

MAA OMWATI DEGREE COLLEGE
HASSANPUR (PALWAL)

NOTES

SUBJECT : Real Analysis

CLASS : B.SC 5th Sem

SESSION : 2021-2022

Chapter - Ist Riemann Integral

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Partition :-

Let $I = [a, b]$ be a finite closed interval. Then by a partition P of I , we mean a finite ordered set $[x_0, x_1, \dots, x_n]$ of points of $[a, b]$ such that

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

For example, $\{0, \frac{1}{2}, 1\}$, $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$\{0, 0.1, 0.2, \dots, 0.9, 1\}$ are all partitions of interval $[0, 1]$. Clearly, an interval can have infinite number of partitions. The family of all partitions of $[a, b]$ is denoted by $P[a, b]$.

Norm of Partition :-

The maximum (or greatest) of the lengths of the sub-intervals of a partition P is called the norm or mesh of the partition P and is denoted by $\|P\|$ or $u(P)$.

$$\text{Thus } \|P\| = \max \{ \delta_n : 1 \leq n \leq n \}$$
$$= \max \{ (x_n - x_{n-1}) : 1 \leq n \leq n \}$$

For exp. $P_1 = \{0, \frac{1}{2}, 1\}$ $P_2 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

are the partitions of interval $[0, 1]$

$$\|P_1\| = \frac{1}{2}, \quad \|P_2\| = \frac{1}{3}$$

Refinement of Partition:-

Let P be a partition of $[a, b]$. Then any partition P' is called a refinement of P if $P' \supseteq P$, every point of P is a point of P' and may have some other points of $[a, b]$ in it.

For exp. Consider the interval $[1, 3]$ then $\{1, 1.5, 2, 3\}$ is a refinement of $\{1, 2, 3\}$ whereas $\{1, 1.4, 1.6, 2.2, 2.6, 3\}$ is not a refinement of $\{1, 2, 3\}$ as 2 is not included in it.

Upper And Lower Sums:-

$$\sum_{r=1}^n m_r \delta x_r = m_1 \delta_1 + m_2 \delta_2 + \dots + m_n \delta_n$$

$$\sum_{r=1}^n M_r \delta x_r = M_1 \delta_1 + M_2 \delta_2 + \dots + M_n \delta_n$$

are respectively called the lower and upper sum of corresponding to the partition P and are denoted by

$$L(F, P) \text{ and } U(F, P)$$

thus

$$L(F, P) = \sum_{r=1}^n m_r \delta x_r$$

$$U(F, P) = \sum_{r=1}^n M_r \delta x_r$$

$$\sum_{r=1}^n f(x_r) \delta x_r = b-a \quad \text{for every partition } P \text{ of } [a, b]$$

Oscillatory Sum:-

$$U(F, P) - L(F, P) = \sum_{r=1}^n M_r \delta x_r - \sum_{r=1}^n m_r \delta x_r = \sum_{r=1}^n (M_r - m_r) \delta x_r$$

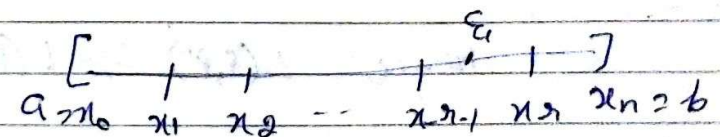
is called the oscillatory sum of f on $[a, b]$ with respect to partition P is denoted by $w(F, P)$

$$w(F, P) = U(F, P) - L(F, P)$$

thm If f is bounded function on $[a, b]$ and P' is a refinement of a partition of $[a, b]$ then

$$(i) \quad L(F, P') \geq L(F, P) \quad \text{and} \quad (ii) \quad U(F, P') \leq U(F, P)$$

Proof:- Let P' be a refinement of the partition $P = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$ of $[a, b]$ such that P' has just one ϵ (say) more than P and $x_{r-1} < \epsilon < x_r$



