MAA OMWATI DEGREE COLLEGE, HASSANPUR (PALWAL)

NOTES

SUBJECT : Numerical Analysis

CLASS : B.SC 5th Sem

SESSION : 2021-2022

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Ch-1

Finite Difference Operators

Forward differences

$$\Delta f(x) = f(x+h) - f(x)$$

Backward differences

$$\nabla f(x) = f(x) - f(x-h)$$

Shift Operators

$$E f(x) = f(x+h)$$
$$E^{-1} f(x) = f(x-h)$$

Central Difference Operator

$$\delta f(x) = f(x+h/2) - f(x-h/2)$$

Average operator

$$\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2) \right]$$

Prove that

$$\Delta \cos(cx+d) = 2\sin\frac{ch}{2}\cos\left(cx+d+\frac{ch+\pi}{2}\right)$$

Pf: $\Delta \cos(cx+d)$.

$$= \cos[c(x+h)+d] - \cos(cx+d)$$

$$= -2\sin\left(cx+d+\frac{ch}{2}\right)\sin\left(-\frac{ch}{2}\right)$$

$$= 2\sin\frac{ch}{2}\cos\left[cx+d+\frac{ch}{2}+\frac{\pi}{2}\right]$$

$$= 2\sin\frac{ch}{2}\cos\left[cx+d+\frac{ch+\pi}{2}\right]$$

* Operator E obeys the law of indices i.e

$$E^m[E^n f(x)] = E^{m+n} f(x)$$

Pf:

$$E f(x) = f(x+h)$$

$$E^2 f(x) = E[E(f(x))]$$

$$E[f(x+h)]$$

$$f(x+2h)$$

Ily,

$$E^n f(x) = f(x+nh)$$

$$E^m[E^n f(x)] = E^m[f(x+nh)]$$

$$f(x+nh+mh)$$

$$f(x+(n+m)h)$$

$$= E^{m+n} f(x)$$

* Prove that $\nabla = \Delta E^{-1}$

Pf: $\nabla f(x) = f(x) - f(x-h)$

$$\Delta f(x-h)$$

$$\Delta E^{-1} f(x)$$

$$\nabla = \Delta E^{-1}$$

Numericals

Q: Locate the error in the following and correct it

1.203, 1.424, 1.681, 2.379, 2.848, 3.429, 4.136.

Sol^n Incorrect difference table is

$10^3 y$	$10^3 \Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1203				
	221			
1424		36		
	257		18	
1681		54		4
	311		22	
1992		76		−16
	387		6	
2379		82		24
	469		30	
2848		112		−16
	581		14	
3429		126		
	707			
4136				

Sum of all value in $\Delta^4 y$ is −4 which is minimum as compared to sum of values in other columns

2379 is the incorrect value

Coefficients of the terms in the expansion $(1-p)^4$ are $1, -4, 6, -4, 1$

Errors in the column $10^3 \Delta^4 y$ are $e, -4e, 6e, -4e, e$

$$6e = 24$$
$$e = 4$$

10^3 (Correct value) $= $ Incorrect value $- e$

$= 2379 - 4$

$= 2375$

Required correct value $= \frac{2375}{10^3}$

$= 2.375$

Q: Given that

$u_0 + u_8 = 1.9243$

$u_1 + u_7 = 1.9590$

$u_2 + u_6 = 1.9823$

$u_3 + u_5 = 1.9956$

Find u_4

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Sol^n $(E-1)^8 u_0 = 0$

$$\left(E^8 - 8E^7 + 28E^6 - 56E^5 + 70E^4 - 56E^3 + 28E^2 - 8E + 1\right) u_0 = 0$$

$$u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$$

$$(u_8 + u_0) - 8(u_7 + u_1) + 28(u_6 + u_2) - 56(u_5 + u_3) + 70u_4 = 0$$

$$1.9243 - 8(1.9590) + 28(1.9823) - 56(1.9956) + 70u_4 = 0$$

$$70u_4 = 69.9969$$

$$u_4 = \frac{69.9969}{70}$$

$$u_4 = 0.999956$$

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Ch-3
Interpolation with Equal Intervals

Let us assume that the function $y = f(x)$ is known for certain values of x say $x_0, x_1, x_2, \ldots x_n$ as $f(x_0), f(x_1), f(x_2) \ldots f(x_n)$.

