

Page : Subhash

Date :

CLASS - Bsc II yr. (4th Sem)

Subject → Special function and
Integral Transformation
(Math)

Power Series-

Power series is an infinite series of form $\sum_{m=0}^{\infty} c_m (x-a)^m = c_0 + c_1 (x-a) + \dots$ (1).

Convergence of power series:-

$$\text{let } S_n(x) = c_0 + c_1 (x-a) + \dots + c_n (x-a)^n.$$

$$\sum_{m=0}^{\infty} c_m (x-a)^m = S_n(x) + R_n(x),$$

$R_n(x) = c_{n+1} (x-a)^{n+1} + \dots$ is called remainder after $c_n (x-a)^n$.

Interval of convergence:- Some time it may happen that it is only value of x for which power series is Cot. $|x-a| < R$

$$\frac{1}{R} = \lim_{m \rightarrow \infty} \left| \frac{c_{m+1}}{c_m} \right| \text{ or } \frac{1}{R} = \lim_{m \rightarrow \infty} \sqrt[m]{|c_m|}$$

Ques Consider the power series

$$\sum_{m=0}^{\infty} \frac{(3m)!}{(m!)^3} x^m.$$

$$\frac{1}{R} = \lim_{m \rightarrow \infty} \left| \frac{c_{m+1}}{c_m} \right| = \lim_{m \rightarrow \infty} \frac{(3m+3)!}{(m+1)!^3} \times \frac{(m!)^3}{(3m)!}$$

$$\lim_{m \rightarrow \infty} \frac{(3m+1)(3m+2)(3m+3)}{(m+1)^3}$$

$$\lim_{m \rightarrow \infty} \frac{(3 + \frac{1}{m})(3 + \frac{2}{m})(3 + \frac{3}{m})}{(1 + \frac{1}{m})^3}$$

! Radius of Convergence

Q7

operation on power series -
Termwise Differentiation -
 $\sum_{m=0}^{\infty} C_m (x-a)^m$ is a power series
which converges for

$$|x-a| < R \quad \text{if} \quad \sum_{m=0}^{\infty} C_m (x-a)^m$$

$$f'(x) = \sum_{m=1}^{\infty} m C_m (x-a)^{m-1} \text{ for } |x-a| < R$$

$$f''(x) = \sum_{m=2}^{\infty} m(m-1) C_m (x-a)^{m-2}$$

$$f'''(x) = \sum_{m=3}^{\infty} m(m-1)(m-2) C_m (x-a)^{m-3}$$

shifting of summation index -

$$S = \sum_{m=3}^{\infty} \frac{m!}{(m+4)!} x^{m+2}$$

$$m+2=R \quad k=1 \quad m=3$$

$$S = \sum_{k=1}^{\infty} \frac{(k+4)!}{(k+1)!} x^k$$

$$\sum_{m=3}^{\infty} \frac{m!}{(m+4)!} x^{m+2} = \frac{1}{72} x + \dots$$

$$\sum_{k=1}^{\infty} \frac{(k+2)!}{(k+1)!} x^k = \frac{3!}{1!} x + \dots$$

Ques write the following sum as a single series
 $x^2 \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} + 2 \sum_{m=1}^{\infty} m C_m x^{m-1}$

$$x^2 \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} \text{ can be written as}$$

$$1 \cdot x^2 \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} + 2 \sum_{m=1}^{\infty} m C_m x^{m-1}$$

$$\text{taking } m=1 \quad m-1=R$$

$$HMS = \sum_{k=2}^{\infty} k(k-1) C_k x^k + \sum_{k=1}^{\infty} 2 C_k x^k$$

$$= \sum_{k=2}^{\infty} k(k-1) C_k x^k + \sum_{k=1}^{\infty} 2 C_k x^k$$

$$= \sum_{k=2}^{\infty} k(k-1) C_k x^k + 2 C_1 x + 2 C_2 x^2 + \dots$$

Ques Express the sum $\sum_{m=2}^{\infty} m(m-1) x^{m-2}$ in term of x^k

$$\text{let } m+3=2k$$

$$\sum_{m=0}^{\infty} \frac{(m+3)!}{(m+2)!} x^{m+3} = \sum_{k=2}^{\infty} \frac{(k+3)!}{(k+2)!} x^k$$

Analytic function

A function defined in an interval containing the point a is said to be analytic at a if it can be represented by a power series of in powers of $x-a$ with radius of convergence $R > 0$.

Ordinary and Singular points of differential equation

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$$

Singular points are of two type

- (a) Regular
- (b) Singular

Ques. show that $x=0$ is an ordinary point of the differential equation

$$(x^2+1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - xy = 0$$

soln

Dividing the given differential eqn by (x^2+1) , we have

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1} \frac{dy}{dx} - \frac{x}{x^2+1} y = 0$$

Comparing it with

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$$

$$P(x) = \frac{x}{x^2+1} \quad Q(x) = -\frac{x}{x^2+1}$$

Existence of power series soln.

$$P(x) = \frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$$

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$$

Ques. find the power series soln of following differential eqn in powers of x

$$(x^2+1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = 0$$

The given differential eqn is

$$(x^2+1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = 0 \quad \text{--- (1)}$$

Clearly $x=0$ is an ordinary point. let $y = \sum_{n=0}^{\infty} a_n x^n$

$$y = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting value of y , $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$

$$\sum_{m=0}^{\infty} (m-1) a_m x^{m-1} = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}$$

shifting index of x to m in each term

$$\sum_{m=2}^{\infty} (m-1)(m-2) a_m x^{m-2} = \sum_{m=2}^{\infty} (m-1)(m-2) a_m x^{m-2}$$

$$-2a_2 + (a_2 - 6a_3)x' = 0$$

Equating to zero the coefficient of various power of x

$$-2a_2 = 0 \quad a_2 = 0$$

$$a_2 - 6a_3 = 0 \quad a_3 = \frac{a_2}{6}$$

$$8a_1 - 18a_4 = 0 \quad a_4 = \frac{4a_1}{6}$$

$$(m-1)(m-2) a_m = 0$$

$$(m-2) a_{m+2} = (m-1) a_{m-2} + (m-1)a'$$

$$a_{m+2} = \frac{m^2 - 2m + 2}{(m-2)(m+1)} a_m$$

$$m = 3, 4, 5$$

$$a_5 = \frac{9-6+2}{5 \cdot 4} a_2 = \frac{5}{20} a_2 = \frac{1}{4} a_2$$

$$a_6 = \frac{16-8+2}{6 \cdot 5} a_3 = \frac{10}{30} a_3 = \frac{1}{3} a_3$$

$$a_7 = \frac{25-10+2}{7 \cdot 6} a_4 = \frac{17}{42} a_4 = \frac{17}{84} a_1$$

Substituting these values in (1)

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y = a_0 + \frac{1}{6} x^3 + \dots + a_1 \left(x + \frac{x^2}{6} + \dots \right)$$

we obtain power series soln of

$$\frac{d^2 y}{dx^2} + (x-2) \frac{dy}{dx} + y = 0 \text{ in power of } (x-2)$$

The given differential Eqn is

$$\frac{d^2 y}{dx^2} + (x-2) \frac{dy}{dx} + y = 0$$

Comparing it with $\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$

$$P(x) = x-2 \quad Q(x) = 1$$

Both $P(x)$ and $Q(x)$ are analytic at $x=2$

$x=2$ is an ordinary point

The series power of $x-2$ The series

$$y = a_0 + a_1(x-2) + \dots = \sum_{m=0}^{\infty} a_m (x-2)^m$$

$$\frac{dy}{dx} = a_1 + 2a_2(x-2) + \dots = \sum_{m=1}^{\infty} m a_m (x-2)^{m-1}$$

$$\frac{dy}{dx} = 2a_2 + 3a_3(x-2)$$

$$= \sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2}$$

Substituting value of y & $\frac{dy}{dx}$ in eqn

$$\sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2}$$

$$2a_2 + 3a_3(x-2) = 0 \quad a_2 = a_3 = 0$$

$$(m+2) a_{m+2} = 0 \quad a_{m+2} = 0$$

$$(m+2) a_{m+2} = 0 \quad a_{m+2} = 0$$

$$a_{m+2} = a_{m+1} - a_m$$

$$a_{m+2} = \frac{a_{m+1} - a_m}{m+2}$$

$$a_{m+2} = \frac{a_{m+1} - a_m}{m+2}$$

which is recurrence relation

$$\text{Putting } n=2,3 \dots \infty$$

$$a_3 = \frac{a_2 - a_1}{3} = \frac{1}{3} \left(\frac{a_1 - a_0}{2} - a_1 \right)$$

$$= \frac{1}{6} a_0 - \frac{1}{2} a_1$$

$$a_4 = \frac{a_3 - a_2}{4} = \frac{1}{4} \left(\frac{1}{6} a_0 - \frac{1}{2} a_1 - \frac{a_1 - a_0}{2} \right)$$

$$= \frac{1}{12} a_0 - \frac{1}{6} a_1$$

$$= \dots$$

Substituting these values in

$$y = a_0 + a_1(x-2) + \frac{a_2(x-2)^2}{2} + \dots$$

$$y = a_0 \left(1 - \frac{1}{2}(x-2) \right)^2 = \dots$$

$$a_1 \left((x-2) + \frac{1}{2}(x-2)^2 \right) = \dots$$

Frobenius method for series when $x=0$ is a regular singularity of the eqn.

$$P(x) \frac{dy}{dx} + Q(x) \frac{dy}{dx} + R(x)y = 0$$

The given differential eqn is

$$P(x) \frac{dy}{dx} + Q(x) \frac{dy}{dx} + R(x)y = 0$$

$$\frac{dy}{dx} + \frac{P(x)}{x} \frac{dy}{dx} + \frac{Q(x)}{x^2} y = 0$$

$x=0$ is regular singularity

$$P(x) = A_0 + A_1 x + \dots$$

$$Q(x) = B_0 + B_1 x + \dots$$

Let the series be

$$y = x^m (a_0 + a_1 x + \dots) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + \dots$$

$$= \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$\frac{dy}{dx} = m(m-1) a_0 x^{m-2} + \dots$$

$$\sum_{n=0}^{\infty} (m+n) (m+n-1) a_n x^{m+n-2}$$

Substituting the value of $y, y', y'', \dots$ in $F(x)$ and $G(x)$ in (1), we have

$$\begin{aligned} & [m(m-1)a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \dots] \\ & + \left[\frac{A_0 + A_1 x + A_2 x^2 + \dots}{x} \right] (m a_0 x^{m-1} + (m+1) a_1 x^m + \dots) \\ & + \left[\frac{(B_0 + B_1 x + B_2 x^2 + \dots)}{x^2} \right] (a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots) = 0 \end{aligned}$$

Multiplying by x^2 , we have

$$\begin{aligned} & x^m [m(m-1)a_0 + (m+1)m a_1 x + \dots] + x^m [A_0 + A_1 x + A_2 x^2 + \dots] [m a_0 x^{m-1} + (m+1) a_1 x^m + \dots] \\ & + (B_0 + B_1 x + B_2 x^2 + \dots) x^m (a_0 + a_1 x + a_2 x^2 + \dots) = 0 \end{aligned}$$

$$\begin{aligned} & 0 + x^m [m(m-1)a_0 + m a_1 A_0 + a_0 B_0] + x^{m+1} [(m+1)m a_1 + m a_1 A_1 + (m+1)a_1 A_0 + (a_1 B_0 + a_0 B_1)] + \dots = 0 \end{aligned}$$

$$\begin{aligned} & 0 + x^m [m(m-1)a_0 + m a_1 A_0 + B_0] a_0 \\ & + x^{m+1} [(m+1)m a_1 + m a_1 A_1 + (m+1)a_1 A_0 + a_1 B_0 + a_0 B_1] + \dots = 0 \end{aligned}$$

which is an identity.
Equating to zero the coefficients of various powers of x , will give us the system of equations involving unknown coefficients $a_0, a_1, a_2, \dots$ etc.

Equating to zero the coefficients of smallest power of x we have

$$m(m-1) + mA_0 + B_0 = 0$$

which is a quadratic in m and is known as indicial equation whose solution will give us two values of m .

Let the roots be m_1 and m_2 . Further solution depends on the nature of roots. We discuss it in different cases, each case followed by an example but before that we give working rule for solving.

$$P(n) \frac{d^2 y}{dx^2} + Q(n) \frac{dy}{dx} + R(n) y = 0$$

about $x=0$ which is a regular singular point.

Solve $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + (1-x^2)y = 0$ in series.

The given differential equation is

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + (1-x^2)y = 0$$

$$\frac{d^2 y}{dx^2} - \frac{1}{x} \frac{dy}{dx} + \frac{(1-x^2)}{x^2} y = 0$$

Comparing it with $\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = -\frac{1}{x} \text{ and } Q(x) = \frac{1-x^2}{x^2}$$

$$(n-0)P(n) = -\frac{1}{2} \text{ and } (n-0)^2 Q(n) = \frac{(n^2-1)}{2}$$

$\therefore$ Both $P(n)$ and $Q(n)$ are analytic at $x=0$.
 $\therefore x=0$ is a regular singular point.

Let the solution be $y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$

$$a_2 x^2 + \dots = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots$$

$$(1) \quad y = \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} (m+r) a_r x^{m+r-1}$$

$$\text{and } \frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} (m+r)(m+r-1) a_r x^{m+r-2}$$

Substituting these values in (1), we have

$$x^m \sum_{r=0}^{\infty} (m+r)(m+r-1) a_r x^{m+r-2} + x^{m+r-2} x \sum_{r=0}^{\infty} (m+r)$$

$$a_r x^{m+r-1} + (1-x^2) \sum_{r=0}^{\infty} a_r x^{m+r} = 0.$$

$$\text{or } \sum_{r=0}^{\infty} (m+r)(m+r-1) a_r x^{m+r} - \sum_{r=0}^{\infty} (m+r)$$

$$a_r x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r} - \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

Shifting the index of last term to $m+r$, we have

$$\sum_{r=0}^{\infty} (m+r)(m+r-1) a_r x^{m+r} - \sum_{r=0}^{\infty} (m+r) a_r x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r} - \sum_{r=2}^{\infty} a_{r-2} x^{m+r} = 0.$$

Collecting the coefficients of various terms, the above equation can be written as

$$[2m(m-1) - m + 1] a_0 x^m + [2(m+1)m - (m+1) + 1] a_1 x^{m+1}$$

$$+ \sum_{r=2}^{\infty} [2(m+r)(m+r-1) a_r - (m+r) a_r + a_{r-2}] x^{m+r} = 0.$$

Equating to zero the coefficient of x^m i.e. lowest power of x , the indicial equation is

$$2m(m-1) - m + 1 = 0$$

$$2m^2 - 3m + 1 = 0 \Rightarrow m = \frac{1}{2}, 1$$

Roots of indicial equation are 1 and $\frac{1}{2}$. Hence the roots are distinct and these do not differ by an integer.

Equating to zero the coefficient of x^{m+1} , we have

$$[2(m+1)m - (m+1) + 1] a_1 = 0 \Rightarrow a_1 = 0$$

In general, equation to zeroth coefficient of x^{m+r} , $r \geq 2$, we have

$$[2(2m+r)(m+r-1) - (m+r+1)] a_r - a_{r-2} = 0, \quad r \geq 2$$

which gives $a_r = \frac{(m+r)(2m+r-3)+1}{(1+r)(2r-1)+1} a_{r-2}$, for $r \geq 2$

If $m=1$: $a_1 = 0$

$$a_r = \frac{a_{r-2}}{(1+r)(2r-1)+1} = \frac{a_{r-2}}{r(2r+1)}$$

Putting $r=2, 3, 4, \dots$ in (3), we get

$$a_2 = \frac{a_0}{2 \cdot 5}$$

$$a_3 = \frac{a_1}{3 \cdot 7} = 0$$

$$a_4 = \frac{a_2}{4 \cdot 9} = \frac{a_0}{(2 \cdot 5)(4 \cdot 9)}$$

$$a_5 = \frac{a_3}{5 \cdot 11} = 0$$

$$a_6 = \frac{a_4}{6 \cdot 13} = \frac{a_0}{(2 \cdot 5)(4 \cdot 9)(6 \cdot 13)} \text{ and so on.}$$

when $m=1$, the particular soln's

$$y = x^m (a_0 + a_1 x + \dots) \\ y = a_0 \left(x + \frac{x^3}{2 \cdot 5} + \dots \right) = u(x)$$

$$\text{If } m=1, \quad a_1 = 0 \\ a_2 = \frac{(1+2)(1-3)+1}{2} = \frac{1-2}{2} = -\frac{1}{2}$$

$$= \frac{2a_{r-2}}{2r(2r-1)} = \frac{a_{r-2}}{r(2r-1)}$$

put $r=2, 3, 4, \dots$ in (1)

$$a_2 = \frac{a_0}{2 \cdot 3}$$

$$a_3 = \frac{a_1}{3 \cdot 5} = 0$$

$$a_4 = \frac{a_2}{4 \cdot 7} = \frac{a_0}{(2 \cdot 3)(4 \cdot 7)}$$

$$a_5 = \frac{a_3}{5 \cdot 9} = 0$$

$$a_6 = \frac{a_4}{6 \cdot 11} = \frac{a_0}{(2 \cdot 3)(4 \cdot 7)(6 \cdot 11)}$$

$$m=1/2$$

$$y = x^{1/2} (a_0 + a_1 x + \dots)$$

$$y = x^{1/2} \left(a_0 + \frac{a_0}{8 \cdot 3} x^2 + \dots \right)$$

$$y = a_0 \left(x^{1/2} + \frac{x^{3/2}}{2 \cdot 3} + \dots \right) = u(x)$$

$$y = 4 \left(x + \frac{x^3}{8 \cdot 5} + \dots \right)$$

$$+ 8 \left(x^{1/2} + \frac{x^{3/2}}{3 \cdot 2} + \dots \right)$$

$$C_1 = 0, C_2 = 8, a_0 =$$

Case II

When the roots of an indicial eqn are $E_1 = m_1$, The Capable $2m_1$ is $y = (x-x_0)^{m_1} + \sum_{n=2}^{\infty} \left(\frac{d^n y}{dx^n} \right)_{m_1}$

C_1 and C_2 are arbitrary constant.