①

Let m_{α}' , m_{α}'' and m_{α} be the g.l.b of f in the intervals $[\alpha_{n-1}, \xi]$, and $[\alpha_{n-1}, \alpha_n]$ respectively. then $m_{\alpha} \leq m_{\alpha}'$ and $m_{\alpha} \leq m_{\alpha}''$

$$L(F, P') - L(F, P) = m_{\alpha}'(\xi - \alpha_{n-1}) + m_{\alpha}''(\alpha_n - \xi) - m_{\alpha}(\alpha_n - \alpha_{n-1})$$

$$= m_{\alpha}'(\xi - \alpha_{n-1}) + m_{\alpha}''(\alpha_n - \xi) - m_{\alpha}(\alpha_n - \xi + \xi - \alpha_{n-1})$$

$$= m_{\alpha}'(\xi - \alpha_{n-1}) + m_{\alpha}''(\alpha_n - \xi) - m_{\alpha}(\alpha_n - \xi) - m_{\alpha}(\xi - \alpha_{n-1})$$

$$= (m_{\alpha}' - m_{\alpha})(\xi - \alpha_{n-1}) + (m_{\alpha}'' - m_{\alpha})(\alpha_n - \xi)$$

$$\Rightarrow L(F, P') - L(F, P) \geq 0$$

$$\because m_{\alpha}' - m_{\alpha} \geq 0, m_{\alpha}'' - m_{\alpha} \geq 0 \text{ and } \xi - \alpha_{n-1} \geq 0, \alpha_n - \xi \geq 0$$

$$\therefore L(F, P') \geq L(F, P)$$

If P' contains ' P ' points more than P , then repeating the above argument P times we get

$$\text{Hence } L(F, P') \geq L(F, P)$$

ii

Let M_{α}' , M_{α}'' and M_{α} denote the l.u.b of f in the intervals $[\alpha_{n-1}, \xi]$, $[\xi, \alpha_n]$ and $[\alpha_{n-1}, \alpha_n]$ resp. then $M_{\alpha}' \leq M_{\alpha}$ and $M_{\alpha}'' \leq M_{\alpha}$

$$U(F, P') - U(F, P) = M_{\alpha}'(\xi - \alpha_{n-1}) + M_{\alpha}''(\alpha_n - \xi) - M_{\alpha}(\alpha_n - \alpha_{n-1})$$

$$= M_{\alpha}'(\xi - \alpha_{n-1}) + M_{\alpha}''(\alpha_n - \xi) - M_{\alpha}(\alpha_n - \xi + \xi - \alpha_{n-1})$$

$$= M_{\alpha}'(\xi - \alpha_{n-1}) + M_{\alpha}''(\alpha_n - \xi) - M_{\alpha}(\alpha_n - \xi) - M_{\alpha}(\xi - \alpha_{n-1})$$

$$= (M_{\alpha}' - M_{\alpha})(\xi - \alpha_{n-1}) + (M_{\alpha}'' - M_{\alpha})(\alpha_n - \xi)$$

$$\leq 0 \quad [\because M_{\alpha}' - M_{\alpha} \leq 0, M_{\alpha}'' - M_{\alpha} \leq 0 \text{ and } \xi - \alpha_{n-1} \geq 0, \alpha_n - \xi \geq 0]$$

$$\Rightarrow U(F, P') - U(F, P) \leq 0$$

$$U(F, P') \leq U(F, P)$$

If P' contains ' P ' points more than P , then repeating the above argument P times,

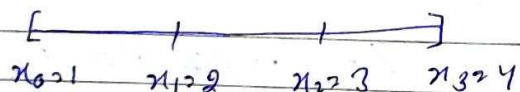
$$\text{Hence } U(F, P') \leq U(F, P)$$

Ex. If $f(x) = \frac{1}{x^2}$ on $[1, 4]$ and $P = \{1, 2, 3, 4\}$ be the partition of $[1, 4]$ then compute $L(f, P)$ and $U(f, P)$ Also find $w(f, P)$

Sol. Here $f(x) = \frac{1}{x^2}$ for all $x \in [1, 4]$

Here $P = \{1, 2, 3, 4\}$ is the partition of $[1, 4]$ which divides it into three sub-interval