The process of finding value of $f(x)$ corresponding to x_i where $x_0 < x_i < x_n$ with help of given value is called interpolation.

The process of finding the values of $f(x)$ corresponding to $x = x_i$ where either $x_i < x_0$ or $x_i > x_n$ is called extrapolation

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* Newton - Gregory formula for forward Interpolation

Pf: Let $y = f(x)$ be a function of x which assumes values $f(a), f(a+h), f(a+2h), \ldots, f(a+nh)$ for $x = a, a+h, a+2h, \ldots, a+nh$.

Let,

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-\overline{a+h}) + A_3(x-a)(x-\overline{a+h})(x-\overline{a+2h}) + \ldots + A_n(x-a)(x-\overline{a+h}) \ldots (x-\overline{a+(n-1)h}) \quad -(1)$$

where $A_0, A_1, \ldots A_n$ are constants to be determined

Put $x=a$

$$f(a) = A_0$$

Put $x = a+h$

$$f(a+h) = A_0 + A_1 h$$

$$\frac{f(a+h) - f(a)}{h} = A_1$$

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$$A_1 = \frac{\Delta f(a)}{h}$$

Put $x = a+2h$

$$f(a+2h) = A_0 + 2h A_1 + 2h^2 A_2$$

$$2h^2 A_2 = f(a+2h) - f(a) - 2h\left[\frac{f(a+h) - f(a)}{h}\right]$$

$$2h^2 A_2 = f(a+2h) - f(a) - 2f(a+h) + 2f(a)$$

$$A_2 = \frac{[f(a+2h) - f(a+h)] - [f(a+h) - f(a)]}{2h^2}$$

$$A_2 = \frac{\Delta f(a+h) - \Delta f(a)}{2!\,h^2}$$

$$A_2 = \frac{\Delta^2 f(a)}{2!\,h^2}$$

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Illy,
$$A_3 = \frac{\Delta^3 f(a)}{3!\, h^3}$$
,
,
$$A_n = \frac{\Delta^n f(a)}{n!\, h^n}$$

Substituting the value of $A_0, A_1, A_2, \ldots A_n$ in eq^n (1)

$$f(x) = f(a) + (x-a)\frac{\Delta f(a)}{h} + \frac{(x-a)(x-\overline{a+h})\,\Delta^2 f(a)}{2!\, h^2} + \ldots$$

$$(x-a)(x-\overline{a+h}) \ldots (x-\overline{a+(n-1)h}) \frac{\Delta^n f(a)}{n!\, h^n}$$

Another form of Newton - Gregory forward interpolation formula.

Put $x = a + uh$

$$u = \frac{x-a}{h}$$

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we obtain

$$f(a+uh) = f(a) + \frac{u^{(1)}}{1!}\Delta f(a) + \frac{u^{(2)}}{2!}\Delta^2 f(a) + \frac{u^{(3)}}{3!}\Delta^3 f(a) + \ldots,$$
$$\frac{u^{(n)}\,\Delta^n f(a)}{n!}$$

Where $u^{(i)}$ denotes factorial polynomial of order i.

** Newton Gregory backward interpolation formula.

$$f(\overline{a+nh}+uh) = f(a+nh) + \frac{u}{1!}\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \ldots + \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(a+nh)$$

$$\frac{1}{2}(q+r) + \frac{1}{16}(q+r-p-s)$$

$$f\left(a+\frac{3}{2}h\right) = A + \frac{1}{16} \times \frac{2}{3} B$$

$$A + \frac{1}{24} B$$

Subdivision of intervals

Let δ and e be the difference and shift operators respectively when interval of differencing be h.

$$a + H = a + nh$$

$$y_{a+H} = y_{a+nh}$$

$$E y_a = e^n y_a$$

$$E = e^n$$

$$1 + \Delta = (1+\delta)^n$$

$$1 + \delta = (1+\Delta)^{1/n}$$

$$\delta = (1+\Delta)^{1/n} - 1$$

$$\left\{1 + \frac{1}{n}\Delta + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2!}\Delta^2 + \dots\right\} - 1$$

$$\frac{1}{n}\Delta + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2!}\Delta^2 + \dots$$

$$\delta^2 = \left\{\frac{1}{n}\Delta + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2!}\Delta^2 + \dots\right\}^2$$

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ch-3

Interpolation with unequal intervals

Thm: Prove that divided difference are symmetric functions of their argument.

