n)$

diverges to zero.

The given infinite product,

$\prod_{n=2}^{\infty}\left(1-\frac{1}{\sqrt{n}}\right)$ diverges to zero.

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Ques Show that $\prod_{n=2}^{\infty}\left(1-\left(1-\frac{1}{n}\right)^{-n} x^{n}\right)$.

Converges absolutely for $|x|>1$

Sol^n The given infinite product

$= \prod_{n=2}^{\infty}\left(1-\left(1-\frac{1}{n}\right)^{-n} x^{-n}\right)$.

$\prod_{n=2}^{\infty}(1+a_n) \qquad a_n = -\left(1-\frac{1}{n}\right)^{-n} x^{-n}$

$|a_n| = \left(1-\frac{1}{n}\right)^{-n} |x|^{-n}$

$|a_{n+1}| = \left(1-\frac{1}{n+1}\right)^{-(n+1)} |x|^{-(n+1)}$

$$\frac{|a_n|}{|a_{n+1}|} = \frac{\left(1-\frac{1}{n}\right)^{-n} |x|^{-n}}{\left(1-\frac{1}{n+1}\right)^{-(n+1)} |x|^{-(n+1)}}$$

$$= \frac{\left(1-\frac{1}{n+1}\right)^{n+1} \cdot |x|^{n+1}}{\left(1-\frac{1}{n}\right)^{n} |x|^{n}}$$

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$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \frac{e}{e}|x| = |x| > 1$$

By Ratio Test $\sum_{n=2}^{\infty} |a_n|$ is cgt.

Infinite product

$\prod_{n=2}^{\infty} (1 + a_n)$ Converges absolutely.

Infinite product $\prod_{n=2}^{\infty} \left(1 - \left(1 - \frac{1}{n}\right)^{x n^2}\right)^{-n}$

Converges absolutely for $|x| > 1$