$I_1 = [1, 2]$ $I_2 = [2, 3]$ and $I_3 = [3, 4]$ each of length 1 thus $\delta_1 = \delta_2 = \delta_3 = 1$



when $x \in I_1$ then $1 \leq x \leq 2$

$$1 \leq x^2 \leq 4$$

$$\frac{1}{4} \leq \frac{1}{x^2} \leq 1$$

$$\therefore m_1 = \text{g.l.b of } f \text{ in } [1, 2] = \frac{1}{4}$$

$$M_1 = \text{l.u.b of } f \text{ in } [1, 2] = 1$$

$$\text{Similarly, } m_2 = \text{g.l.b of } f \text{ in } [2, 3] = \frac{1}{9}$$

$$M_2 = \text{l.u.b of } f \text{ in } [2, 3] = \frac{1}{4}$$

$$m_3 = \text{g.l.b of } f \text{ in } [3, 4] = \frac{1}{16}$$

$$M_3 = \text{l.u.b of } f \text{ in } [3, 4] = \frac{1}{9}$$

Now,

$$L(f, P) = \sum_{i=1}^3 m_i \delta_i = m_1 \delta_1 + m_2 \delta_2 + m_3 \delta_3$$

$$\frac{1}{4} \cdot 1 + \frac{1}{9} \cdot 1 + \frac{1}{16} \cdot 1 = \frac{61}{144}$$

$$U(f, P) = \sum_{i=1}^3 M_i \delta_i =$$

$$M_1 \delta_1 + M_2 \delta_2 + M_3 \delta_3$$

$$1 \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{9} \cdot 1 = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$$

$$w(f, P) = U(f, P) - L(f, P)$$

$$= \frac{49}{36} - \frac{61}{144} = \frac{135}{144}$$

$$\text{Hence } L(f, P) = \frac{61}{144}, U(f, P) = \frac{49}{36}$$

$$\text{and } w(f, P) = \frac{135}{144}$$

Ex. Show that the function f defined on $[0, 1]$ by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$ is bounded but not $\mathbb{R}$ -integrable.

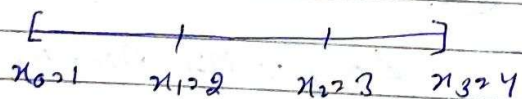
Sol. Clearly the function f is bounded since $|f(x)| = 1$ for all x

Ex. If $f(x) = \frac{1}{x^2}$ on $[1, 4]$ and $P = [1, 2, 3, 4]$ be the partition of $[1, 4]$ then compute $L(f, P)$ and $U(f, P)$ also find $w(f, P)$

Sol. Here $f(x) = \frac{1}{x^2}$ for all $x \in [1, 4]$

Here $P = [1, 2, 3, 4]$ is the partition of $[1, 4]$ which divides it into three sub-interval

$I_1 = [1, 2]$ $I_2 = [2, 3]$ and $I_3 = [3, 4]$ each of length 1 thus $\delta_1 = \delta_2 = \delta_3 = 1$



when $x \in I_1$ then $1 \leq x \leq 2$

$$1 \leq x^2 \leq 4$$

$$\frac{1}{4} \leq \frac{1}{x^2} \leq 1$$

$$\therefore m_1 = \text{g.l.b of } f \text{ in } [1, 2] = \frac{1}{4}$$

$$M_1 = \text{l.u.b of } f \text{ in } [1, 2] = 1$$

$$\text{Similarly, } m_2 = \text{g.l.b of } f \text{ in } [2, 3] = \frac{1}{9}$$

$$M_2 = \text{l.u.b of } f \text{ in } [2, 3] = \frac{1}{4}$$

$$m_3 = \text{g.l.b of } f \text{ in } [3, 4] = \frac{1}{16}$$

$$M_3 = \text{l.u.b of } f \text{ in } [3, 4] = \frac{1}{9}$$

Now,

$$L(f, P) = \sum_{i=1}^3 m_i \delta_i = m_1 \delta_1 + m_2 \delta_2 + m_3 \delta_3$$