Pf: The first divided difference on x_0, x_1 are given by

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0)$$

$$f(x_0, x_1) = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1}$$

$$= \sum \frac{f(x_0)}{x_0 - x_1}$$

$f(x_0, x_1)$ is a symmetric function.

The second divided difference are given by

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$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0} \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right]$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$\sum \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

Thus the second divided difference $f(x_0, x_1, x_2)$ is a symmetric function and is independent of the order of the argument x_0, x_1, x_2.

eg: Given the following data, find $f(x)$ as a polynomial in powers of $x-5$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		11			
2	26		7	1	
		32			
3	58		11	1	0
		54			
4	112	118	16	1	0
7	466	228	22		
9	922				

By Newton's divided difference formula.

$$f(x) = f(0) + x\Delta f(0) + x(x-2)\Delta^2 f(0)$$

$$+ x(x-2)(x-3)\Delta^3 f(0)$$

$$4 + 11x + 7x(x-2) + 1x(x-2)(x-3)$$

$$4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6x$$

$$x^3 + 2x^2 + 3x + 4$$

5	1	2	3	4
		5	35	190
	1	7	38	194
		5	60	
	1	12	98	
		5		

$$(x-5)^3 + 17(x-5)^2 + 98(x-5) + 194$$

* Lagrange's Interpolation formula

Let $f(x)$ be the nth degree polynomial of x and $x_0, x_1, x_2, \ldots x_n$ be the values of arguments not equally spaced and corresponding entries be $f(x_0), f(x_1), f(x_2), f(x_3) \ldots f(x_n)$

Consider the function $f(x)$ as

$$f(x) = A_0(x-x_1)(x-x_2)\ldots(x-x_n)$$
$$+ A_1(x-x_0)(x-x_2)\ldots(x-x_n) + \ldots$$
$$\ldots A_n(x_n-x_0)(x_n-x_2)\ldots(x_n-x_{n-1})$$

Put $x = x_0$

$$f(x_0) = A_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

$$A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$

Put $x = x_1$

$$f(x_1) = A_1(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)$$

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

$$A_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}f(x_1)$$

$$+ \dots \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}f(x_n)$$

$$\frac{f(x)}{(x-x_0)(x-x_1)\dots(x-x_n)} = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x-x_0)}$$

$$+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)(x-x_1)}$$

$$+ \dots \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})(x-x_n)}$$

eg: The values of the function $f(x)$ for values of x are given as

$f(1)=4$, $f(2)=5$, $f(7)=5$, $f(8)=4$

Find the value of $f(6)$ and also values of x for which $f(x)$ is maximum or minimum.

Solⁿ

$$\frac{f(x)}{(x-1)(x-2)(x-7)(x-8)} = \frac{f(1)}{(1-2)(1-7)(1-8)(x-1)} + \frac{f(2)}{(2-1)(2-7)(2-8)(x-2)} + \frac{f(7)}{(7-1)(7-2)(7-8)(x-7)} + \frac{f(8)}{(8-1)(8-2)(8-7)(x-8)}$$

$$\frac{4}{-42(x-1)} + \frac{5}{30(x-2)} + \frac{5}{-30(x-7)} + \frac{4}{42(x-8)}$$

$$\frac{2}{21}\left(\frac{1}{x-8} - \frac{1}{x-1}\right) + \frac{1}{6}\left(\frac{1}{x-2} - \frac{1}{x-7}\right)$$

$$\frac{2}{3(x-1)(x-8)} \qquad \frac{-5}{6(x-2)(x-7)}$$

$$f(x) = \frac{2}{3}(x-2)(x-7) - \frac{5}{6}(x-1)(x-8)$$

$$\frac{2}{3}(x^2-9x+14) - \frac{5}{6}(x^2-9x+8)$$

$$\frac{1}{6}(-x^2+9x+16)$$

$$f(6) = \frac{1}{6}(-36+54+16)$$

$$\frac{36}{6} = 5.66$$

For maximum and minimum value of $f(x)$

$$f'(x) = 0$$

$$\frac{1}{6}(-2x+9) = 0$$

$$x = 9/2 = 4.5$$

Ch. 4

Central Difference Interpolation Formula

* Gauss Forward Interpolation formula.