Thus find the series soln of following diff. eqn about 0.

$$(x-x_0) \frac{dy}{dx} + (1-5x) \frac{dy}{dx} - 4y = 0$$

The given differential eqn is,

$$(x-x_0) \frac{dy}{dx} + (1-5x) \frac{dy}{dx} - 4y = 0$$

$x=0$ is a regular singular point

$$y = x^m (a_0 + a_1 x + \dots)$$

$$= \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

put value of y $\frac{dy}{dx}$ $\frac{dy}{dx}$

$$\begin{aligned}
 & (x-k) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (m+j) (m+i) a_{k+i} x^{m+i} a_{k+j} x^{m+j} \\
 & + (1-k) \sum_{i=0}^{\infty} (m+i) a_{k+i} x^{m+i} \\
 & = 4 \sum_{k=0}^{\infty} a_{k+m} x^{2m}
 \end{aligned}$$

shifting the index of x to $m+k$

$$\sum_{k=0}^{\infty} (m+k+1) (m+k) a_{k+m} x^{m+k} + \sum_{k=0}^{\infty} (m+k) a_{k+m} x^{m+k} = 0$$

$$(m+m-1) a_0 x^{m-1} + \sum_{k=1}^{\infty} (m+k+1) (m+k) a_{k+m} x^{m+k} + \sum_{k=0}^{\infty} (m+k) a_{k+m} x^{m+k} = 0$$

$$- \sum_{k=1}^{\infty} (m+k) (m+k-1) a_{k+m} x^{m+k} + \sum_{k=0}^{\infty} (m+k) a_{k+m} x^{m+k} = 0$$

$$\left[m a_{m-1} + \sum_{k=1}^{\infty} (m+k) a_{k+m} \right] x^{m-1} + \sum_{k=0}^{\infty} (m+k) a_{k+m} x^{m+k} = 0$$

$$(m(m-1) + m) a_{m-1} + \sum_{k=1}^{\infty} (m+k) a_{k+m} x^{m+k} = 0$$

equating coefficient of x^{m-1} to zero

$$(m(m-1) + m) a_{m-1} + \sum_{k=1}^{\infty} (m+k) a_{k+m} x^{m+k} = 0$$

$$(m(m-1) + m) a_{m-1} + \sum_{k=1}^{\infty} (m+k) a_{k+m} x^{m+k} = 0$$

$$(m+m+1) a_{m+1} = (m+m) (m+m-1) + 5 (m+m) + 4 a_0$$

$$(m+m+1) a_{m+1} = (m+m) a_{m+1} + 4 (m+m) + 4 a_0$$

$$a_{m+1} = \frac{(m+m+1) a_{m+1}}{(m+m+1)} + 4 a_0 \quad k \geq 0 \quad \text{--- (1)}$$

$$a_1 = \frac{(m+2) a_1}{m+1} + 4 a_0 \quad \text{in (1)}$$

$$a_2 = \frac{(m+3) a_2}{m+2} + 4 a_1$$

$$a_3 = \frac{(m+4) a_3}{m+3} + 4 a_2 = \left(\frac{m+4}{m+1} \right) a_0$$

The series is

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$y(x) = x^m \left(a_0 + \sum_{k=1}^{\infty} \left(\frac{m+k}{m+1} \right)^k a_0 x^k \right)$$

put $m=0$ in (5)

$$y(x) = a_0 (1 + x + x^2 + \dots) = a_0 \sum_{k=0}^{\infty} x^k$$

$$\left(\frac{dy}{dx} \right) = a_0 x^m \log x \left(1 + \frac{m+2}{m+1} x + \dots \right)$$

$$+ a_0 x^m \left[\frac{(m+2)(1)}{(m+1)^2} x + \dots \right]$$

$$\left(\frac{dy}{dx} \right)_{m=0} = a_0 \log x (1 + 2x + \dots)$$

$$+ 2a_0 \left[\frac{2(1-x)}{1} x + 3(1-2)x^2 + \dots \right]$$

$$a_0 (1.2x + 2.3x^2 + \dots)$$

$$= a_0 (1.2x + 2.3x^2 + \dots)$$

The Complete solⁿ is

$$y = C_1 y_{hom} + C_2 \left(\frac{dy}{dx} \right)_{m=0}$$

$$y = C_1 a_0 u(x) + C_2 a_0 v(x)$$

$$y = C_1 u(x) + C_2 v(x)$$

$$u(x) = (1+2x) \dots$$

$$v(x) = x \log x - 2(1+2x) + 2.3x^2 + \dots$$

Case II When roots of indicial Equation are unequal and differ by an integer and making coefficients of y infinite

Step 2. Let the solⁿ $y = x^m (a_0 + a_1 x + \dots)$
Step 3. $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Step 3. obtain the indicial Equation by Equating the lowest powers of x to zero.

Step 4. Let the root of indicial Equation be m_1 and m_2 i.e. there differ by an integer $m_2 - m_1$

Step 5. Equate coefficient of other power of x to zero to obtain coefficients $a_1, a_2, \dots$ and then solⁿ in terms of a_0

Step 6. observe that some coefficient of y become infinite when we put $m = m_2$

Step 7. Replace a_0 by $b_0 (m - m_2)$ now put $m = m_2$. This gives modified form of $y = u(x)$. as a solⁿ. Solⁿ is obtained from $\left(\frac{dy}{dx} \right)_{m=m_2}$ at $m = m_2$.

Ques find the Series solution of following differential Equation about 0.

$$x \frac{dy}{dx} - 3 \frac{dy}{dx} = y \ln x$$

$$\frac{dy}{dx} - 3 \frac{dy}{dx} = y \ln x$$

The given Equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$

$$\frac{dy}{dx} - 3 \frac{dy}{dx} = y \ln x$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = -3, Q(x) = y \ln x$$

It is not an ordinary point of O . It is not analytic at O .

It is not analytic at O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

It is a regular singular point of O .

$$-3 \sum_{k=0}^{\infty} (m+k) a_k x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} (m+k) (m+k-1) a_k x^{m+k-2} + \sum_{k=0}^{\infty} a_k x^{m+k-1} = 0$$

$$(m-1) a_0 x^{m-1} + (m-1) (m-2) a_1 x^m + \dots = 0$$

$$\sum_{k=2}^{\infty} (m+k) (m+k-1) a_k x^{m+k-2} + \sum_{k=0}^{\infty} a_k x^{m+k-1} = 0$$

Equating Coefficient of lowest power of x to zero.

$$(m-1) a_0 = 0, (m-1) a_1 = 0$$

which is identical eqn whose root are $0, 1$

Equating Coeff of next lowest power $a_1 = 0$

Equating to zero. the coeff of x^{m+k-1}

$$(m+k) (m+k-1) a_k + a_{k-1} = 0$$

$$a_k = -\frac{a_{k-1}}{(m+k) (m+k-1)}$$

$$a_2 = -\frac{a_1}{(m+2) (m+1)}$$

$$a_3 = -\frac{a_2}{(m+3) (m+2)}$$

$$a_4 = -\frac{a_3}{(m+4) (m+3)}$$

$$a_5 = -\frac{a_4}{(m+5) (m+4)}$$

$$a_6 = -\frac{a_5}{(m+6) (m+5)}$$

$$a_7 = -\frac{a_6}{(m+7) (m+6)}$$

$$a_8 = -\frac{a_7}{(m+8) (m+7)}$$

$$a_9 = -\frac{a_8}{(m+9) (m+8)}$$

$$a_{10} = -\frac{a_9}{(m+10) (m+9)}$$

$$a_{11} = -\frac{a_{10}}{(m+11) (m+10)}$$

$$a_{12} = -\frac{a_{11}}{(m+12) (m+11)}$$

Substituting these value in (3)

$$y = x^m (a_0 + 0 \cdot x - \frac{a_0}{(m+2)(m-2)} x^2 + \dots)$$

$$y(x) = b_0 x^m \left(m - \frac{m x^2}{(m+2)(m-2)} + \dots \right)$$

$$b_0 = 1$$

$$y(x) = x^m \left(m - \frac{m x^2}{(m+2)(m-2)} + \frac{x^4}{(m+2)(m-2)(m-4)} \right)$$

$$\text{Let } u(x) = y(x)_{m=0}$$

$$u(x) = x^0 \left(0 - 0 + \frac{x^4}{(-2) \cdot 4} - \frac{x^8}{(-2) \cdot 2 \cdot 4 \cdot 6} + \dots \right)$$

$$u(x) = -\frac{x^4}{2 \cdot 4} + \frac{x^8}{2 \cdot 4 \cdot 6} + \dots$$

Differentiating $y(x)$ partially w.r.t m .

$$\frac{\partial y(x)}{\partial m} = x^m \log x \left(m - \frac{m x^2}{(m+2)(m-2)} + \frac{x^4}{(m+2)(m-2)(m-4)} \right)$$

$$v(x) = \left(\frac{\partial y}{\partial m} \right)_{m=0}$$

$$v(x) = u(x) \log x + 1 + \frac{4x^2}{2 \cdot 2 \cdot 2} + \dots$$

$$u(x) = u(x) \log x + 1 + \frac{1}{2 \cdot 2} x^2 + \dots$$

Case IV

Ans

When the root of identical Eqn are unequal and differ by an integer and making a coefficient indeterminate.

The coefficient by putting $m=m_1$ a coefficient of y become indeterminate, the complete soln is obtained by put $m=m_2$ in y .

By putting $m=m_1$, we get series soln which is a constant multiple of one of the series obtained in first soln.

$$\text{Solve } (1-x^2) \frac{dy}{dx} - x \frac{dy}{dx} + 4xy = 0 \text{ in } \frac{dy}{dx}$$

Series about $x=0$

$$\text{The given Eqn is } (1-x^2) \frac{dy}{dx} - x \frac{dy}{dx} + 4xy = 0$$

Here $x=0$ is an ordinary point of (1)

$$\text{Let the soln be } y = x^m (a_0 + a_1 x + \dots) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k) a_k x^{m+k-1}$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k) (m+k-1) a_k x^{m+k-2}$$

$$\text{Substituting these value in (1)} \\ (1-x^2) \sum_{k=0}^{\infty} (m+k) (m+k-1) a_k x^{m+k-2}$$

$$-x \sum_{k=0}^{\infty} (m+k) a_k x^{m+k-1} + 4 \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} (m+k) (m+k-1) a_k x^{m+k-2} - \sum_{k=0}^{\infty} (m+k) a_k x^{m+k} + 4 \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$m(m-1) a_0 x^{m-2} + (m-1) m a_1 x^{m-1} + \dots + 4 a_0 x^m + 4 a_1 x^{m+1} + \dots = 0$$

Equating to zero
 $m(m-1) = 0$ $m=1$ or $m=0$
 Equating coefficient of x^{m-1} to 0.
 $(m+1) a_1 = 0$

a_1 to be finite and find all other coefficient in terms of a_0 and a_1

Equating the coefficient of general term to zero.

$$(m+k+2) (m+k+1) a_{k+2} - (m+k) (m+k-1) a_k + 4 (m+k-1) a_k = 0$$

$$a_{k+2} = \frac{(m+k+2) (m+k-1) a_k}{(m+k+2) (m+k+1)}$$

for $m=0$ $a_{k+2} = \frac{k(k-1)}{(k+2) (k+1)}$

$$a_2 = \frac{-4 a_0}{2} = -2 a_0$$

$$a_3 = \frac{-3}{3 \cdot 2} a_1 = -\frac{1}{2} a_1$$

$$a_4 = \frac{0}{4 \cdot 3} a_2 = 0$$

$$a_5 = \frac{5}{5 \cdot 4} a_3 = \frac{1}{4} a_3 = -\frac{1}{8} a_1$$

$$a_6 = \frac{12}{6 \cdot 7} a_4 = 0$$

$$a_7 = \frac{21}{7 \cdot 6} a_5 = \frac{1}{2} a_5 = -\frac{1}{16} a_1$$

$$y = x^0 (a_0 + a_1 x - 2 a_0 x^2 - \frac{1}{2} a_1 x^3 - \frac{1}{8} a_1 x^5 - \frac{1}{16} a_1 x^7 + \dots)$$

Ques Solve $x \frac{dy}{dx} - y - x = 0$ in power series of $(x+2)$

A The given differential eqn is $x \frac{dy}{dx} - y - x = 0$

Put $x = z$ $x = 1 + z$ $\frac{dz}{dx} = 1$

$x \frac{dy}{dx} - y - x = 0$

$z \frac{dy}{dz} - y - z = 0$

$(1+z) \frac{dy}{dz} - y - z = 0$

Its series soln about $z=0$ is an ordinary point let soln be

$y = \sum_{n=0}^{\infty} a_n z^n$

$\frac{dy}{dz} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$

$(1+z) \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} z^n = 0$

$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} z^{n+1} - \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} z^n = 0$

$(1+z) \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} z^n = 0$

$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} z^{n+1} - \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} z^n = 0$

$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=1}^{\infty} n a_n z^n - \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} z^n = 0$

$-(z+2) = 0$

$[a_1 \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=1}^{\infty} n a_n z^n - \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} z^n] = 0$

$- [a_0 + \sum_{n=1}^{\infty} a_n z^n] - (z+2) = 0$

$a_1 - a_0 + \sum_{n=1}^{\infty} (n+1) a_{n+1} z^n + n a_n z^n - a_n z^n - z - 2 = 0$

$(a_1 - a_0 - 2) + \sum_{n=1}^{\infty} (n+1) a_{n+1} z^n$

$+ (n+1) a_{n+1} z^n - z - 2 = 0$

Equating the constant term to zero

$a_1 - a_0 - 2 = 0$ $a_1 = a_0 + 2$

$2a_2 - 2 = 0$ $a_2 = 1$

Equating the coefficient of z to zero we have

$$2g_1 = 0 \quad g_2 = \frac{1}{2}$$

Equating coefficient of general

term $\sum m$ to zero.

$$(m+1)am + (m+1)am = 0$$

$$\lim_{m \rightarrow \infty} = -\frac{(m+1)}{(m+1)} a_m$$

$$m = 2, 3, 4, \dots$$

$$g_3 = -\frac{1}{3!} a_3 = -\frac{1}{3!} = \frac{(-1)^3}{3!}$$

$$g_4 = -\left(\frac{1}{4}\right) a_4 = \frac{1}{4!} = \frac{(-1)^4}{4!}$$

Put these value of coeff in (3)

$$y = a_0 + (a_0 + 2)z + \frac{1}{2}z^2 + (-1)^3 \frac{1}{3!}z^3$$

$$+ (-1)^4 \frac{1}{4!}z^4 + (-1)^5 \frac{1}{5!}z^5$$

$$y = a_0 + (a_0 + 2)z + \frac{1}{2}z^2 + \frac{(-1)^3}{3!}z^3 + \frac{(-1)^4}{4!}z^4 + \frac{(-1)^5}{5!}z^5$$

Bessel's Equation and Bessel's function.

Beta function

The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$

Properties of Beta function

$$(1) B(m, n) = B(n, m)$$

$$(2) B(m, n) = \frac{(m-1)! (n-1)!}{(m+n-1)!}$$

$$(3) B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$(4) B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$(5) B(p, q+1) = \frac{B(p, q+1)}{p}$$

Gamma function

The integral $\int_0^\infty x^{n-1} e^{-x} dx$ is function of n .