$$\frac{1}{4} \cdot 1 + \frac{1}{9} \cdot 1 + \frac{1}{16} \cdot 1 = \frac{61}{144}$$

$$U(f, P) = \sum_{i=1}^3 M_i \delta_i =$$

$$M_1 \delta_1 + M_2 \delta_2 + M_3 \delta_3$$

$$1 \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{9} \cdot 1 = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$$

$$w(f, P) = U(f, P) - L(f, P)$$

$$= \frac{49}{36} - \frac{61}{144} = \frac{135}{144}$$

$$\text{Hence } L(f, P) = \frac{61}{144}, U(f, P) = \frac{49}{36}$$

$$\text{and } w(f, P) = \frac{135}{144}$$

Ex.

Show that the function f defined on $[0, 1]$ by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$ is bounded but not $\mathbb{R}$ -integrable.

Sol.

Clearly the function f is bounded since $|f(x)| = 1$ for all x

Consider is Partition $P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} \right\}$ of $[0, 1]$, where

length of each sub interval $[x_{r-1}, x_r]$ is $\frac{1}{n}$ and $\delta x = \frac{1}{n}$

$$m_r = \text{g.l.b of } f \text{ in } [x_{r-1}, x_r] = -1$$

$$M_r = \text{L.u.b of } f \text{ in } [x_{r-1}, x_r] = 1$$

$$L(P, P) = \sum_{r=1}^n m_r \delta x = \sum_{r=1}^n (-1) \frac{1}{n}$$

$$= -\frac{1}{n} \sum_{r=1}^n 1 = -\frac{1}{n} \times n$$

$$= -1$$

$$U(P, P) = \sum_{r=1}^n M_r \delta x = \sum_{r=1}^n (1) \times \frac{1}{n} = 1$$

The values of $L(P, P)$ and $U(P, P)$ are constants, therefore

$$\int_a^b f(x) dx = \text{L.u.b } \{L(P, P)\} = -1$$

$$\int_a^b f(x) dx = \text{g.l.b } \{U(P, P)\} = 1$$

$$\int_0^1 f(x) dx = \int_0^1 f(x) dx$$

Hence f is not integrable

The necessary and sufficient condition for a bounded function f to be integrable on $[a, b]$ is that to each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P) - L(P) < \epsilon$

Proof:- Condition is necessary:-

Suppose f is integrable on $[a, b]$ so that

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \int_a^b f(x) dx$$

Let $\epsilon > 0$

Since f is integrable on $[a, b]$ and f is bounded, therefore, there exists partition P_1 and P_2 of $[a, b]$ such that

$$L(f, P_1) > \int_a^b f(x) dx - \frac{\epsilon}{2} = \int_a^b f(x) dx - \frac{\epsilon}{2}$$

$$U(f, P_2) < \int_a^b f(x) dx + \frac{\epsilon}{2} = \int_a^b f(x) dx + \frac{\epsilon}{2}$$

Let $P = P_1 \cup P_2$ then

$$U(f, P) \leq U(f, P_2) < \int_a^b f(x) dx + \frac{\epsilon}{2}$$

$$< L(f, P_1) + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\leq L(F, P) + \epsilon$$

$$\Rightarrow U(F, P) - L(F, P) < \epsilon \text{ for a Partition } P$$

Condition is sufficient:-

Let $\epsilon > 0$ be given and P be a partition of $[a, b]$ such that $U(F, P) - L(F, P) < \epsilon$

$$\text{Since } L(F, P) \leq \int_a^b f(x) dx \leq \int_a^b f(x) dx \leq U(F, P)$$

$$\therefore \int_a^b f(x) dx - \int_a^b f(x) dx \leq U(F, P) - L(F, P) < \epsilon$$