∴ Newton's forward interpolation formula is

$$y_x = y_0 + u_{(1)}\Delta y_0 + u_{(2)}\Delta^2 y_0 + u_{(3)}\Delta^3 y_0 + u_{(4)}\Delta^4 y_0 + \dots$$

$$u = \frac{x - x_0}{h}$$

$$u_{(i)} = \frac{u(u-1)(u-2)\dots(u-i+1)}{i!}$$

Use, $\Delta^{n+1} y_{-1} = \Delta^n y_0 - \Delta^n y_{-1}$

$$\Delta^n y_0 = \Delta^n y_{-1} + \Delta^{n+1} y_{-1}$$

Hence.

$$\Delta^2 y_0 = \Delta^3 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$$

So,

$$y_x = y_0 + u_{(1)}\Delta y_0 + u_{(2)}(\Delta^2 y_{-1} + \Delta^3 y_{-1})$$

$$+ u_{(3)}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + u_{(4)}(\Delta^4 y_{-1} + \Delta^5 y_{-1})$$

$$u_{(i-1)} + u_i = (u+1)_{(i)}$$

$$y_x = y_0 + u_{(1)}\Delta y_0 + u_{(2)}\Delta^2 y_{-1}$$

$$+ (u+1)_{(3)}\Delta^3 y_{-1} + (u+1)_{(4)}\Delta^4 y_{-1} + \ldots$$

Again diminshing the Subscript

$$y_x = y_0 + u_{(1)}\Delta y_0 + u_{(2)}\Delta^2 y_{-1} +$$

$$(u+1)_{(3)}\Delta^3 y_{-1} + (u+1)_{(4)}\Delta^4 y_{-2} + \ldots$$

★ Gauss Backward Interpolation Formula

$$y_x = y_0 + u_{(1)}\Delta y_{-1} + (u+1)_{(2)}\Delta^2 y_{-1}$$

$$+ (u+1)_{(3)}\Delta^3 y_{-2} + (u+2)\Delta^4 y_{2} + \ldots$$

★ Sterling Formula.

Sterling formula is the average of Gauss forward & backward formula

Gauss forward formula

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1}$$

$$+ \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$+ \ldots$$

Gauss backward formula

$$y_x = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1}$$

$$+ \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \ldots$$

Taking ① & ② mean.

$$y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1})$$

$$+\frac{1}{2}\frac{u}{2!}\left[(u-1)+(u+1)\right]\Delta^2 y_{-1}$$

$$+\frac{1}{2}\frac{(u+1)u(u-1)}{3!}\left[\Delta^3 y_{-1}+\Delta^3 y_{-2}\right]$$

$$+\frac{1}{2}\frac{(u+1)u(u-1)}{4!}\left[u-2+u+2\right]\Delta^4 y_{-2}+\dots$$

$$y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2!}\Delta^2 y_{-1}$$

$$+\frac{u(u^2-1)}{3!}\frac{1}{2}\left[\Delta^3 y_{-1}+\Delta^3 y_{-2}\right]$$

$$+\frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}+\dots \quad -(3)$$

which is Sterling Formula.

* Bessel's formula.

Bessel's formula can be obtained by taking mean of Gauss backward (after shifting origin to 1) and Gauss forward formula.

Gauss backward formula.

$$y_x = y_0 + \frac{u^{(1)}}{1!}\Delta y_{-1} + \frac{(u+1)^{(2)}}{2!}\Delta^2 y_{-1}$$

$$+\frac{(u+1)^{(3)}}{3!}\Delta^3 y_{-2} + \frac{(u+2)^{(4)}}{4!}\Delta^4 y_{-2}+\dots$$

After shifting origin

$$y_x = y_1 + \frac{(u-1)^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0$$

$$+\frac{u^{(3)}}{3!}\Delta^3 y_{-1} + \frac{(u+1)^{(4)}}{4!}\Delta^4 y_{-1}+\dots$$

Gauss forward formula.

$$y_x = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_{-1}$$

$$+\frac{(u+1)^{(3)}}{3!}\Delta^3 y_{-1} + \frac{(u+1)^{(4)}}{4!}\Delta^4 y_{-2} + \ldots$$