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx$$

Recurrence formula for Gamma function.

$$\Gamma(n+1) = n \Gamma(n)$$

$$\Gamma(n) = n(n-1) \Gamma(n-1)$$

$$= n(n-1)(n-2) \Gamma(n-2)$$

Continuing like this.

$$\Gamma(n+1) = n(n-1)(n-2) \dots \Gamma(1)$$

$$= n! \quad \Gamma(1) = 1$$

$$\Gamma(n) = (n-1) \Gamma(n-1) \text{ and } \Gamma(n) = (n-1)!$$

Relation b/w Beta Gamma function

$$\Gamma(m) \Gamma(n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

Some important results

$$\Gamma(1/2) = \sqrt{\pi}$$

$$(1) \int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}$$

$$(2) \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$(3) \int_0^{\pi/2} \sin(x) \cos^2(x) dx = \frac{1}{2} \left[\frac{\sin^2(x)}{2} \right]_0^{\pi/2}$$

Bessel's Equation

The differential eqn of form

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$$

Soln of Bessel's Equation.

Bessel's eqn is $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$

$$y = x^n (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$y = \sum_{k=0}^{\infty} a_k x^{n+k}$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} (n+k) a_k x^{n+k-1}$$

$$\frac{d^2 y}{dx^2} = \sum_{k=0}^{\infty} (n+k-1)(n+k) a_k x^{n+k-2}$$

$$\frac{dy}{dx} = \sum_{h=0}^{\infty} a_h (n+1)^h$$

$$\frac{dy}{dx} = \sum_{h=0}^{\infty} a_h (n+1)^h$$

$$y = \sum_{h=0}^{\infty} \frac{a_h}{h!} (n+1)^h$$

$$n^2 \sum_{h=0}^{\infty} a_h (n+1)^h = \sum_{h=0}^{\infty} a_h (n+1)^{h+2}$$

$$+ n \sum_{h=0}^{\infty} a_h (n+1)^h = \sum_{h=0}^{\infty} a_h (n+1)^{h+1}$$

$$\sum_{h=0}^{\infty} a_h n^2 (n+1)^h = \sum_{h=0}^{\infty} a_h n^2 (n+1)^{h+2}$$

$$\sum_{h=0}^{\infty} a_h (n+1)^h = \sum_{h=0}^{\infty} a_h (n+1)^{h+2} + \sum_{h=0}^{\infty} a_h (n+1)^{h+1}$$

$$+ \sum_{h=0}^{\infty} a_h n^2 (n+1)^h = \sum_{h=0}^{\infty} a_h n^2 (n+1)^{h+2} + \sum_{h=0}^{\infty} a_h n^2 (n+1)^{h+1}$$

$$\sum_{h=0}^{\infty} (n+1)^h (n+1)^h = \sum_{h=0}^{\infty} (n+1)^{h+2} + \sum_{h=0}^{\infty} (n+1)^{h+1}$$

$$a_h n^2 (n+1)^h + \sum_{h=0}^{\infty} a_h n^2 (n+1)^{h+2} = \sum_{h=0}^{\infty} a_h n^2 (n+1)^{h+2}$$

$$\sum_{h=0}^{\infty} (n+1)^h = \sum_{h=0}^{\infty} (n+1)^{h+2} + \sum_{h=0}^{\infty} (n+1)^{h+1}$$

$$(n^2 - n^2) a_h n^2 + (n+1)^h = \sum_{h=0}^{\infty} a_h n^2 (n+1)^{h+2}$$

$$+ \sum_{h=0}^{\infty} (n+1)^h - n^2 a_h + a_h (n+1)^h = 0$$

Equating coefficient of lowest power of n to zero.

$$n^2 - n^2 = 0$$

$$(n+1)^2 - n^2 = 0$$

Equating the coefficient of general term

$$(n+1)^2 - n^2 + a_h = 0$$

$$a_h = 0$$

$$(n+1)^2 - n^2$$

$$h = 2, 3, 4, \dots$$

$$a_h = \frac{a_0}{(n+1)^2 - n^2}$$

$$a_h = \frac{a_1}{(n+1)^2 - n^2} = 0$$

Substituting these values in (2)

$$y = a_0 n^2 \left(1 - \frac{1}{(n+1)^2 - n^2} \right)$$

putting $m = n$ in (4)

$$y = u(x) = a_0 x^n \left(1 - \frac{1}{4(n+1)} x^2 \right)$$

putting $m = n$ in (4)

$$y = u(x) = a_0 x^n \left(1 - \frac{1}{4(n+1)} x^2 \right)$$

$$y = \dots$$

If n is not an integer, series is

The series of Bessel's Equation

$$y = y_1 u(x) + y_2 v(x)$$

y_1 and y_2 are arbitrary

Bessel's function

Can be if n is not an integer, then the complete series of Bessel eqn.

$$y = y_1 u(x) + y_2 v(x)$$

$$a_0 = \frac{1}{2^n} \times \frac{1}{\Gamma(n+1)}$$

$$a_0 = \frac{1}{2^n(n+1)!} \text{ in (5)}$$

$$J_n(x) = \left(\frac{x}{2} \right)^n \left(\frac{1}{\Gamma(n+1)} \right) \left(1 - \frac{1}{4(n+1)} x^2 + \dots \right)$$

$$\left(\frac{x}{2} \right)^n \left[\frac{1}{\Gamma(n+1)} - \dots \right]$$

$$J_n(x) = \left(\frac{x}{2} \right)^n \left(\frac{1}{\Gamma(n+1)} - \frac{1}{4(n+1)} x^2 + \dots \right)$$

$$J_{-n}(x) = \left(\frac{x}{2} \right)^{-n} \left(\frac{1}{\Gamma(n+1)} - \frac{1}{4(n+1)} x^2 + \dots \right)$$

$$J_n(x) = \frac{1}{\Gamma(n+1)} \left(\frac{x}{2} \right)^n - \frac{1}{4(n+1)} \left(\frac{x}{2} \right)^{n+2} + \dots$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+1-k+1)} \left(\frac{x}{2} \right)^{n+2k}$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+1-k+1)} \left(\frac{x}{2} \right)^{n+2k}$$

Thm To show that $J_n(x) = (-1)^n J_0(x)$ when n is any integer

Proof If n is a positive integer

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$$J_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k} + \sum_{k=n}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$$J_n(x) = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

Let $n+k$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$$(1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

Deduction of Bessel functions

$J_0(x)$ $J_1(x)$ in form of series

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$n=0$ in (1)

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$$J_0(x) = 1 - \frac{(x)^2}{2^2} + \frac{(x)^4}{2^4} - \frac{(x)^6}{2^6} + \dots$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4} - \frac{x^6}{2^6} + \dots$$

$n=1$ in (1)

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(n-k+1)} \left(\frac{x}{2}\right)^{n-2k}$$

$$J_1(x) = \frac{x}{2} \left(1 - \frac{(x)^2}{2^2} + \frac{(x)^4}{2^4} - \dots \right)$$

Putting $n = \frac{1}{2}$ in (1)

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+1/2}}{\sqrt{1+x^2}}$$

$$J_{1/2}(x) = \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+1}}{\sqrt{1+x^2}}$$

$$J_{1/2}(x) = \frac{1}{\sqrt{x}} \left[\frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]$$

$$= \frac{1}{\sqrt{x}} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right]$$

$$= \frac{1}{\sqrt{x}} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right]$$

$$= \frac{1}{\sqrt{x}} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right]$$

$$= \frac{1}{\sqrt{x}} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$n = \frac{1}{2} \text{ in (2)}$$

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+1/2}}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{x}} \left(\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)$$

$$= \frac{1}{\sqrt{x}} \left(\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \right)$$

$$= \frac{1}{\sqrt{x}} \left(\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \right)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Reurrence relation for Bessel function

To show that $\frac{d}{dx} (x^n J_n(x))$

$$= x^n J_{n-1}(x)$$

$$\frac{d}{dx} (x^n J_n(x)) = x^n J_{n+1}(x)$$

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$J_n'(x) = \frac{n}{x} J_n(x) + J_{n-1}(x)$$

$$J_n'(x) = \frac{1}{2} (J_{n-1}(x) - J_{n+1}(x))$$

$$\frac{1}{2} J_n'(x) = J_{n+1}(x) + J_{n-1}(x)$$

$$J_n'(x) = \frac{n}{x} (J_n(x) + J_{n-1}(x))$$

$$J_{n+1}(x) = \frac{n}{x} (J_n(x) - J_{n+1}(x))$$

$$\frac{d}{dx} (x^n J_n(x)) = \frac{d}{dx} (x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x}{2}\right)^{2k+n})$$

$$= \frac{d}{dx} \left(\frac{x^{2n+2}}{2^{n+1}} \right)$$

$$= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x^{2k+n+1}}{2^{n+1}} \right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x^{2k+n}}{2^{n+1}} \right)$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x^{2k}}{2^{n+1}} \right)$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x^{2k}}{2^{n+1}} \right)$$

$$\frac{d}{dx} (x^n J_n(x)) = x^n J_{n+1}(x)$$

Ques Show that

① $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

A $\frac{d}{dx} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$

$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

put $n = 1/2$ in (1)

$J_{3/2}(x) = \frac{1}{2} \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

$= \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

② $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{3-\cos x}{x^2} \sin x - \frac{3}{x} \cos x \right)$

A $n = 3/2$ in (1)

$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$

$= \frac{3}{x} \left(\sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) \right) - \sqrt{\frac{2}{\pi x}} \sin x$

$= \sqrt{\frac{2}{\pi x}} \left(\frac{3-\cos x}{x^2} \sin x - \frac{3}{x} \cos x \right)$

③ $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$
 $n = 1/2$

$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$

$= -\frac{1}{x} \sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$

$= \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$

Generating func for $J_n(x)$.

To prove that $e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) \cdot t^n$.

$$e^{x/2(t-1/t)} = e^{x/2} \cdot e^{-x/2t}$$

$$= \left[1 + \frac{x}{2}t + \frac{1}{2!} \left(\frac{x}{2}t \right)^2 + \dots \right] \times \left[1 - \frac{x}{2t} + \frac{1}{2!} \left(\frac{x}{2t} \right)^2 - \dots \right]$$

$$= \left[1 + \frac{x}{2}t + \dots + \frac{1}{n!} \left(\frac{x}{2} \right)^n t^n + \dots \right] \left[1 - \frac{x}{2t} + \frac{1}{2!} \left(\frac{x}{2} \right)^2 \frac{1}{t^2} - \dots \right]$$

Multiplying and coefficient of

$$\text{Coeff of } t^0 = 1 - \frac{x}{2} + \frac{1}{2!} \left(\frac{x}{2} \right)^2 - \dots$$

$$= J_0(x)$$

$$\text{Coeff of } t^n = \frac{1}{n!} \left(\frac{x}{2} \right)^n + \frac{1}{(n+1)!} \left(\frac{x}{2} \right)^{n+1} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x}{2} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x}{2} \right)^n$$

$$\text{Coeff of } \frac{1}{t^n} = \frac{(-1)^n}{n!} \left(\frac{x}{2} \right)^n + \dots$$

$$= (-1)^n \left(\frac{1}{n!} \left(\frac{x}{2} \right)^n + \frac{(-1)^{n+1}}{(n+1)!} \left(\frac{x}{2} \right)^{n+1} + \dots \right)$$

$$= (-1)^n \sum_{n=0}^{\infty} \frac{(-1)^k}{n!} \frac{1}{(n+k)!} \left(\frac{x}{2} \right)^{n+k}$$

$$= (-1)^n J_n(x) = J_{-n}(x)$$

$$e^{x/2(t-1/t)} = J_0(x) + J_1(x)t + \dots$$

$$= \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Representation of $J_n(x)$ in integral

To prove that $J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta) \cos(x \sin \theta) d\theta$

$$= x \sin \theta d\theta$$

∴ integral value of n

$$e^{x/2} \left(t - \frac{1}{t} \right) J_0(x) + J_1(x)t + \dots \quad (1)$$

$$J_n(x) = (-1)^n J_n(x)$$

$$J_{-n}(x) = (-1)^n J_n(x)$$

we it in (1)

$$e^{x/2} \left(t + \frac{1}{t} \right) = J_0(x) + J_1(x)t + \dots$$

$$= J_0(x) + \left(t - \frac{1}{t} \right) J_1(x) + \dots \quad (2)$$

$$t = \cos \theta + i \sin \theta, \quad \frac{1}{t} = \cos \theta - i \sin \theta$$

$$t - \frac{1}{t} = 2i \sin \theta \quad (3)$$

$$t^n = \cos n\theta + i \sin n\theta$$

$$\frac{1}{t^n} = \cos n\theta - i \sin n\theta$$

$$t^n + \frac{1}{t^n} = 2 \cos n\theta \quad (4)$$

$$t^n - \frac{1}{t^n} = 2i \sin n\theta \quad (5)$$

we (3) (4) and (5)

$$e^{x/2} 2i \sin \theta = J_0(x) + 2i \sin \theta J_1(x) + 2 \cos 2\theta J_2(x) + \dots$$

$$\cos(x \sin \theta) + i \sin(x \sin \theta) = J_0(x) + 2(J_2(x) \cos 2\theta + \dots)$$

Equating Real and Imaginary part -

$$\cos(x \sin \theta) = J_0(x) + 2J_2(x) \cos 2\theta + \dots$$

$$\sin(x \sin \theta) = 2J_1(x) \sin \theta + 2J_3(x) \sin 3\theta + \dots$$

(2) and (3) are Jacobi series

Multiplying Both Side of (2) by $\cos n\theta$ and integrating within limit 0 to π

$$\int_0^\pi \cos(x \sin \theta) \cos n\theta d\theta =$$

$$\int_0^\pi J_0(x) \cos n\theta + 2J_2(x) \cos 2\theta + \dots$$

$$= \int_0^\pi J_0(x) \cos n\theta d\theta + 2 \int_0^\pi J_2(x) \cos 2\theta d\theta + \dots$$

$$\int_0^\pi J_0(x) \cos n\theta d\theta = \int_0^\pi J_0(x) \sin n\theta d\theta = 0$$

$$\int_0^\pi J_2(x) \cos 2\theta d\theta = \frac{1}{2} \int_0^\pi J_2(x) \cos 2\theta d\theta = \frac{1}{2} \int_0^\pi J_2(x) \cos 2\theta d\theta$$

$$\int_0^{\pi} \cos(n \sin \theta) d\theta = \pi J_0(n)$$

Ques Establish the Jacobie Series

$$\cos(n \cos \theta) = J_0(n) - 2J_2(n) \cos 2\theta + 2J_4(n) \cos 4\theta - \dots$$

Ans We know that

$$\cos(n \sin \theta) = J_0(n) + 2J_2(n) \cos 2\theta - 2J_4(n) \sin 2\theta + \dots$$

In Both of these Changing θ to $\frac{\pi}{2} - \theta$

$$\cos(n \cos \theta) = J_0(n) + 2J_2(n) \cos 2\theta - 2J_4(n) \sin 2\theta + \dots$$

$$\sin(n \cos \theta) = 2J_1(n) \sin \theta - 2J_3(n) \sin 3\theta + \dots$$

$$\cos(n \cos \theta) = J_0(n) - 2J_2(n) \cos 2\theta + 2J_4(n) \cos 4\theta - \dots$$

$$\sin(n \cos \theta) = 2J_1(n) \cos \theta - 2J_3(n) \cos 3\theta + \dots$$

$$J_3(n) \cos 3\theta - \dots$$

Equations reducible to Bessel's Equation.