since $\epsilon > 0$ is arbitrary, therefore

$$\int_a^b f(x) dx - \int_a^b f(x) dx = 0$$

$$\Rightarrow \int_a^b f(x) dx = \int_a^b f(x) dx$$

Riemann Sum:-

$$\sum_{i=1}^n f(x_i) \delta x_i \text{ is called}$$

Riemann Sum of f on $[a, b]$ corresponding partition P and denoted by $R(F, P)$

Second definition of integrability:-

A function f is said to be integrable on $[a, b]$ if there exists a number I such that for each $\epsilon > 0$, there exists $\delta > 0$ such that for every partition $P = [a = x_0, x_1, x_2, \dots, x_n = b]$ of $[a, b]$ with $\|P\| < \delta$ and $\xi_i \in [x_{i-1}, x_i]$

$$\left| \sum_{i=1}^n f(\xi_i) \delta x_i - I \right| < \epsilon$$

where the number $I = \int_a^b f(x) dx$

Ex. Using definition, evaluate $\int_0^{\pi/2} \sin x dx$

Sol. $f(x) = \sin x$ for all $x \in [0, \pi/2]$

Consider a partition $P = \{0 = x_0, x_1, \dots, x_n = \pi/2\}$

of $[0, \pi/2]$ which divides it into n equal sub intervals each of length $\delta x = h = \frac{\pi}{2n}$ or $\|P\| \rightarrow 0$ as $n \rightarrow \infty$

$$x_i = \frac{i\pi}{2n} ; \delta x = \frac{\pi}{2n} \quad (i = 1, 2, \dots, n)$$

$$F(u) = \sin u \quad F(\xi_n) = \sin \xi_n$$

$$\therefore F(\xi_n) = \sin \frac{2\pi}{2n}$$

$$\int_0^{\pi/2} \sin u \, du = \lim_{n \rightarrow \infty} \sum_{i=1}^n F(\xi_i) \delta x_i$$

$$\text{Taking } \xi_n = x_n = \frac{2\pi}{2n}$$

$$\int_0^{\pi/2} \sin u \, du = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{2\pi}{2n} \cdot \frac{\pi}{2n}$$

$$= \lim_{n \rightarrow \infty} \frac{\pi}{2n} \left[\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \dots + \sin \frac{n\pi}{2n} \right]$$

$$\text{using } \sin \alpha + \sin(\alpha + \beta) + \dots + \sin(\alpha + (n-1)\beta)$$

$$= \frac{\sin \left(\frac{\text{first angle} + \text{last angle}}{2} \right) \sin \left[\frac{n \cdot \text{diff of angle}}{2} \right]}{\sin \left[\frac{\text{diff of angle}}{2} \right]} \quad \text{Sub.}$$

$$\sin \left(\frac{\alpha + (n-1)\beta}{2} \right) \sin \frac{n\beta}{2}$$

$$\sin \frac{\beta}{2}$$

$$\int_0^{\pi/2} \sin u \, du = \lim_{n \rightarrow \infty} \frac{\pi}{2n} \frac{\sin \left(\frac{\pi}{2n} + \frac{n\pi}{2n} \right) \sin \left(\frac{n}{2} \cdot \frac{\pi}{2n} \right)}{\sin \left(\frac{\pi}{2n} \right)}$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \cdot \frac{\sin \left(\frac{\pi}{2n} + \frac{\pi}{2} \right) \sin \frac{\pi}{4}}{\sin \frac{\pi}{2n}}$$

$$= \frac{1}{\cancel{2}} \lim_{n \rightarrow \infty} \frac{\sin \left(\frac{\pi}{2n} + \frac{\pi}{2} \right)}{\sin \frac{\pi}{2n}} \times \frac{\pi}{2n}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\sin \frac{\pi}{2}}{1} \times \frac{\pi}{2n} \times \frac{2n}{\cancel{2}}$$

$$= \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times 2 = \frac{1}{\cancel{2} \times \cancel{2}} = 1$$