Taking mean we get

$$y_x = \frac{1}{2}(y_1+y_0) + \frac{1}{2}\left[(u-1)^{(1)} + u^{(1)}\right]$$

$$\Delta y_0 + \frac{u^{(2)}}{2!}\frac{1}{2}\left[\Delta^2 y_0 + \Delta^2 y_{-1}\right]$$

$$+\frac{1}{3!}\frac{1}{2}\left[u^{(3)} + (u+1)^{(3)}\right]\Delta^3 y_{-1}$$

$$+\frac{(u+1)^{(4)}}{4!}\frac{1}{2}\left[\Delta^4 y_{-1} + \Delta^4 y_{-2}\right] + \ldots$$

$$y_x = \frac{1}{2}(y_1+y_0) + \left(u-\tfrac{1}{2}\right)\Delta y_0$$

$$+\frac{u(u-1)}{2!}\frac{1}{2}\left[\Delta^2 y_0 + \Delta^2 y_{-1}\right] +$$

$$\frac{\left(u-\tfrac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} +$$

$$\frac{(u+1)u(u-1)(u-2)}{4!}\frac{1}{2}\left[\Delta^4 y_{-1} + \Delta^4 y_{-2}\right]$$

$$+ \ldots$$

which is Bessel's formula.

Eg: Apply Bessel's formula to obtain y_{25} given.

$y_{20} = 2854$ $\quad y_{24} = 3162$

$y_{28} = 3544$ $\quad y_{32} = 3992$

Sol^n $\quad u = \frac{x-a}{h} = \frac{25-24}{4} = 0.25$

u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
-1	2854			
		308		
0	3162		74	
		382		-8
1	3544		66	
		448		
2	3992			

Bessal's formula is

$$y_x = \frac{1}{2}(y_0+y_1) + \left(u-\tfrac{1}{2}\right)\Delta y_0$$

$$+\, u\frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2}\frac{(u-1)}{2!}$$

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$$+\frac{(u-\tfrac{1}{2})u(u-1)}{3!}\Delta^3 y_{-1}+\ldots$$

$$\frac{1}{2}(3162+3544)+\left(\tfrac{1}{4}-\tfrac{1}{2}\right)382$$

$$+\frac{\tfrac{1}{4}\left(\tfrac{1}{4}-1\right)}{2!}\left(\frac{74+66}{2}\right)$$

$$+\frac{\left(\tfrac{1}{4}-\tfrac{1}{2}\right)\tfrac{1}{4}\left(\tfrac{1}{4}-1\right)}{3!}(-8)$$

$3353-95.5-6.5625-0.0625$

$=3250.875$

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Ch-5

Probability Distribution.

Mean & Variance of random variable

$$\text{Mean}= E(x)=\sum_{i=1}^{n} x_i\,p(x_i)$$

$$\text{Variance}= E(x^2)-[E(x)]^2$$

$$\text{where } E(x^2)=\sum_{i=1}^{n} x^2 p(x)$$

Binomial Distribution

A r.v X is said to follow binomial distribution if it assumes only non-negative integers and its p.m.f is given by

$$P(X=r)={}^{n}C_{r}\,p^{r}q^{n-r}.$$

where p is probability of success

q " " " failure

$p+q=1$

Mean $= np$

Variance $= npq$

Poisson distribution

A r.v X is said to follow poisson distⁿ if its p.m.f is given by

$$P(X=r) = \frac{e^{-m} m^r}{r!}$$

$$r = 0, 1, 2, \ldots \infty$$

mean $= m$

Variance $= m$

Normal distⁿ.

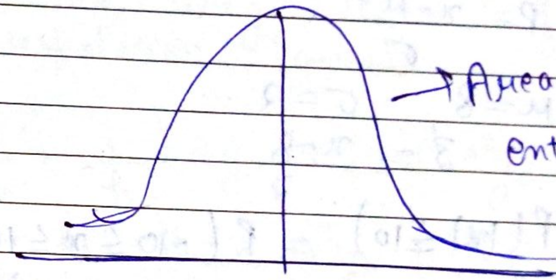
A r.v X is said to follow normal distⁿ if its p.d.f is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$-\infty < x < \infty$$

Mean $= \mu$

Variance $= \sigma^2$



$x = \mu$

↓

mean, mode, median coincide

Standard normal variate

Let $Z = \frac{x-\mu}{\sigma}$ where Z is $N(0,1)$ ie, mean zero and variance one.