To reduce the differential Eqⁿ

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

The given differential Eqⁿ is in the form of Bessel's Eqⁿ

$$\text{put } t = x^2 \quad \frac{dt}{dx} = 2x$$

$$\frac{dy}{dx} = 2x \frac{dy}{dt} \quad \frac{d^2 y}{dx^2} = 2 \frac{dy}{dt} - t \frac{d^2 y}{dt^2}$$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

$$\text{Eqⁿ (1) } t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - n^2)y = 0$$

$$t(t^2 - n^2)y = 0$$

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - n^2)y = 0$$

Which is Bessel's Eqⁿ of order n

$$y = C_1 J_n(t) + C_2 J_n(t)$$

$$= C_1 J_n(t) + C_2 J_n(t)$$

Soln of Eqn is

$$y = C_1 J_n(x) + C_2 J_n(x)$$

$$= C_1 J_n(x) + C_2 J_n(x)$$

Reduction of two standard form

$$x \frac{dy}{dx} + C \frac{dy}{dx} + k^2 y = 0$$

which can be written as

$$x^2 \frac{dy}{dx^2} + (1-2\alpha) x \frac{dy}{dx} + k^2 x y = 0$$

put $y = u x^\alpha$ where $-2\alpha = 2$

$$\frac{dy}{dx} = \mu \alpha x^{\alpha-1} = x^\alpha \frac{d\mu}{dx}$$

$$\frac{dy}{dx^2} = \mu \alpha (\alpha-1) x^{\alpha-2} + \frac{d\mu}{dx} x^\alpha$$

$$+ x^\alpha \frac{d^2 \mu}{dx^2} + \alpha x^{\alpha-1} \frac{dy}{dx}$$

Substituting these in (1) we get

$$(1-\alpha(\alpha-1)) x^\alpha + \frac{d\mu}{dx} x^{\alpha+1} + x^{\alpha+2} \frac{d^2 \mu}{dx^2}$$

$$+ x^{\alpha+1} \frac{d\mu}{dx} + (1-2\alpha) (x^\alpha x^\alpha + x^{\alpha+2} \frac{d\mu}{dx})$$

$$+ x^{\alpha+2} \frac{d^2 \mu}{dx^2} = 0$$

which can be written as

$$x^2 \frac{d^2 \mu}{dx^2} + x \frac{d\mu}{dx} + (\alpha^2 - \alpha + 1 - 2\alpha) x^\alpha \mu = 0$$

$$+ (\alpha^2 - \alpha + 1 - 2\alpha + k^2 x^2) x^\alpha \mu = 0$$

Dividing by x^α we get

$$x^2 \frac{d^2 \mu}{dx^2} + x \frac{d\mu}{dx} + (k^2 x^2 - \alpha^2) \mu = 0$$

Further by substituting $t = kx$
 $x^2 \frac{d^2 \mu}{dx^2} + x \frac{d\mu}{dx} + (t^2 - \alpha^2) \mu = 0$

which is Bessel's Equation of order α

$$\mu = C_1 J_\alpha(kx) + C_2 Y_\alpha(kx)$$

$$= C_1 J_\alpha(kx) + C_2 Y_\alpha(kx)$$

$$\mu = \frac{y}{x^\alpha}$$

$$y = x^\alpha (C_1 J_\alpha(kx) + C_2 Y_\alpha(kx))$$

$$y = x^\alpha (C_1 J_\alpha(kx) + C_2 Y_\alpha(kx))$$

$$y = C_1 J_n(t) + C_2 Y_n(t)$$

$$= C_1 J_n(t) + C_2 Y_n(t)$$

Soln of Eqn is

$$y = C_1 J_n(x) + C_2 Y_n(x)$$

$$= C_1 J_n(x) + C_2 Y_n(x)$$

Reduction of two standard form.

$$x \frac{d^2 y}{dx^2} + C \frac{dy}{dx} + k^2 y = 0$$

which can be written as

$$x^2 \frac{d^2 y}{dx^2} + (1-2\alpha) x \frac{dy}{dx} + k^2 x^2 y = 0$$

put $y = u x^\alpha$ where $1-2\alpha = 2\beta$

$$\frac{dy}{dx} = \mu \alpha x^{\alpha-1} - x^\alpha \frac{d\alpha}{dx}$$

$$\frac{d^2 y}{dx^2} = \mu \alpha (\alpha-1) x^{\alpha-2} + \frac{d\alpha}{dx} x^\alpha + x^\alpha \frac{d^2 \alpha}{dx^2}$$

$$+ x^\alpha \frac{d^2 \alpha}{dx^2}$$

$$\frac{d^2 y}{dx^2} + \alpha x^{\alpha-1} \frac{dy}{dx}$$

Substituting these in (1) we get

$$\begin{aligned} & (\mu \alpha (\alpha-1) x^\alpha + \frac{d^2 \alpha}{dx^2} x^{\alpha+2} + x^{\alpha+2} \frac{d^2 \alpha}{dx^2}) \\ & + (1-2\alpha) x^\alpha (\mu \alpha x^{\alpha-1} + x^{\alpha+1} \frac{d\alpha}{dx}) \\ & - \mu x^{\alpha+1} \frac{d\alpha}{dx} = 0 \end{aligned}$$

which can be written as

$$x^{\alpha+2} \frac{d^2 \alpha}{dx^2} + x^{\alpha+1} \frac{d\alpha}{dx} (\alpha+1-2\alpha)$$

$$+ (\alpha^2 - \alpha + \alpha - 2\alpha^2 + k^2 x^2) x^\alpha \mu = 0$$

Dividing by x^α we get

$$x^2 \frac{d^2 \alpha}{dx^2} + x \frac{d\alpha}{dx} + (k^2 x^2 - \alpha^2) \mu = 0$$

Further by substituting $t = kx$
 $+ 2 \frac{d^2 \alpha}{dt^2} + t \frac{d\alpha}{dt} + (t^2 - \alpha^2) \mu = 0$

which is Bessel's Equation of order α

$$\mu = C_1 J_\alpha(kx) + C_2 Y_\alpha(kx)$$

$$= C_1 J_\alpha(kx) + C_2 Y_\alpha(kx)$$

$$\mu = \frac{y}{x^\alpha}$$

$$y = x^\alpha (C_1 J_\alpha(kx) + C_2 Y_\alpha(kx))$$

$$y = x^\alpha (C_1 J_\alpha(kx) + C_2 Y_\alpha(kx))$$

Ques Find the soln of $x \frac{dy}{dx} + \frac{dy}{dx} + \frac{1}{y} = 0$

In form of Bessel's function.

The given differential eqn is $x \frac{dy}{dx} + \frac{dy}{dx} + \frac{1}{y} = 0$

$$\text{put } t = x^{1/2} \quad \frac{dt}{dx} = \frac{1}{2x} \Rightarrow \frac{1}{2t} = \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{2t}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{t} \frac{dy}{dt} + \frac{1}{t^2} \right) + \left(\frac{1}{t^2} \right) \frac{dt}{dx}$$

$$\frac{dy}{dt} = \frac{1}{2} \left(\frac{1}{t} \frac{dy}{dt} + \frac{1}{t^2} \right) - \frac{1}{t^2} \frac{dy}{dt}$$

$$\frac{1}{4t^2} \frac{dy}{dt} - \frac{1}{4t^3} \frac{dy}{dt}$$

using (1) and (2) in (1), we have

$$t^2 \left(\frac{1}{4t^2} \frac{dy}{dt} - \frac{1}{4t^3} \frac{dy}{dt} \right) + \frac{1}{2t} \frac{dy}{dt} + \frac{1}{y} = 0$$

$$\frac{1}{4} \frac{dy}{dt} - \frac{1}{4t} \frac{dy}{dt} + \frac{1}{2t} \frac{dy}{dt} + \frac{1}{y} = 0$$

$$t \frac{dy}{dt} + \frac{dy}{dt} + ty = 0$$

$$t^2 \frac{dy}{dt} + t \frac{dy}{dt} + ty = 0$$

$$x^2 \frac{d^2y}{dx^2} + t \frac{dy}{dx} + (t^2 - 1)y = 0$$

which is a Bessel's Equation of order 0

Its soln is $y = C_1 J_0(t) + C_2 Y_0(t)$

$$y = C_1 J_0(\sqrt{x}) + C_2 Y_0(\sqrt{x})$$

Ques Show that under transformation $y = u/\sqrt{x}$ the Bessel's Equation becomes

$$\frac{d^2y}{dx^2} + \left(1 + \frac{1-4n^2}{4x^2} \right) y = 0 \quad \text{Hence find the soln of this eqn}$$

The Bessel eqn is

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

$$\text{put } y = u/\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} \left(-\frac{1}{2} u \right) x^{-3/2} + x^{-1/2} \frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{1}{2} u \right) \left(-\frac{3}{2} x^{-5/2} \right) + \left(-\frac{1}{2} \right) x^{-3/2} \frac{du}{dx} + x^{-1/2} \frac{d^2u}{dx^2}$$

put value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1)

$$x^2 \left(\frac{3}{4} x^{-5/2} u + \left(-\frac{1}{2} \right) x^{-3/2} \frac{du}{dx} \right) + x^{-1/2} \frac{d^2u}{dx^2} + \left(-\frac{1}{2} \right) x^{-3/2} \frac{du}{dx} + \left(x^2 - n^2 \right) \frac{u}{\sqrt{x}} = 0$$

$$+ x \left(4(-1/2) x^{-3/2} + x^{-1/2} \frac{dy}{dx} \right) + (x^2 - n^2) u x^{-1/2}$$

$$u^{3/2} \frac{d^2 y}{dx^2} + u^{1/2} \left(\frac{3}{4} - \frac{1}{2} + x^2 - n^2 \right) u$$

$$\frac{dy}{dx} + \frac{4}{x} \left(x^2 + \frac{1}{4} n^2 \right) u$$

Multiply by x^2

$$x^2 \frac{d^2 y}{dx^2} + \left(x^2 + \frac{1}{4} n^2 \right) u = 0$$

Comparing this equation is

$$x^2 \frac{d^2 y}{dx^2} + (1 - 2n^2) x \frac{dy}{dx} + (3/4 - n^2) x^2 u +$$

$$(x^2 - n^2) u = 0$$

$$x^2 = 1/2, \quad 1/2 = 1/2$$

$$\text{put } y = x^{1/2} u \text{ so that } u = x^{-1/2} y$$

$$x^2 = 1, \quad x' = x$$

$$\frac{dy}{dx} = x^{1/2} \frac{dy}{dx} + \frac{1}{2} x^{-1/2} y \quad \frac{d^2 y}{dx^2} =$$

$$\text{So } \frac{d^2 y}{dx^2} =$$

$$\frac{dy}{dx} = x^{1/2} \frac{dy}{dx} + \frac{1}{2} x^{-1/2} y$$

$$x^{1/2} \frac{dy}{dx} + \frac{1}{2} x^{-1/2} y$$

$$\frac{d^2 y}{dx^2} = x^{1/2} \frac{d^2 y}{dx^2} + \frac{1}{2} x^{-1/2} \frac{dy}{dx}$$

$$+ \frac{1}{2} x^{-1/2} \frac{dy}{dx} - \frac{1}{4} x^{-3/2} y$$

$$= x^{1/2} \frac{d^2 y}{dx^2} + \frac{1}{2} x^{-1/2} \frac{dy}{dx} + \frac{1}{2} x^{1/2}$$

$$\frac{dy}{dx} - \frac{1}{4} x^{-3/2} y$$

$$= x^{1/2} \frac{d^2 y}{dx^2} + x^{1/2} \frac{dy}{dx} - \frac{1}{4} x^{-3/2} y$$

$$x^2 \frac{d^2 y}{dx^2} = x^{3/2} \frac{d^2 y}{dx^2} + x^{3/2} \frac{dy}{dx}$$

$$= \frac{1}{4} x^{1/2} y$$

$$x^2 \frac{d^2 y}{dx^2} + \left(x^2 + \frac{1}{4} - n^2 \right) y =$$

$$x^{3/2} \frac{d^2 y}{dx^2} + x^{3/2} \frac{dy}{dx} - \frac{1}{4} x^{1/2} y$$

$$= \left(x^2 + \frac{1}{4} - n^2 \right) x^{1/2} y$$

$$= x^{1/2} \left(x^2 - \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - \frac{1}{4} y + x y \right)$$

$$+ \frac{1}{4} y - x^2 y$$

As $x = x$

$$x^2 \frac{d^2 y}{dx^2} + \left(x^2 + \frac{1}{4} - n^2 \right) y$$

$$y_2 \left(x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y \right) = 0$$

$$\text{It gives } x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

which is a Bessel equation of

order n

its solⁿ is

$$y_2 = c_1 J_n(x) + c_2 J_{-n}(x) \text{ if } n \text{ is non integral}$$

$$y_2 = c_1 J_n(x) + c_2 Y_n(x) \text{ if } n \text{ is an integral}$$

$$u = \int x (c_1 J_n(x) + c_2 J_{-n}(x)) dx \text{ if } n \text{ is non integral}$$

$$u = \int x (c_1 J_n(x) + c_2 Y_n(x)) dx \text{ if } n \text{ is an integral}$$

Orthogonality relation of Bessel's funⁿ.
If α and β are the roots of $J_n(x) = 0$ then,

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0 & \alpha \neq \beta \\ \frac{1}{2} [J_n'(\alpha)]^2 & \alpha = \beta \end{cases}$$

Consider the Bessel's equation.

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad (1)$$

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + (\beta^2 x^2 - n^2)u = 0 \quad (2)$$

We know $u = J_n(\alpha x)$ and $v = J_n(\beta x)$

Multiply (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$

$$xv \frac{d^2 y}{dx^2} + v \frac{dy}{dx} + \frac{v}{x} (x^2 x^2 - n^2)u = 0$$

$$xu \frac{d^2 v}{dx^2} + u \frac{dv}{dx} + \frac{u}{x} (\beta^2 x^2 - n^2)v = 0$$

subtracting (1) from (2)

$$x \left(v \frac{d^2 y}{dx^2} - u \frac{d^2 v}{dx^2} \right) + \left(v \frac{dy}{dx} - u \frac{dv}{dx} \right) + \frac{v}{x} (x^2 - \beta^2 x^2) - \frac{u}{x} (\beta^2 x^2 - x^2) = 0$$

$$+ xuv (x^2 - \beta^2 x^2) = 0$$

$$\frac{d}{dx} \left(x \left(v \frac{dy}{dx} - u \frac{dv}{dx} \right) + (v^2 - u^2) x \right) = 0$$

Integrating e.g. within limit a to b

$$\left(x \left(v \frac{dy}{dx} - u \frac{dv}{dx} \right) + (v^2 - u^2) x \right) \Big|_a^b = 0$$

$$\left[\frac{v \frac{dy}{dx} - u \frac{du}{dx}}{dx} \right]_{\alpha}^{\beta} = 0 + \int_{\alpha}^{\beta} (x^2 - \beta^2) x u dx$$

$$(x^2 - \beta^2) \int_{\alpha}^{\beta} x u dx = \left(u \frac{du}{dx} - \frac{u^2}{2} \right)_{\alpha}^{\beta}$$

$$u = J_n(x)$$

$$\frac{dy}{dx} = \frac{d}{dx} J_n(x) = x J_n'(x)$$

$$v = J_n(\beta x)$$

$$\frac{du}{dx} = \beta J_n'(\beta x)$$

$$\int_{\alpha}^{\beta} x u dx = \beta \int_{\alpha}^{\beta} J_n(\beta x) J_n'(x) dx = \beta \int_{\alpha}^{\beta} J_n(x) J_n'(x) dx$$

$$x^2 - \beta^2$$

$$J_n(x) = 0, J_n(\beta) = 0, J_n(\beta x) = 0$$

$$\int_{\alpha}^{\beta} x u dx = 0$$

$$\int_{\alpha}^{\beta} x u dx$$

$$\int_{\alpha}^{\beta} x J_n(x) J_n(\beta x) dx = 0$$

If $\alpha = \beta$,

Right Hand side of 7 is $\int_{\alpha}^{\alpha} \frac{d}{dx} \left(\frac{u^2}{2} \right) dx$

Also a part of Eqn $J_n(x) = 0$

Let $\beta = \alpha + h$

$$\text{put } \beta = \alpha + h$$

$$\lim_{h \rightarrow 0} \int_{\alpha}^{\alpha+h} J_n(x) J_n(\alpha+x+h) dx$$

$$= \lim_{h \rightarrow 0} \left(-\alpha J_n(\alpha+h) J_n'(\alpha) \right)$$

$$= \lim_{h \rightarrow 0} -\alpha \left[J_n(\alpha+h) J_n'(\alpha) \right]$$

$$\int_{\alpha}^{\alpha+h} x J_n^2(\alpha+x) dx = \lim_{h \rightarrow 0} \frac{\alpha J_n'(\alpha+h) J_n(\alpha)}{2\alpha+h}$$

$$\frac{1}{2} (J_n'(\alpha))^2 = \frac{1}{2} J_n(\alpha) J_n'(\alpha)$$

Legendre's Eqn

Legendre's Eqn is the differential eqn of the form $(1-x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + n(n+1)y = 0$

is called Legendre Eqn. where n is called parameter.