Eg: If x is normal variate with mean 8 and s.d 2; find $P(|x| \leq 10)$

Sol^n $z = \frac{x-\mu}{\sigma}$

$\mu = 8 \quad \sigma = 2$

$z = \frac{x-8}{2}$

$P(|x| \leq 10) = P(-10 \leq x \leq 10)$

$P(-10-8 \leq x-8 \leq 10-8)$

$P\left(\frac{-18}{2} \leq \frac{x-8}{2} \leq \frac{2}{2}\right)$

$P(-9 \leq z \leq \phi)$

$P(-9 \leq z \leq 0) + P(0 \leq z < 1)$

$P(0 \leq z \leq 9) + P(0 \leq z \leq 1)$

$0.5 + 0.3413 = 0.8413$

Ch-6.

Numerical Differentiation

Derivative using Newton's forward interpolation formula.

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right]$$

Derivative using Newton's backward interpolation formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \right]$$

Derivatives using Sterling Formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{3u^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{6}\right) + \frac{2u^3-u}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{6u^2-1}{12}\Delta^4 y_{-2} + \cdots\right]$$

Derivatives using Bessel's formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{3u^2-3u+\frac{1}{2}}{3!}\Delta^3 y_{-1} + \frac{4u^3-6u^2-2u+2}{4!}\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right) + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} - \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{24}\left(\Delta^4 y_{-1} + \Delta^4 y_{-2}\right) + \cdots\right]$$

Eg: Find the first derivative of function $y = f(x)$ tabulated below at the point $x = 10$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
3	-13	18		
5	23	146	16	1
11	899	1026	40	1
27	17315	2613	69	
34	35606			

By Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 f(x_0) + \cdots$$

Differentiating both side w.r.t x we get:

$$f'(x) = \Delta f(x_0) + (2x - x_0 - x_1)\Delta^2 f(x_0)$$
$$+ (3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0)\Delta^3 f(x_0)$$

$$f'(10) = 18 + (2\times10 - 3 - 5)16$$
$$+ \{3\times100 - 20(3+5+11) + 15 + 55 + 33\}$$
$$18 + 12\times16 + 23$$
$$= 233.$$

Ch-7

Eigen Value Problems

$A - \lambda I$ is characteristic matrix of A

$|A - \lambda I|$ is characteristic polynomial of A

$|A - \lambda I| = 0$ is characteristic eq^n of A.

The value of λ is called characteristic roots or eigen value or eigen roots

Power method to find Eigen values or eigen roots

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$

Let the initial eigen vector be

$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ax_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 5 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 6\begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix}$$

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$$X^{(1)} = \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix}$$

$$AX^{(1)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.166 \\ 2.336 \\ 8.003 \end{bmatrix} = 8.003 \begin{bmatrix} 0.0208 \\ 0.292 \\ 1 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 0.021 \\ 0.292 \\ 1 \end{bmatrix}$$

$$AX^{(2)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.021 \\ 0.292 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.145 \\ 0.252 \\ 6.002 \end{bmatrix} = 6.002 \begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix}$$

$$6.002\, X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 2.065 \\ -0.068 \\ 6.272 \end{bmatrix} = 6.272 \begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix}$$

$$6.272\, X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.362 \\ 0.272 \\ 6.941 \end{bmatrix}$$

$$= 6.941 \begin{bmatrix} 0.340 \\ 0.039 \\ 1 \end{bmatrix}$$

$$= 6.941\, X^{(5)}$$

Hence largest eigen value is 6.941 and corresponding eigen vectors is

$$\begin{bmatrix} 0.340 \\ 0.039 \\ 1 \end{bmatrix}.$$

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Eg: Using Jacobi's method, find all the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Sol^n: Largest non-diagonal element of A is

$$a_{13} = a_{31} = 2$$

Also $a_{11} = 1$, $a_{33} = 1$

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2 \times 2}{1-1} = \infty$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

To annihilate $a_{13} = a_{31}$, we choose orthogonal matrix B_1 in plane [1, 3]

$$B_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

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$D_1 = B_1' A B_1$

$B_1' A B_1$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Largest non-diagonal element

$$a_{12} = a_{21} = 2$$

$a_{11} = 3$, $a_{22} = 3$

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2 \times 2}{3-3} = \infty$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

To annihilate $a_{12} = a_{21}$ we choose orthogonal matrix

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B_2 in the plane $[1,2]$

$$B_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_2 = B_2^{-1} D_1 B_2$$

$$B_2 D_1 B_2$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$