Solution of Legendre's Equation:-

The Equation $(1-x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + n(n+1)y = 0$

parameter n is a real no. using Frobenius method

at point $x=0$ let $y = x^m (a_0 + a_1x + a_2x^2 + \dots)$

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Substituting these in (1)

$$(1-x^2) \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2} + 2x \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1} + n(n+1) \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$m(m-1) a_0 x^{m-2} + (m-1) m a_1 x^{m-1} + \sum_{k=2}^{\infty} (m+k)(m+k-1) a_k x^{m+k-2} - \sum_{k=2}^{\infty} (m+k)(m+k-1) + 2(m+k-1) a_{k-1} x^{m+k-1} - n(n+1) \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$m(m-1) a_0 x^{m-2} + (m-1) m a_1 x^{m-1} + \sum_{k=2}^{\infty} (m+k+2)(m+k+1) a_{k+2} x^{m+k} - \sum_{k=2}^{\infty} (m+k)(m+k-1) + 2(m+k-1) a_{k-1} x^{m+k-1} - n(n+1) \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

Equating coeff. of x^{m-2} to zero.

$$m(m-1) a_0 = 0$$

$$a_1 \neq 0 \quad m(m-1) = 0 \quad \text{at } m=0$$

Equating coeff. of x^{m+k} to 0

$$(m+k+2)(m+k+1) a_{k+2} - (m+k)(m+k-1) a_k + 2(m+k-1) a_{k-1} - n(n+1) a_k = 0$$

$$a_{k+2} = \frac{(m+k)(m+k-1) - n(n+1)}{(m+k+2)(m+k+1)} a_k$$

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k$$

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k$$

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k$$

$$a_2 = \frac{-n(n-1)a_0}{2!}$$

$$a_3 = \frac{-n(n-1)(n-2)a_1}{3!}$$

$$a_4 = \frac{(n-2)(n-1)(n-2)(n-3)a_0}{4!}$$

$$a_5 = \frac{(n-3)(n-1)(n-2)(n-2)(n-3)a_1}{5!}$$

Substituting these values in ②

$$y = a_0 \left(1 + \frac{(n-1)(n-2)}{2!} x^2 + \frac{(n-1)(n-2)(n-3)}{3!} x^3 + \dots \right)$$

$$y = a_0 \left(1 + \frac{(n-1)(n-2)}{2!} x^2 + \frac{(n-2)(n-1)(n-2)(n-3)}{4!} x^4 + \dots \right)$$

$$y = a_0 y_1(x) + a_1 y_2(x)$$

Radius of convergence of $y_1(x)$ and

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{-(n-2m)(n+2m)}{(2m+2)(2m+1)} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \left(1 - \frac{2}{2m}\right) \left(1 + \frac{2}{2m}\right) \right| = 1$$

∴ Radius of convergence is 1

General soln is (gt for $|x| < 1$)

Legendre's polynomial 1-

Legendre's eqn is linear
Combination of two independent
soln $y_1(x)$ and $y_2(x)$

$$y_1(x) = \frac{(1-n)(n-1)}{2!} x^2 + \frac{(n-2)(n-1)(n-2)(n-3)}{4!} x^4$$

$$y_2(x) = x - \frac{(n-1)(n-2)}{3!} x^3 + \dots$$

Legendre polynomial - polynomial soln
with a_0 or a_1 chosen in such a
way that the value of the polynomial
becomes 1 at $x=1$

Rodrigue's formula

To prove that $P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^2-1)^n$

Proof

Let $y = (x^2-1)^n$

$$\frac{dy}{dx} = n(x^2-1)^{n-1} \cdot 2x$$

$$(n^2-1) \frac{dy}{dx} = 2nx (x^2-1)^n$$

$$(n^2-1) \frac{dy}{dx} = 2nxy$$

$$(n-1) y_1 = 2nxy$$

Differentiating n times by
derivative is $\frac{d^n}{dx^n}$

$$n(n-1) \frac{d^2 y}{dx^2} + n(n-1) \frac{dy}{dx} + n(n-1)y = 0$$

$$y = (x^2-1)^n \Rightarrow y' = 2nx(x^2-1)^{n-1}$$

$$y'' = 2n(x^2-1)^{n-1} + 2n(n-1)x^2(x^2-1)^{n-2}$$

$$y'' - 2n(x^2-1)^{n-1} - 2n(n-1)x^2(x^2-1)^{n-2} = 0$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

Diff. the RHS by product using Leibnitz's Th^m

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

Derivation of Legendre's polynomial from Rodrigues' formula:-

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_1(x) = \frac{1}{2!} \frac{d}{dx} (x^2-1)^1 = x$$

$$P_2(x) = \frac{1}{2!} \frac{d^2}{dx^2} (x^2-1)^1 = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{3!} \frac{d^3}{dx^3} (x^2-1)^1 = \frac{1}{2}(5x^3-3x)$$

Similarly we can find that

$$P_4(x) = \frac{1}{4!} \frac{d^4}{dx^4} (x^2-1)^1 = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{5!} \frac{d^5}{dx^5} (x^2-1)^1 = \frac{1}{8}(63x^5-70x^3+15x)$$

$$P_6(x) = \frac{1}{6!} \frac{d^6}{dx^6} (x^2-1)^1 = \frac{1}{16}(231x^6-315x^4+105x^2-5)$$

Leibnitz T_n

$$n_0 y_{nt2} (x^2-1) + n_1 y_{nt1} 2x + n_2 y_{nt0} 2 + (2-2n) (n_0 y_{nt2} + n_1 y_{nt1} - 2n y_{nt0}) = 0$$

$$y_{nt2} (x^2-1) + x y_{nt1} (2n+2-2n) + y_{nt0} \left(\frac{n(n-1)}{2} 2 + 2n - 2n^2 - 2n \right) = 0$$

$$-(1-x^2) y_{nt2} + 2x y_{nt1} - n(n+1) y_{nt0} = 0$$

$$(1-x^2) y_{nt2} - 2x y_{nt1} + n(n+1) y_{nt0} = 0$$

Ques Show that $P_{2n+1}(0) = 0$

$$P_{2n+1}(0) = \frac{(-1)^n}{2^{n+1}} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} = (-1)^n \frac{2n!}{(n!) 2^n}$$

$$\text{Sol}^n \sum_{k=0}^n P_k(x) = (1-x^2)^{1/2} \frac{d}{dx} (1-x^2)^{1/2} = -x(1-x^2)^{-1/2}$$

$$\text{Put } x=0 \text{ in (1)}$$

$$\sum_{k=0}^n P_k(0) = (1+(-1)^2)^{1/2} \frac{d}{dx} (1-x^2)^{1/2} = 1 + \frac{1}{2}(-2) + \frac{(-1/2)(-3/2)}{2} (-x^2) - \dots$$

$$\sum_{k=0}^n P_k(0) = 1 + \frac{1}{2}(-2) + \frac{(-1/2)(-3/2)}{2} (-x^2) - \dots$$

$$\sum_{k=0}^n P_k(0) = 1 + \frac{1}{2}(-2) + \frac{(-1/2)(-3/2)}{2} (-x^2) - \dots$$

(1) Equating the coefficient of x^{2n} on both side. $P_{2n+1}(0) = 0$

(2) Equating the coefficient of x^{2n} on both side.

$$P_{2n+1}(0) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} = (-1)^n \frac{2n!}{(n!) 2^n}$$

$$= (-1)^n \frac{2n!}{(n!) 2^n} = (-1)^n \frac{2n!}{(n!) 2^n}$$

Thus verify that Legendre polynomial $P_{2n+1}(x) = \frac{1}{2^n} (5x^2-3n)$ satisfies the Legendre Equation when the parameter n is equal to 3.

$$\text{Legendre Equation with parameter 3}$$

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 3 \cdot 4 y = 0$$

$$y = \frac{1}{2} (5x^2-3)$$

$$\frac{dy}{dx} = \frac{1}{2} (10x) = 5x$$

Substituting the value of $\frac{dy}{dx}$ and by in ①

$$(1-x^2)15x - 2x \cdot \frac{1}{2} (15x^2 - 3) = 0$$

$$+12 \cdot \frac{1}{2} (15x^3 - 3x) = 0$$

$$15x - 15x^3 = 15x^3 + 3x + 30x^3 - 18x = 0 \Rightarrow 0 = 0$$

which is true

$P_2(x)$ is sum of Legendre E_n

Recurrence Relation
To show that

$$(n+1) P_{n+1}(x) + n P_{n-1}(x) = (2n+1) x P_n(x)$$

$$\text{L.H.S. } (1-2xt+t^2)^{1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Diff. both side w.r.t t

$$-\frac{1}{2} (1-2xt+t^2)^{-1/2} (2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$= \sum_{n=0}^{\infty} n t^n P_n(x) \quad \text{--- ②}$$

$$(n+1) (1-2xt+t^2)^{1/2} = \sum_{n=0}^{\infty} n t^n P_n(x)$$

Multiply both side by $(1-2xt+t^2)$

$$(n+1) (1-2xt+t^2)^{3/2} = (1-2xt+t^2) \sum_{n=0}^{\infty} n t^n P_n(x) \quad \text{--- ③}$$

$$\sum_{n=0}^{\infty} n t^{n+1} P_n(x) = \text{--- ④}$$

Equating coefficient of t^n on both side

$$n P_n(x) - P_{n+1}(x) = (n+1) P_{n+1}(x) - 2x n P_n(x)$$

$$P_n(x) + (n+1) P_{n+1}(x)$$

$$= (n+1) P_{n+1}(x) - 2x n P_n(x)$$

$$P_n(x) + n P_n(x) = (n+1) P_{n+1}(x) - 2x n P_n(x)$$

$$(n+1) P_{n+1}(x) + n P_n(x) = (2n+1) x P_n(x)$$

(i) $P_n(x) = x P_n'(x) - P_{n+1}'(x)$

(ii) $(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

(iii) $P_n(x) = P_{n+1}'(x) - x P_n'(x)$

(iv) $(1-x^2) P_n'(x) = n P_{n+1}(x) - x P_n(x)$

(v) $(1-x^2) P_n'(x) = (n+1) P_n(x)$

Ques show that $(1-x^2) P_n'(x) = (n+1) P_n(x)$

Ans By the recurrence relations:

$$(n+1) P_{n+1}(x) = (1-x^2) P_n'(x) + n P_n(x)$$

$$(1-x^2) P_n'(x) = (n+1) P_{n+1}(x) - n P_n(x)$$

Adding (i) and (ii) we get:

$$(n+1) P_{n+1}(x) + n P_{n+1}(x) + (1-x^2) P_n'(x) = (2n+1) P_{n+1}(x) + n P_n(x) - n P_n(x)$$

$$(1-x^2) P_n'(x) = (2n+1) P_{n+1}(x) - n P_n(x)$$

$$(1-x^2) P_n'(x) = (2n+1) P_{n+1}(x) - n P_n(x)$$

$$P_n(x) = (n+1) P_{n+1}(x)$$

$$(1-x^2) P_n'(x) = x (2n+1) P_{n+1}(x) - n P_n(x)$$

$$= (n+1) P_{n+1}(x)$$

$$= (n+1) P_{n+1}(x) - (n+1) P_{n+1}(x)$$

$$= (n+1) P_{n+1}(x) - (n+1) P_{n+1}(x)$$

Ques Prove that $P_n(x) = P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x)$

Ans By the recurrence relations:

$$(n+1) P_{n+1}(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

$$(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)$$

Multiplying (i) by (ii)

$$(2n+1) P_n(x) = 2 P_{n+1}'(x) - 2x P_n'(x)$$

Subtracting (i) from (ii)

$$P_n(x) = P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x)$$

orthogonality of Legendre polynomials

To Prove that $\int_{-1}^1 P_m(x) P_n(x) dx = 0$

$$\int_{-1}^1 (P_n(x))^2 dx = \frac{2}{2n+1} \text{ when } m=n$$

Proof Consider when $m \neq n$

Consider Legendre Equation

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

These are respectively satisfied by $P_m(x)$ and $P_n(x)$

$$(1-x^2) P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x) = 0 \quad \text{--- (1)}$$

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 \quad \text{--- (2)}$$

Multiplying (1) by $P_n(x)$ and (2) by $P_m(x)$ and subtracting

$$(1-x^2) (P_m(x) P_n''(x) - P_n(x) P_m''(x)) - 2x (P_m(x) P_n'(x) - P_n(x) P_m'(x)) = 0$$

$$-2x (P_m(x) P_n'(x) - P_n(x) P_m'(x)) + m(m+1) P_m(x) P_n(x) - n(n+1) P_m(x) P_n(x) = 0$$

$$\frac{d}{dx} (1-x^2) P_m(x) P_n'(x) - P_n'(x) P_m(x) = 0$$

Integrating both side w.r.t x b/w limit -1 to 1

$$(1-x^2) P_m(x) P_n'(x) - P_n'(x) P_m(x) \Big|_{-1}^1 = 0$$

$$= \int_{-1}^1 (m-m) (m+n+1) P_m(x) P_n(x) dx$$

$$0 = (m-m) (m+n+1) \int_{-1}^1 P_m(x) P_n(x) dx$$

$$\text{As } m \neq n \Rightarrow \int_{-1}^1 P_m(x) P_n(x) dx = 0$$

Condition when $m=n$

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

Multiplying (1) and (2)

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

Integrating Both side w.r.t x

$$\sum_{n=0}^{\infty} \int_1^{\infty} P_n(x) t^{nx} dx =$$

$$\int_1^{\infty} \frac{dx}{1-2xt+t^2} \quad (3)$$

$$\int_1^{\infty} P_n(x) dx = 0.$$

$$\sum_{n=0}^{\infty} \int_1^{\infty} t^{nx} P_n(x) dx = \int_1^{\infty} \frac{dx}{1+t^2-2xt}$$

$$= \int_1^{\infty} \frac{\log(1+t^2-2xt)}{-2t} dx$$

$$= \frac{\log(1+t^2-2t)}{-2t} - \frac{\log(1+t^2+2t)}{-2t}$$

$$\sum_{n=0}^{\infty} \int_1^{\infty} t^{nx} P_n(x) dx =$$

$$= \frac{1}{2t} \log(1-t)^2 - \log(1+t)^2$$

$$= -\frac{1}{2t} [2 \log(1-t) - \log(1+t)]$$

$$= \frac{1}{t} [\log(1+t) - \log(1-t)]$$

$$\frac{1}{t} \left(t - \frac{t^2}{2} + \frac{t^3}{3} \right) - \left(-t - \frac{t^2}{2} + \frac{t^3}{3} \right)$$

$$= \frac{2}{t} \left(t + \frac{t^3}{3} + \frac{t^5}{5} \right)$$

$$\sum_{n=0}^{\infty} \int_1^{\infty} t^{2n} P_n^2(x) dx = 2 \left(1 + \frac{t^2}{3} + \frac{t^4}{5} \right)$$

Equating Coefficient of t^{2n} on Both sides.

$$\int_1^{\infty} P_n^2(x) dx = \frac{2}{2n+1}$$

Laplace integral representation of Legendre's polynomial

Laplace's first integral for $P_n(x)$ is stated: when x is a positive integer, then

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} (x \pm \sqrt{x^2-1} \cos \phi)^n d\phi$$

Proof In our previous classes we have that

$$\int_0^{\pi} d\phi = \pi$$

$$\int_0^{\pi} x^2 \cos \phi d\phi = \frac{\pi}{\sqrt{x^2-1}}$$

put $\alpha^2 = 1-tx$ $\beta = t\sqrt{x^2-1}$
 $\alpha^2 - \beta^2 = (1-tx)^2 - t^2(x^2-1)$
 $= 1 - 2xt + t^2$

putting these values of $\alpha\beta$
 and $\alpha^2 - \beta^2$ in (1)

$$\pi(1-2xt+t^2)^{-1/2} = \int_0^{\pi} \frac{(1-tx \pm t\sqrt{x^2-1})}{\cos \phi} d\phi$$

$$= \int_0^{\pi} (1-t \pm t\sqrt{x^2-1} \cos \phi) d\phi$$

$$= \int_0^{\pi} (1-ty) d\phi$$

$$\pi \int_0^{\pi} t^n P_n(x) = \int_0^{\pi} (1+ty+ty^2 \dots)$$

Equating Coefficient of y^n on both side

$$\pi P_n(x) = \int_0^{\pi} y^n d\phi$$

$$= \int_0^{\pi} (x \pm \sqrt{x^2-1} \cos \phi)^n d\phi$$

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} (x \pm \sqrt{x^2-1} \cos \phi) d\phi$$

Ques Prove that $\int_{-1}^1 x P_n(x) P_{n+1}(x) dx = \frac{2n}{4n^2-1}$

Solⁿ
 By the recurrence relation
 $(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x)$

which may be written as
 $xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)$

Multiplying Both side by $P_{n+1}(x)$
 and integrating b/w limit -1 to 1

$$\int_{-1}^1 x P_n(x) P_{n+1}(x) dx = \frac{n+1}{2n+1} \int_{-1}^1 P_{n+1}^2(x) dx + \frac{n}{2n+1} \int_{-1}^1 P_{n-1}(x) P_{n+1}(x) dx$$

$$\int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m=n \end{cases}$$

making use of it in (1)

$$\int_{-1}^1 x P_n(x) P_{n+1}(x) dx = 0 + \frac{n}{2n+1} \times \frac{2}{2n+1}$$

$$= \frac{2n}{(2n+1)^2} = \frac{2n}{4n^2+4n+1}$$

Ques Prove that $\int_{-1}^1 (1-x^2) P_m(x) P_n(x) dx$

Hermite's Equation.

The differential Eqn of the form $\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0$.

Soln of Hermite Eqn

The Eqn $\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0$

Let the soln

$$y = x^m (a_0 + a_1 x + \dots)$$

$$\frac{dy}{dx} = x^m \sum_{h=0}^{\infty} a_h (m+h) x^{m+h-1}$$

$$\frac{d^2y}{dx^2} = \sum_{h=0}^{\infty} a_h (m+h) (m+h-1) x^{m+h-2}$$

Substituting these values in (1)

$$\sum_{h=0}^{\infty} a_h (m+h) (m+h-1) x^{m+h-2} - 2x \sum_{h=0}^{\infty} a_h (m+h) x^{m+h-1} + 2n \sum_{h=0}^{\infty} a_h x^{m+h} = 0$$

$$- 2n \sum_{h=0}^{\infty} a_h (m+h) x^{m+h-1} + 2n \sum_{h=0}^{\infty} a_h x^{m+h} = 0$$

Equating the coefficient of lowest degree

$$a_0 m(m-1) = 0$$

$$m(m-1) = 0$$

$$m = 0, 1$$

The roots of indicial Eqn are integers

Eqn of the coefficient of x^{m-1}

$$a_1 m(m-1) = 0$$

Equating the coefficient of x^{m+2} to zero

$$a_{r+2} (m+r+2) (m+r+1) - 2a_r (m+r-1) = 0$$

$$a_{r+2} = \frac{2a_r (m+r-1)}{(m+r+2) (m+r+1)} \quad r \geq 0$$

for $m=0$.

$$a_{r+2} = \frac{2(r-1)}{(r+2) (r+1)} a_r$$

Putting $r=0, 2, 4, \dots$

$$a_2 = \frac{2(-1)}{2!} a_0 = \frac{(-1)^1 2!}{2!} a_0$$

$$a_4 = \frac{(-1)^2 4!}{4!} a_0$$

Hermit's Equation.

The differential Eqⁿ of the form $\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0$.

Solⁿ of Hermitic Eqⁿ

The Eqⁿ $\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0$

Let the solⁿ

$$y = x^m \left(a_0 + a_1 x + \dots \right)$$

$$\frac{dy}{dx} = x \sum_{h=0}^{\infty} a_h (m+h) x^{m+h-1}$$

$$\frac{d^2y}{dx^2} = \sum_{h=0}^{\infty} a_h (m+h) (m+h-1) x^{m+h-2}$$

Substituting these values in (1)

$$-2x \sum_{h=0}^{\infty} a_h (m+h) x^{m+h-1} + 2n \sum_{h=0}^{\infty} a_h x^{m+h} = 0$$

Equating the coefficient of lowest degree

$$a_0 m(m-1) = 0$$

$$m(m-1) = 0$$

$$m = 0, 1$$

The root of indicial Eqⁿ are integer

Eqⁿ the coefficient of x^{m-1}

$$a_1 m(m-1) = 0$$

Equating the coefficient of x^{m+r} to zero.

$$a_{r+2} (m+r+2) (m+r+1)$$

$$- 2 a_r (m+r-n) = 0$$

$$a_{r+2} = \frac{2 a_r (m+r-n)}{(m+r+2) (m+r+1)} \quad r \geq 0$$

for $m=0$.

$$a_{r+2} = \frac{2(r-n)}{(r+2)(r+1)} a_r$$

Take $r=0, 2, 4, \dots$

$$a_2 = \frac{2(-n)}{2 \cdot 1} a_0 = \frac{(-1)^1 2^1}{2!} a_0$$

$$a_4 = \frac{(-1)^2 n(n-2) 2^2}{4!} a_0$$

$$a_3 = \frac{(-1)^1 (n-1)}{3!} a_1$$

$$a_5 = \frac{(-1)^2 (n-1)(n-3)}{5!} a_1$$

$$a_{2k} = \frac{(-1)^k n(n-2)(n-4)\dots(n-2k+2)}{(2k)!} a_1$$

$$y = a_0 + a_1 x + \dots$$

$$y = (a_0 + a_2 x^2 + \dots) + (a_1 x + a_3 x^3 + \dots)$$

$$y = a_0 \left(1 + \frac{(-1) 2n}{n!} x^2 + \dots \right)$$

$$y = a_0 y_1(x) + a_1 y_2(x)$$

$$y = a_0 y_1(x)$$

$$= a_0 \left(1 + \frac{(-1) 2n}{2!} x^2 + \dots \right)$$

$$y = a_1 y_2(x)$$

$$a_1 \left(x + \frac{(-1)^1 2(n-1)}{2!} x^3 + \dots \right)$$

Hermite's polynomial
n as an even integer and

$$a_0 = \frac{(-1)^{n/2} n!}{(n/2)!}$$

$$\text{Coeff. of } x^n = \frac{(-1)^{n/2} n!}{(n/2)!} \frac{(-1)^{n/2} 2^{n/2} n!}{n!}$$

$$= 2^n$$

$$\text{Coeff of } x^{n-2} = \frac{-n(n-1) 2^{n-2}}{1!}$$

$$\text{Coeff of } x^{n-4} = \frac{n(n-1)(n-2)(n-3) 2^{n-4}}{2!}$$

$$y = \frac{(-1)^{n/2} 2^{n/2} n!}{(n/2)!} + \dots - 2^n x^n$$

Writing it in term

$$y = (2^n)^{n/2} \frac{(-1)^{n/2} n!}{(n/2)!}$$

$$H_0(x) = (2x)^0 = \frac{0!}{0!} = 1$$

Hermite's polynomial
Hermite's polynomial $H_n(x)$

$$H_n(x) = \sum_{r=0}^{n/2} \frac{n!}{r!(n-2r)!} (-1)^r (2x)^{n-2r}$$

$$\left(\frac{n}{2}\right) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$H_n(-x) = (-1)^n H_n(x)$$

Hermite's polynomial for
Some value of n

$$H_n(x) = \sum_{r=0}^{n/2} \frac{n!}{r!(n-2r)!} (-1)^r (2x)^{n-2r}$$

$$\left(\frac{n}{2}\right) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

n=0 in ①

$$H_0(x) = (-1)^0 \frac{1}{0!} (2x)^0 = 1$$

Taking $n=1$

$$H_1(x) = \frac{1}{0!1!} 2x = 2x$$

Taking $n=2$

$$H_2(x) = \frac{1}{0!2!} 4x^2 - \frac{2!}{1!1!} 2x = 4x^2 - 2$$

generating function for Hermite's

To show that $e^{2xt - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$

Proof

$$e^{2xt - t^2} = e^{2xt} e^{-t^2} = \sum_{r=0}^{\infty} \frac{(2x)^r}{r!} t^r \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} t^{2s}$$

Let $s + 2r = n$ so $n - 2r$

$$t^n = \sum_{r=0}^{n/2} \frac{(2x)^r}{r!} \frac{(-1)^{n-2r}}{(n-2r)!} t^n$$

Left of t^n is $H_n(x)$

$$\frac{1}{n!} = \begin{cases} \frac{1}{n!} & \text{if } n \text{ is even} \\ \frac{1}{n!} & \text{if } n \text{ is odd} \end{cases}$$

$$\frac{1}{n!} \sum_{k=0}^n \frac{(-1)^k n! (2x)^{n-k}}{k! (n-k)!}$$

$H_n(x)$

$$e^{x^2 - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

Reduigues formula for $H_n(x)$

To show that $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

Proof $e^{x^2 - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$

$$e^{2xt - t^2} = \frac{e^{x^2} e^{-x^2}}{0!} + \frac{H_1(x)}{1!} t + \dots$$

Laplace Transform

Let $f(t)$ be a function defined $t \geq 0$. Then the Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$.

$$\int_0^{\infty} e^{-st} f(t) dt$$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

Laplace Transform of some Elementary functions

① $\mathcal{L}\{1\} = \frac{1}{s}, s > 0$

② $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0$

③ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a$

④ $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}, s > 0$

⑤ $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, s > 0$

⑥ $\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{s^2 - a^2 + b^2}, s > |a|$

⑦ $\mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{s^2 - a^2 + b^2}, s > |a|$

Linearity property of Laplace Transformation

To show that $L(a f(t) + b g(t)) = a L f(t) + b L g(t)$

where a, b are constants and $f(t), g(t)$ are functions of t .

$$\text{Proof } L(a f(t) + b g(t)) = L(a f(t) + b g(t))$$

$$= \int_0^{\infty} e^{-st} (a f(t) + b g(t)) dt$$

$$= a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt$$

$$= a L f(t) + b L g(t)$$

Ques find the Laplace Transform of

A $L(\sin t - \cos t)^2 = L(\sin^2 t + \cos^2 t - 2 \sin t \cos t)$

$$= L(1 - \sin 2t) = L(1) - L(\sin 2t)$$

$$= \frac{1}{s} - \frac{2}{s^2 + 4} = \frac{s^2 + 4 - 2s}{s(s^2 + 4)}$$

Q1 find the Laplace transform of the function $\sin 3t$

A $L(\sin 3t) = L\left(\frac{e^{3it} - e^{-3it}}{2i}\right)$

$$= L\left(\frac{1}{8} (e^{6t} - 3e^{2t} + 3e^{4t} - e^{6t})\right)$$

$$= \frac{1}{8} L(e^{6t}) - 3L(e^{2t}) + 3L(e^{4t}) - L(e^{6t})$$

$$= \frac{1}{8} \left[\frac{1}{s-6} - 3 \frac{1}{s-2} + 3 \frac{1}{s-4} - \frac{1}{s-6} \right]$$

$$= \frac{1}{8} \left(\frac{1}{s-6} - \frac{1}{s-6} \right) + 3 \left(\frac{1}{s-4} - \frac{1}{s-2} \right)$$

$$= \frac{1}{8} \left(\frac{s+6-s-6}{s^2-36} + 3 \frac{(s-2)-(s-4)}{(s-2)(s-4)} \right)$$

$$= \frac{1}{8} \left(\frac{18}{s^2-36} - \frac{6}{s^2-4} \right)$$

$$= \frac{18}{8} \int_0^{\infty} \frac{e^{-st} - e^{-3t}}{(s^2-36)} dt - \frac{6}{8} \int_0^{\infty} \frac{e^{-st} - e^{-2t}}{(s^2-4)} dt$$

first shifting property :-
if $L\{f(t)\} = F(s)$ then $L\{e^{at}f(t)\} = F(s-a)$

$$L\{e^{at}f(t)\} = F(s-a)$$

$$L\{e^{at}f(t)\} = \int_0^{\infty} e^{-st} e^{at} f(t) dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt = F(s-a)$$

$$P(s) = F(s-a)$$

Shifting property

$$L\{e^{at}f(t)\} = F(s-a)$$

$$L\{e^{at}f(t)\} = \frac{F(s-a)}{(s-a)^n}$$

$$L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$$

$$L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}$$

$$L\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2 - b^2}$$

change of scale property
To show that

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$L\left\{f\left(\frac{t}{a}\right)\right\} = a F(as)$$

$$\text{Proof: } L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$at = u, \quad a dt = du$$

$$L\{f(at)\} = \int_0^{\infty} e^{-st} f(at) dt$$

$$L\left\{f\left(\frac{t}{a}\right)\right\} = a F(as)$$

$$L\left\{f\left(\frac{t}{a}\right)\right\} = \int_0^{\infty} e^{-st} f\left(\frac{t}{a}\right) dt$$

$$t/a = u, \quad dt = a du$$

$$L\left\{f\left(\frac{t}{a}\right)\right\} = \int_0^{\infty} e^{-su} f(u) a du$$

$$= a \int_0^{\infty} e^{-su} f(u) du$$

$$= a F(as)$$

Ques find the Laplace transform of $e^{at} \sinh bt$

$L(\sinh bt) = \frac{1}{s^2 + b^2}$

$$L(e^{at} \sinh bt) = \frac{b}{s^2 - (a^2 + b^2)}$$

$$= \frac{b}{(s-a)^2 - b^2} = \frac{b}{(s-a-b)^2 - 0}$$

Ques If Laplace transform $L(\cosh at) = f(s)$

$L(e^{at} f(s)) = f(s-a)$

$L(e^{-at} f(s)) = f(s+a)$

$$L(\cosh at) = \frac{1}{s^2 - a^2} = \frac{1}{(s-a)(s+a)}$$

$$= \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right)$$

$$= \frac{1}{2} \left(f(s-a) + f(s+a) \right)$$

function of exponential order

A function $f(t)$ is said to be of exponential order $\alpha > 0$ if $|f(t)| \leq M e^{\alpha t}$

$$|e^{\alpha t} f(t)| \leq M \quad \forall t \geq T$$

$$|f(t)| \leq M e^{-\alpha t} \quad \forall t \geq T$$

unit step function

If $\alpha > 0$ then function of t defined as below and denoted by $u(t-a)$

$$u(t-a) = \begin{cases} 0 & \text{if } 0 \leq t < a \\ 1 & \text{if } t \geq a \end{cases}$$

Second shifting property

If $f(t)$ is a function of t for $t \geq 0$ whose Laplace transform is $F(s)$ then for any constant $\alpha > 0$

$$g(t) = f(t-a) \quad u(t-a) \Rightarrow \int_a^\infty f(t-a) e^{-st} dt$$

has Laplace Transform

Proof $L(g(t)) = F(s) = \int_0^\infty e^{-st} f(t) dt$

$$= \int_a^\infty e^{-st} f(t-a) dt$$

$$\int_0^a g(t) dt + \int_a^\infty e^{-st} g(t) dt$$

$$\int_0^a 0 dt + \int_a^\infty e^{-st} g(t) dt$$

$$\int_a^\infty e^{-st} f(t-a) dt$$

$$\text{put } t-a = u \quad dt = du$$

$$L(g(t)) = \int_0^\infty e^{-st} g(t) dt$$

$$= e^{-as} \int_0^\infty e^{-s(t-a)} g(t-a) dt$$

Ques. find the Laplace transform of $g(t)$

$$= \int_0^3 \cos\left(t - \frac{2\pi}{3}\right) dt + \int_3^\infty 0 dt$$

$$\text{By defn } \int_0^3 \cos\left(t - \frac{2\pi}{3}\right) dt + \int_3^\infty 0 dt$$

$$L(g(t)) = \int_0^\infty e^{-st} g(t) dt$$

$$= \int_0^3 e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt + \int_3^\infty 0 dt$$

$$= \left[\frac{e^{-st}}{s^2+1} \left(-s \cos\left(t - \frac{2\pi}{3}\right) + \sin\left(t - \frac{2\pi}{3}\right) \right) \right]_0^3$$

$$= \frac{e^{-3s}}{s^2+1} \left(-s \cos\left(3 - \frac{2\pi}{3}\right) + \sin\left(3 - \frac{2\pi}{3}\right) \right) + \frac{1}{s^2+1}$$

$$= \frac{e^{-3s}}{s^2+1}$$

Laplace Transform of Derivatives

Thm For $t > 0$, let $f(t)$ be a continuous func of exponential order α on $[0, \infty)$ and $f'(t)$ be also of exponential order and is continuous. $L\{f'(t)\} = sF(s) - f(0)$ for $s > \alpha$ $L\{f(t)\} = F(s)$

Proof

$$\text{Let } f(t) \text{ be continuous on } [0, \infty)$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt$$

$$= \left(e^{-st} f(t) \right) \Big|_0^\infty + s \int_0^\infty e^{-st} f(t) dt$$

$$= \lim_{t \rightarrow \infty} e^{-st} f(t) - f(0) + sF(s)$$

$f(t)$ is continuous exponential order α $|f(t)| \leq M e^{\alpha t}$

$$|e^{-st} f(t)| \leq e^{-st} M e^{\alpha t} = M e^{-(s-\alpha)t}$$

$$\lim_{t \rightarrow \infty} e^{-(s-\alpha)t} \rightarrow 0 \quad t \rightarrow \infty$$

$$\lim_{t \rightarrow \infty} e^{-st} f(t) \rightarrow 0 \quad L\{f'(t)\} = sF(s) - f(0)$$

Laplace Transform of n th order derivative of $f(t)$ be $\det f(t), f'(t), \dots, f^{(n)}(t)$ are continuous on $[0, \infty)$ and are of exponential order, then $\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \dots - f^{(n-1)}(0)$.

Proof $\mathcal{L}\{f^{(n)}(t)\} = \int_0^\infty e^{-st} f^{(n)}(t) dt$

Integrating by parts using general rule $\mathcal{L}\{f^{(n)}(t)\} = (e^{-st} f^{(n-1)}(t)) \dots$

Assuming that

$\lim_{t \rightarrow \infty} e^{-st} f^{(n-1)}(t) = 0, \dots, \lim_{t \rightarrow \infty} e^{-st} f(t) = 0$

$\mathcal{L}\{f^{(n)}(t)\} = -f^{(n-1)}(0) - \dots + s^n \int_0^\infty e^{-st} f(t) dt$
 $= s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$

Ques Evaluate $\mathcal{L}\{e^{at}\}$

solⁿ $f(t) = e^{at}$ $f'(t) = a e^{at}$

$f(0) = 1$ $\mathcal{L}\{f(t)\} = s \mathcal{L}\{f'(t)\} - f(0)$

$\mathcal{L}\{a e^{at}\} = s \mathcal{L}\{e^{at}\} - 1$

$a \mathcal{L}\{e^{at}\} = s \mathcal{L}\{e^{at}\} - 1$

$\frac{1}{2} (s-a) \mathcal{L}\{e^{at}\} = -1$ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

Ques Evaluate $\mathcal{L}\left\{\frac{e^{at} \sin t}{t}\right\}$

solⁿ $\mathcal{L}\{e^{at} \sin t\} = \frac{1}{(s+1)^2 + 1^2}$

$\mathcal{L}\left\{\frac{e^{at} \sin t}{t}\right\} = \int_0^\infty \frac{e^{-(s+1)t} \sin t}{t} dt$

$= 2 \left[\tan^{-1}(s+1) \right]_0^\infty$

$= 2 \left[\frac{\pi}{2} - \tan^{-1}(s+1) \right] = \pi - 2 \tan^{-1}(s+1)$

Laplace Transformation of a periodic det a function $f(t)$ be periodic with period ω so that $f(t+\omega) = f(t)$

ex 1, 2, 3 $\mathcal{L}\{f(t)\} = \int_0^\omega \frac{e^{-st} f(t)}{1 - e^{-s\omega}} dt$

$\mathcal{L}\{f(t)\} = \int_0^\omega e^{-st} f(t) dt$

$= \int_0^\omega e^{-st} f(t) dt + \int_\omega^{2\omega} e^{-st} f(t) dt + \dots$

$\mathcal{L}\{f(t)\} = \sum_{n=0}^\infty \int_n \omega^{n-1} e^{-st} f(t) dt$

put $t = n\omega + u$ $dt = du$

$t = n\omega$ $n=0$ $t = (n+1)\omega$ $n=\infty$

$\mathcal{L}\{f(t)\} = \sum_{n=0}^\infty \int_0^\omega e^{-s(n\omega+u)} f(n\omega+u) du$

$$= \frac{1}{1-e^{sT}} \int_0^T e^{st} f(t) dt = \int_0^\infty \frac{e^{st} f(t)}{1-e^{sT}} dt$$

Ques find Laplace Transform of $f(t)$.

$$f(t) = kt, \quad 0 < t < T, \quad f(t+T) = f(t)$$

Ans. As the function is periodic with T ,

$$L(f) = \int_0^T \frac{e^{st} f(t)}{1-e^{sT}} dt$$

$$= \left(\frac{1}{1-e^{sT}} \right) \int_0^T e^{st} kt dt$$

$$= \left(\frac{1}{1-e^{sT}} \right) \cdot \frac{k}{s} \left[1 + \left(\frac{e^{sT}}{s} \right) - 1 \right] \left(\frac{e^{sT}}{s^2} \right)$$

$$= \frac{k}{s} \left(\frac{1}{1-e^{sT}} \right) \left[-\frac{T e^{sT}}{s} + \frac{e^{sT}}{s} + \frac{1}{s} \right]$$

$$= \frac{k}{s} \left(\frac{1}{1-e^{sT}} \right) \left[-\frac{T e^{sT}}{s} + \frac{e^{sT}}{s} + \frac{1}{s} \right] (1-e^{sT})$$

$$= \frac{k e^{sT}}{s(1-e^{sT})} + \frac{k}{s^2 T}$$

Laplace Transform of Integral.

$$\text{If } L(f(t)) = f(s) \text{ then}$$

$$L\left(\int_0^t f(u) du\right) = \frac{f(s)}{s}$$

$$\text{Ans } L(f(t)) = f(s)$$

$$\phi(t) = \int_0^t f(u) du$$

$$t=0 \quad \phi(0) = 0$$

Leibnitz rule of differentiation under integral sign.

$$\phi(t) = \int_0^t f(u) du + \frac{d}{dt} \int_0^t f(u) du$$

$$= 0 + f(t) = f(t)$$

$$L(\phi(t)) = \int_0^\infty e^{-st} f(t) dt$$

$$\phi(s) = \frac{1}{s} L(f(t))$$

$$L(\phi(t)) = \frac{1}{s} L(f(t))$$

$$L\left(\int_0^t f(u) du\right) = \frac{1}{s} f(s)$$

Evaluation of Integral.

$$\text{If } L(f(t)) = f(s), \quad \int_0^\infty e^{-st} f(t) dt = f(s)$$

$$\text{Take limit as } s \rightarrow 0$$

$$\int_0^\infty f(t) dt = f(0)$$

Ans Evaluate $\int_0^\infty t e^{st} \cos t \, dt$

Ans $\int_0^\infty e^{st} \cos t \, dt = \int_0^\infty e^{st} t \cos t \, dt$

$= L(t \cos t)$

$= -\frac{d}{ds} \left(\frac{s}{s^2+1} \right)$

$= - \left(\frac{s^2+1}{(s^2+1)^2} - \frac{s \cdot 2s}{(s^2+1)^2} \right) = \frac{1-s^2}{(s^2+1)^2}$

$= \frac{s^2-1}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2} = \frac{3}{(s^2+1)^2} = \frac{3}{25}$

Therefore $J_0(t) = \frac{1}{\sqrt{1+s^2}}$

To find $L(J_0 \cos t) = \frac{1}{\sqrt{1+s^2}}$

To find $L(t J_0 \cos t) = \frac{s}{(s^2+1)^2}$

$L \frac{2}{\sqrt{1+s^2}} \int_0^\infty e^{st} \cos t \, dt = \frac{1}{\sqrt{1+s^2}}$

$L \int_0^\infty \frac{e^{-x}}{x} dx = \frac{1}{s} \log(s+1)$

$L \int_0^\infty \frac{\sin x}{x} dx = \frac{1}{s} \tan^{-1} \frac{1}{s}$

$L \int_0^\infty \frac{\cos x}{x} dx = \frac{1}{2s} \log(s^2+1)$

Ans Prove that $\int_0^\infty J_0(t) \cos t \, dt = \log 2$

$L(t \cos t) = \frac{1}{s^2+1} - \frac{s}{s^2+1}$

$L \left(\frac{J_0(t) - \cos t}{t} \right) = \int_0^\infty \left(\frac{1}{s^2+1} - \frac{s}{s^2+1} \right)$

$= \log 2 + \log \left(\frac{s^2+1}{s^2+1} \right)$

$= \log 2 + \log \left(\frac{s^2+1}{s^2+1} \right)$

Put $s=0$ in (1)

$\int_0^\infty \frac{J_0(t) - \cos t}{t} dt = \log 2 + \log 1$

$= \log 2$

Inverse Laplace Transforms.

$$1) \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$2) \mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$3) \mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \frac{1}{2a} (1 - \sin at)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{2a} (1 + \sin at)$$

Ques find inverse Laplace transform of $\frac{1}{s^2+2}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2}\right) = \frac{1}{\sqrt{2}} \sin \sqrt{2}t$$

$$= \frac{1}{\sqrt{2}} \sin \sqrt{2}t = \frac{1}{\sqrt{2}} \sin \sqrt{2}t$$

Ques find $\mathcal{L}^{-1}\left(\frac{1}{s} \sin t\right)$

$$\mathcal{L}^{-1}\left(\frac{1}{s} \sin t\right) = \frac{1}{s} \sin t = \frac{1}{s} \sin t$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2} + \frac{1}{s^4} + \frac{1}{s^6}\right) = \frac{t}{1!} + \frac{1}{3!} \frac{t^3}{1!} + \frac{1}{5!} \frac{t^5}{1!}$$

Ques find the inverse Laplace transform of $\frac{2s-5}{4s^2+25} + \frac{4s-18}{9s^2-4}$

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{2s-5}{4s^2+25} + \frac{4s-18}{9s^2-4}\right) &= \mathcal{L}^{-1}\left(\frac{2s-5}{4(s^2+\frac{25}{4})} + \frac{4s-18}{9(s^2-\frac{4}{9})}\right) \\ &= \frac{1}{4} \mathcal{L}^{-1}\left(\frac{2s-5}{s^2+\frac{25}{4}}\right) - \frac{2}{9} \mathcal{L}^{-1}\left(\frac{4s-18}{s^2-\frac{4}{9}}\right) \\ &= \frac{1}{4} \left(2 \cos \frac{5t}{2} - \sin \frac{5t}{2}\right) - \frac{2}{9} \left(4 \cosh \frac{2t}{3} - 18 \sinh \frac{2t}{3}\right) \\ &= \frac{1}{2} \cos \frac{5t}{2} - \frac{1}{2} \sin \frac{5t}{2} - \frac{8}{9} \cosh \frac{2t}{3} + \frac{4}{3} \sinh \frac{2t}{3} \end{aligned}$$

other method

If $L^{-1}(F(s)) = f(t)$ then
 $L^{-1}(F(s)e^{-as}) = e^{-at} f(t)$

If $L^{-1}(F(s)) = f(t)$ and $f(t) \propto$
 then $L^{-1}(sF(s)) = \frac{d}{dt} f(t)$

$L^{-1}(s^n F(s)) = \frac{d^n}{dt^n} f(t)$

$L^{-1}(F(s)) = f(t)$ then $L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau) d\tau$

$L^{-1}(F(s)) = f(t)$ then $L^{-1}\left(-\frac{d}{ds} F(s)\right) = t f(t)$

If $L^{-1}(F(s)) = f(t)$ then $L^{-1}\left(\int_0^\infty F(s) ds\right) = \frac{f(t)}{t}$

provided inverse transform of $\int_0^\infty F(s) ds$ can be calculate

Ques And inverse Laplace transform of

P. $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

$L^{-1}\left(\frac{1}{s^3(s^2+1)}\right) = \int_0^t \int_0^\tau \sin u du d\tau$

$= \int_0^t \int_0^\tau (1-\cos u) du d\tau$

$= \int_0^t \left(\tau - \sin \tau \right) d\tau$

$= \left(\frac{\tau^2}{2} + \cos \tau \right) \Big|_0^t = \frac{t^2}{2} + \cos t - 1$

Convolution Thm :

statement : If $L^{-1}(F(s)) = f(t)$ and $L^{-1}(G(s)) = g(t)$ then

$L^{-1}(F(s)G(s)) = \int_0^t f(\tau)g(t-\tau) d\tau$

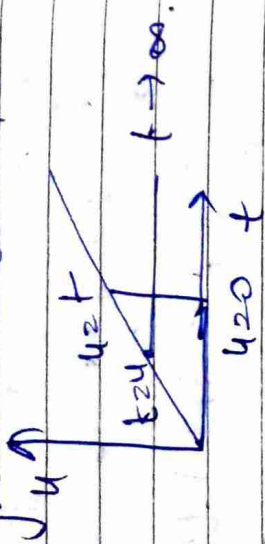
ex. det $\phi(t) = \int_0^t f(\tau)g(t-\tau) d\tau$

$L(\phi(t)) = L\left(\int_0^t f(\tau)g(t-\tau) d\tau\right)$

$= \int_0^\infty e^{-st} \left(\int_0^t f(\tau)g(t-\tau) d\tau \right) dt$

$= \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau) dt d\tau$

Integration under the area



$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} f(t) dt$$

$$= f(s) \cdot g(s)$$

Now Apply Convolution Thm to Evaluate

$$L\left\{\left(\frac{s}{s^2+a^2}\right)\right\} = \frac{s}{s^2+a^2}$$

$$L\{f(s)\} = L\left\{\left(\frac{s}{s^2+a^2}\right)\right\} = \cos at$$

$$L\{g(s)\} = L\left\{\left(\frac{1}{s^2+a^2}\right)\right\} = \frac{1}{a} \sin at$$

Key Convolution Thm

$$L\left\{\left(\frac{s}{s^2+a^2}\right)\right\} = L\left\{\left(\frac{s}{s^2+a^2}\right)\right\}$$

$$\frac{1}{a} \int_0^t \cos au \sin (t-u) du$$

$$\frac{1}{2a} \int_0^t 2 \cos au \sin (t-u) du$$

$$\frac{1}{2a} \int_0^t \sin at - \sin (2a u - at) du$$

$$\frac{1}{2a} \left(4 \sin at + \cos (2au - at) \right)$$

$$\frac{1}{2a} \sin at$$

Solutions of linear differential Eq with constant coefficient by transform method

Step 1. Take Laplace transform of both side of differential Eq using method of transform of derivatives

Step 2. Step 1 gives an algebraic Eqn

Step 3. Divide by coefficient of y which is used in place of $f(s)$

Step 4. Take inverse transform of both side.

$$L\{f(t)\} = s f(s) - f(0)$$

$$L\{t^n f(t)\} = s^n f(s) - \dots$$

$$L\left\{\frac{dy}{dt}\right\} = sy - y(0)$$

Chen solve $\frac{d^4 y}{dt^4} - 16y = 0$ by transform method, where.

$$y(0) = 1, y'(0) = y''(0) = y'''(0) = 0.$$

The given Eqn is $\frac{d^4 y}{dt^4} - 16y = 0$

Taking Laplace Transform of both side

$$(s^4 y - s^3 y(0) - \dots) - 16y = 0.$$

$$(s^4 - 16)y = s^3$$

$$y = \frac{s^3}{s^4 - 16}$$

$$\frac{s^3}{s^4 - 16} = \frac{(s-k)(s+k)(s^2+k^2)}{s^4 - 16}$$

$$= \frac{A}{(s-k)} + \frac{B}{(s+k)} + \frac{Cs+D}{s^2+k^2}$$

Multiplying both side by $(s-k)(s+k)(s^2+k^2)$

$$s^3 = A(s+k)(s^2+k^2) + B(s-k)(s^2+k^2) + (Cs+D)(s^2+k^2)$$

put $s=k$ in (3)
 $R_3 = A(2k)(2k) = A = 1/4$

$$s = -k \quad R^3 = B(-2k)(2k) = 0 = 1/4$$

$$1 = A + B + C \quad \text{--- (4)}$$

$$0 = AR^3 - BR^3 - CR^2 \quad \text{--- (5)}$$

$$C = 1 - A - B = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

$$DR^2 = R^3(A-B) = R^3\left(\frac{1}{4} - \frac{1}{4}\right)$$

$D > 0$

Substituting these value in (3)
 $\frac{s^3}{s^4 - k^4} = \frac{1}{4} \cdot \frac{1}{s-k} + \frac{1}{4} \cdot \frac{1}{s+k} + \frac{1}{2} \cdot \frac{1}{s^2 + k^2}$

Taking inverse Laplace transform

$$L^{-1}\left(\frac{s^3}{s^4 - k^4}\right) = \frac{1}{4}e^{kt} + \frac{1}{4}e^{-kt} + \frac{1}{2}t \cos kt$$

$$= \frac{1}{2}(\cos kt + t \cos kt)$$

ser of ordinary diff. eqn with variable coeff. by Transform method

$$(x^m) \frac{d^m}{dx^m} (y(x))$$

Ques solve $x \frac{dy}{dx} + (1-x) \frac{dy}{dx} = 2y = 0$

$$y(0) = 1 \quad y'(0) = 2$$

The given eqn is $x \frac{dy}{dx} + (1-x) \frac{dy}{dx} = 2y$

Taking Laplace transform of both side

$$L\left(x \frac{dy}{dx}\right) + L\left((1-x) \frac{dy}{dx}\right) = 2L(y)$$

$$-s \frac{d}{ds} (sY - y(0)) - sY(0) + (1-s) \frac{d}{ds} (sY - y(0)) = 2sY$$

$$+ 2 \frac{d}{ds} (sY - y(0)) - 2y(0)$$

$$-s^2 \frac{dY}{ds} + 2sY - (1-s)Y = 2sY$$

$$-s^2 \frac{dY}{ds} + 2sY - Y + sY = 2sY$$

$$(s^2 - 2s) \frac{dx}{ds} - 15x = 0$$

$$(s^2 - 2s) \frac{dy}{ds} = -15y$$

$$\frac{dy}{y} = \frac{-15}{s^2 - 2s} ds$$

$$\log y = -\log(s-2) + \log C$$

$$\log y = \log \frac{C}{s-2}$$

$$y = \frac{C}{s-2}$$

$$y = 4e^{2s}$$

$$C = 4$$

$$y = e^{2s} \text{ is soln.}$$

soln of simultaneous linear eqn with constant coefficients by Transform method.

Qus. Solve the following simultaneous eqn

$$3 \frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$$

$$\frac{dx}{dt} + 4 \frac{dy}{dt} + 13y = 0$$

$$\text{where } x(0) = 3, y(0) = 0$$

The given eqn are

$$3 \frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$$

$$\frac{dx}{dt} + 4 \frac{dy}{dt} + 13y = 0$$

Taking Laplace transform of (1) and (2)

$$3(s\bar{x} - x(0)) + (s\bar{y} - y(0)) + 2\bar{x} = \frac{1}{s}$$

$$s\bar{x} - x(0) + 4(s\bar{y} - y(0)) + 13\bar{y} = 0$$

$$x(0) = 3 \text{ and } y(0) = 0$$

$$3(s\bar{x} - 3) + (s\bar{y} - 0) + 2\bar{x} = \frac{1}{s}$$

$$s\bar{x} = 3 + 4s\bar{y} + 3\bar{y} - \frac{1}{s}$$

$$(3s + 2)\bar{x} + s\bar{y} = 3 + \frac{1}{s}$$

$$s\bar{x} + (4s + 2)\bar{y} = 3$$

multiplying (3) by $(4s+3)$ and (4) by $(3s+2)$

$$(3s+2)(4s+3)\bar{x} + s(4s+3)\bar{y} = 9(4s+3) + \frac{4s+3}{s}$$

$$s\bar{x} + s(4s+3)\bar{y} = 3s$$

Subtracting (6) from (5), we have

$$(3s+2)(4s+3)\bar{x} - s(4s+3)\bar{y} = 9(4s+3) + \frac{4s+3}{s}$$

$$(11s^2 + 17s + 6)\bar{x} = 9(4s+3) + \frac{4s+3}{s}$$

$$(s+1)(11s+6)\bar{x} = 9(4s+3) + \frac{4s+3}{s}$$

$$\bar{x} = \frac{1}{2s} + \frac{1}{s+1} + \frac{33}{2(11s+6)}$$

Taking inverse transform

$$x = \frac{1}{2}e^{1/s} + e^{1/(s+1)} + \frac{3}{2}e^{1/(11s+6)}$$

Again multiplying (3) by s and (4) by $(3s+2)$

$$s(3s+2)\bar{x} + 2s\bar{y} = 9s + 1$$

$$s(3s+2)\bar{x} + (4s+3)(3s+2)\bar{y} = 3(3s+2)$$

Subtracting Eqn from Eqn

$$(4s+3)(3s+2)\bar{y} - s(4s+3)\bar{y} = 9s + 6 - 9s + 1$$

$$(s+1)(11s+6)\bar{y} = \frac{7}{s}$$

$$\bar{y} = \frac{7}{s(s+1)(11s+6)} = \frac{1}{s} + \frac{11}{s+1} + \frac{11}{11s+6}$$

Taking inverse transform

$$y = 7\left(\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+\frac{6}{11}}\right)$$

The required soln is $x = \frac{1}{2}e^{1/s} + e^{1/(s+1)} + \frac{3}{2}e^{1/(11s+6)}$ and $y = e^{1/s} + e^{1/(s+1)} + e^{1/(11s+6)}$

fourier transform

fourier transform

let $f(t)$ be a function defined for $t \in (-\infty, \infty)$

$$\int_{-\infty}^{\infty} f(t) e^{-ist} dt$$

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-ist} dt$$

$$f(t) = F^{-1}(F(s)) = \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

The inverse formula

$$f(t) = F^{-1}(F(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{ist} ds$$

fourier sine transform

$$F_s(s) = F_s(f(t)) = \int_0^{\infty} f(t) \sin st dt$$

The inverse formula

$$f(t) = F_s^{-1}(F_s(s)) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin st ds$$

Ans do — (4)

fourier cosine transform

$$F_c(s) = F_c(f(t)) = \int_0^{\infty} f(t) \cos st dt$$

The inverse formula

$$f(t) = F_c^{-1}(F_c(s)) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos st ds$$

Properties of fourier

(1)

Linear property

If $f(x)$ and $g(x)$ are any two functions whose fourier transforms exist and C_1 and C_2 are arbitrary constants

$$F(C_1 f(x) + C_2 g(x)) = C_1 F(f(x)) + C_2 F(g(x))$$

(2)

$$F(C_1 f(x) + C_2 g(x)) = C_1 F(f(x)) + C_2 F(g(x))$$

(3)

$$F(C_1 f(x) + C_2 g(x)) = C_1 F(f(x)) + C_2 F(g(x))$$

(4)

change of scale property

(1)

If $f(x)$ is the fourier transform of $f(x)$ then the fourier transform of $f(ax)$ is $\frac{1}{|a|} F\left(\frac{s}{a}\right)$

② If the Fourier sine transform of $f(x)$ is $F_s(s)$ then Fourier sine transform of $f(x)$ is $\frac{1}{a} F_s\left(\frac{s}{a}\right)$

③ If the Fourier cosine transform of $f(x)$ is $F_c(s)$ then the Fourier cosine transform of $f(ax)$ is $\frac{1}{a} F_c\left(\frac{s}{a}\right)$.

Shifting property

If $f(x)$ is Fourier transform of $F(s)$, then show that $e^{-as} f(x)$ is the Fourier transform of $F(s-a)$.

Modulation property

If $f(x)$ is the infinite Fourier transform of $F(s)$ then Fourier transform of $f(x) \cos ax$ is given by $\frac{1}{2} [F(s-a) + F(s+a)]$.

Ques Find the Fourier transform of $f(x) = e^{-x}$

A $\int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \int_{-\infty}^{\infty} e^{-ax} e^{isx} dx$$

$$= \int_{-\infty}^0 e^{-ax} e^{isx} dx + \int_0^{\infty} e^{-ax} e^{isx} dx$$

$$= \int_{-\infty}^0 e^{ax} e^{-isx} dx + \int_0^{\infty} e^{-ax} e^{isx} dx$$

$$= \int_{-\infty}^0 e^{(a-is)x} dx + \int_0^{\infty} e^{-(a+is)x} dx$$

$$= \left[\frac{e^{(a-is)x}}{a-is} \right]_{-\infty}^0 + \left[\frac{e^{-(a+is)x}}{-(a+is)} \right]_0^{\infty}$$

$$= \frac{1}{a-is} + \frac{1}{a+is}$$

$$= \frac{2ay}{a^2 + b^2}$$

Some Based on Fourier Sine and Cosine Transform

Ques find the Fourier Sine transform of e^{-ax}/x

A $f(x) = \frac{e^{-ax}}{x}$

$\bar{f}_s(s) = \int_0^\infty f(x) \sin sx \, dx$

$= \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$

Sol. where using Leibnitz rule

$\frac{d}{ds} (\bar{f}_s(s)) = \int_0^\infty \frac{\partial}{\partial s} \left(\frac{e^{-ax}}{x} \sin sx \right) dx$

$= \int_0^\infty e^{-ax} \cos sx \, dx$

$= \frac{a}{a^2 + s^2}$

Integrating w.r.s we have

$\bar{f}_s(s) = \frac{1}{a} \tan^{-1} \frac{s}{a} + C$

$= \tan^{-1} \frac{s}{a} + C$

Put $s=0$ in (1)

$\bar{f}_s(0) = C$

put $s=0$ in (2)

$\bar{f}_s(s) = C$

$C=0$

from (2)

$\bar{f}_s(s) = \tan^{-1} \frac{s}{a}$

Ques find $f(x)$ given that $\bar{f}_s(s) = \frac{e^{-as}}{s}$

and deduce $\bar{f}_s\left(\frac{1}{s}\right)$

$\bar{f}_s(s) = \frac{e^{-as}}{s}$

Sine inversion formula

$f(x) = \frac{2}{\pi} \int_0^\infty \frac{e^{-as}}{s} \sin sx \, ds$

Diff both w.r to x

$\frac{df}{dx} = \frac{2}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left(\frac{e^{-as}}{s} \sin sx \right) ds$

$= \frac{2}{\pi} \int_0^\infty e^{-as} \cos sx \, ds$

$= \frac{2}{\pi} \left[\frac{e^{-as} \sin sx}{-a \cos sx} + \frac{e^{-as} \cos sx}{a \sin sx} \right]_0^\infty$

$$= \frac{2}{\pi} \left(e^{-\frac{1}{a^2+x^2}} - e^{-a^2} \right)$$

$$\frac{d}{dx} = \frac{2}{\pi} \left(\frac{a}{a^2+x^2} \right)$$

Integrate w.r.t. x

$$f(x) = \frac{2a}{\pi} \int_a^\infty \frac{1}{a} \left(\tan^{-1} u \right) du$$

$$f(0) = 0$$

$$f(x) = \frac{2}{\pi} \tan^{-1} u$$

$$f(x) = \frac{2}{\pi} \tan^{-1} u$$

Convolution +

The Convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$

$$f * g = \int_{-\infty}^{\infty} f(x-u) g(u) du$$

$$f * g = g * f$$

put $x-u = v$ then $du = -dv$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u) g(u) du$$

$$= \int_{-\infty}^{\infty} g(u) f(x-u) du$$

$$= (g * f)(x)$$

$$f * g = g * f$$

Fourier Transform of the derivative

If the Fourier Transform of $f(x)$ is $F(s)$ then the Fourier Transform of $f'(x)$ is given by $is F(s)$

$$\text{Proof: } \int_{-\infty}^{\infty} f'(x) e^{-isx} dx = \int_{-\infty}^{\infty} f(x) (-is) e^{-isx} dx$$

$$= (e^{15n} f(n)) - \int_{-\infty}^{\infty} e^{-15n} (-15) f(n) dn$$

$$= 0 + 15 \int_{-\infty}^{\infty} e^{-15n} f(n) dn$$

$$= 15 F(s)$$

extension of above
if f(n) and first (n-1) derivatives
of f(n) vanish identically as n approaches

$$F\left(\frac{d^m f}{dn^m}\right) = (15)^m F(s)$$

$$\text{Prp } F\left(\frac{d^m f}{dn^m}\right) = \int_{-\infty}^{\infty} e^{-15n} \frac{d^m f}{dn^m} dn$$

Integrate by parts

$$F\left(\frac{d^m f}{dn^m}\right) = \left(e^{-15n} \frac{d^{m-1} f}{dn^{m-1}} \right) - \int_{-\infty}^{\infty} e^{-15n} (-15) \frac{d^{m-1} f}{dn^{m-1}} dn$$

$$= 0 + 15 \int_{-\infty}^{\infty} e^{-15n} \frac{d^{m-1} f}{dn^{m-1}} dn$$

Again integrate

$$F\left(\frac{d^m f}{dn^m}\right) = 15 \int_{-\infty}^{\infty} e^{-15n} \frac{d^{m-1} f}{dn^{m-1}} dn$$

$$= \int_{-\infty}^{\infty} e^{-15n} (-15) \frac{d^{m-2} f}{dn^{m-2}} dn$$

$$= 15 \left(0 + 15 \int_{-\infty}^{\infty} e^{-15n} \frac{d^{m-2} f}{dn^{m-2}} dn \right)$$

$$= (15)^2 F\left(\frac{d^{m-2} f}{dn^{m-2}}\right)$$

Apply general rule of integration

$$F\left(\frac{d^m f}{dn^m}\right) = \int_{-\infty}^{\infty} e^{-15n} f(n) dn$$

$$F\left(\frac{d^m f}{dn^m}\right) = (15)^m F(s)$$

Relation B/w Fourier and Laplace Transform

$$f(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \phi(\omega) d\omega \quad t > 0$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

$$= \int_{-\infty}^0 e^{-i\omega t} f(t) dt + \int_0^{\infty} e^{-i\omega t} f(t) dt$$

$$= \int_{-\infty}^0 e^{-i\omega t} f(t) dt + \int_0^{\infty} e^{-i\omega t} f(t) dt$$

$$= \int_{-\infty}^0 e^{-i\omega t} f(t) dt + \int_0^{\infty} e^{-i\omega t} f(t) dt$$

$$= \int_{-\infty}^0 e^{-i\omega t} f(t) dt + \int_0^{\infty} e^{-i\omega t} f(t) dt$$

$$= L\{\phi(\omega)\}$$

$$F(\omega) = L\{\phi(\omega)\}$$

Pareyval's identity for Fourier Transform

If $f(x)$ and $g(x)$ are Fourier Transform of $f(x)$ and $g(x)$ then

$$\int_{-\infty}^{\infty} f(x) g(x) dx = \int_{-\infty}^{\infty} F(\omega) G(\omega) d\omega$$

$$\int_{-\infty}^{\infty} f(x) g(x) dx = \int_{-\infty}^{\infty} f(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega x} d\omega dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \left(\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) F(\omega) d\omega$$

Notes: Find the Fourier sine and cosine transforms of the function e^{-mx} and by using its second derivative

Let $f(x) = e^{-mx}$ for $m > 0$

$$f'(x) = -m e^{-mx}$$

$$f''(x) = m^2 e^{-mx}$$

$$F(f''(x)) = -m^2 F(f(x)) = -m^2 F(e^{-mx})$$

$$F(f''(x)) = -m^2 F(f(x)) = -m^2 F(e^{-mx})$$

$$F(f''(x)) = -m^2 F(f(x)) = -m^2 F(e^{-mx})$$

$$F(f''(x)) = -m^2 F(f(x)) = -m^2 F(e^{-mx})$$

$$F(f''(x)) = -m^2 F(f(x)) = -m^2 F(e^{-mx})$$

The finite sine transform

$$f_s(f(x)) = \int_0^b f(x) \sin \frac{n\pi x}{b} dx$$

The finite cosine transform

$$f_c(f(x)) = \int_0^b f(x) \cos \frac{n\pi x}{b} dx$$

Ques Find the finite sine transform of $f(x) = 2x$ $0 < x < 4$

$$f_s(f(x)) = \int_0^4 2x \sin \frac{n\pi x}{4} dx$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} dx$$

$$= 2 \left(\frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right) - 2 \left(\frac{-\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right)$$

$$= 2 \left(\frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right) - 2 \left(\frac{-\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right)$$

$$= 0 - 2 \left(\frac{\sin 0}{\frac{n\pi}{4}} \right)$$

$$= \frac{-32}{1\pi} (-1)^n$$

Soln of diff eqn by fourier

$$(1) \frac{dy}{dx} = 2 \frac{dy}{dx}$$

$$(2) \frac{dy}{dx} = 2 \frac{dy}{dx}$$

Method to Solve Above type of Equation

$$(1) \frac{dy}{dx} = 2 \frac{dy}{dx}$$

(2) If $\left(\frac{dy}{dx} \right)_{x=0}$ is given then we use infinite cosine transform

$$\text{Ques Solve } \frac{dy}{dx} = 2 \frac{dy}{dx}$$

$$(1) y(0) = 2 \quad y(2) = 2 \quad y(4) = 2$$

$$\text{Soln } y = 2 \frac{dy}{dx}$$

Ques Both side of $y = 2 \frac{dy}{dx}$ multiply and integrate from 0 to 2

$$\int_0^2 y dy = \int_0^2 2 \frac{dy}{dx} dx$$

$$\frac{d}{ds} \frac{1}{s} = -\frac{1}{s^2} \int_0^\infty \frac{1}{s} \cos s x dx$$

$$\frac{d}{ds} \frac{1}{s} = -\frac{1}{s^2} \int_0^\infty \frac{1}{s} \cos s x dx$$

$$= -\int_0^\infty \frac{1}{s^2} \cos s x dx$$

$$\frac{d}{ds} \frac{1}{s} = -\int_0^\infty \frac{1}{s^2} \cos s x dx$$

$$\frac{d}{ds} \frac{1}{s} = -\int_0^\infty \frac{1}{s^2} \cos s x dx$$

$$\frac{d}{ds} \frac{1}{s} = -\int_0^\infty \frac{1}{s^2} \cos s x dx$$

$$\text{Integrating by } \frac{1}{s} = -\int_0^\infty \frac{1}{s^2} \cos s x dx + \log A$$

$$\log \frac{1}{s} = -\int_0^\infty \frac{1}{s^2} \cos s x dx$$

$$\frac{1}{s} = A e^{-s^2}$$

$$\frac{1}{s} (s^2) = A e^{-s^2}$$

$$u(s^2) = 2x$$

$$\text{The function}$$

$$u(s^2) = \int_0^\infty \frac{1}{s^2} \cos s x dx$$

$$u(s^2) = \int_0^\infty \frac{1}{s^2} \cos s x dx$$

$$A = 2 \int_0^\infty \frac{1}{s} \sin s x dx$$

$$A = 2 \int_0^\infty \frac{1}{s} \left(\frac{\cos s x}{s} - \frac{\sin s x}{s^2} \right) dx$$

$$A = 2 \int_0^\infty \left(\frac{\cos s x}{s} - \frac{\sin s x}{s^2} \right) dx$$

$$A = -\frac{2x}{s} (-1)^s$$

$$u(s^2) = -\frac{2x}{s} (-1)^s e^{-s^2}$$

Inverse Fourier Transform

$$u(s^2) = -\frac{2x}{s} \sum_{s=1}^\infty \frac{1}{s} \sin s x$$

$$u(s^2) = -\frac{2x}{s} \sum_{s=1}^\infty \frac{1}{s} \sin s x$$