$$\text{Hence } \int_0^{\pi/2} \sin u \, du = 1$$

$$\text{Evaluate } \int_1^3 (x^2 + 2x + 3) \, dx \text{ by}$$

using the limit of Riemann sums

$$F(u) = x^2 + 2x + 3$$

clearly $F(u)$ is monotonic increasing on $[1, 3]$

consider a partition $P = \{1 = x_0, x_1, \dots, x_n = 3\}$

of $[1, 3]$ which divides it into n equal sub-interval each of length

$$\delta_n = \frac{2}{n} \text{ so that } \|P\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$x_n = 1 + \frac{2j}{n}, \quad F(x_n) = x_n^2 + 2x_n + 3$$

$$\int_1^3 P(x) dx = \lim_{n \rightarrow \infty} \sum_{j=1}^n F(x_{j-1}) \delta x_j$$

$$\text{Take } x_{j-1} = x_{j-1} = 1 + \frac{2(j-1)}{n}$$

$$\int_1^3 (x^2 + 2x + 3) dx$$

$$\leq \sum_{j=1}^n \left[\left(1 + \frac{2(j-1)}{n}\right)^2 + 2 + \frac{4(j-1)}{n} + 3 \right] \frac{2}{n}$$

$$= \frac{2}{n} \sum_{j=1}^n \left[1 + \frac{4(j-1)}{n} + \frac{4(j-1)^2}{n^2} + \frac{4(j-1)}{n} + 5 \right]$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n} \left[\sum_{j=1}^n 6 + \sum_{j=1}^n \frac{4(j-1)}{n} + \sum_{j=1}^n \frac{4(j-1)^2}{n^2} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n} \left[6n + \frac{4}{n} \cdot \frac{n(n-1)(n+1)}{6} + \frac{4}{n^2} \cdot \frac{n(n-1)}{2} \right]$$

$$= 12 + \lim_{n \rightarrow \infty} \left[\frac{4}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) + 8 \left(\frac{1}{n}\right) \right]$$

$$= 12 + \frac{8}{3} + 8$$

$$= \frac{26 + 8 + 24}{3} = \frac{68}{3}$$

If F is bounded and integrable on $[a, b]$, then $|F|$ is also bounded and integrable on $[a, b]$. Moreover,

$$\left| \int_a^b F(x) dx \right| \leq \int_a^b |F(x)| dx$$

Proof:

As function F is bounded on $[a, b]$ therefore there exists a positive number k such that

$$|F(x)| \leq k \text{ for all } x \in [a, b]$$

$$|F(x)| \leq k \text{ for all } x \in [a, b]$$

2) $|F|$ is bounded on $[a, b]$

Let $\epsilon > 0$ be given.

Since F is integrable on $[a, b]$ therefore exists a partition

$$P = \{a = x_0, x_1, \dots, x_n = b\}$$

Such that

$$U(F, P) - L(F, P) < \epsilon \quad \text{--- (1)}$$

Let M_j, m_j and M, m be the bounds of F and $|F|$ respectively on the sub-interval

$$[x_{j-1}, x_j]$$

$$[1, 2, \dots, n]$$

Now for all $x', x'' \in [x_{n-1}, x_n]$

$$|f(x'') - f(x')| = |f(x'')| - |f(x')|$$

$$\leq |f(x'') - f(x')|$$

$$|f(x'') - f(x')| \leq M_n' - m_n'$$

$$(n=1, 2, \dots, n)$$

$$M_n - m_n \leq M_n' - m_n'$$

$$\sum_{n=1}^n (M_n - m_n) \delta x \leq \sum_{n=1}^n (M_n' - m_n') \delta x$$

$$u(f, P) - L(f, P) < \epsilon$$

$$u(|f|, P) - L(|f|, P) < \epsilon$$

$\Rightarrow |f|$ is integrable on $[a, b]$

$$f(x) - f(y) \leq |f(x)|$$

$$\geq |f(x)| \text{ for all } x \in [a, b]$$

$$\int_a^b f(x) dx \leq \int_a^b |f(x)| dx \quad \text{--- (2)}$$

$$-\int_a^b f(x) dx \leq \int_a^b |f(x)| dx$$

$$\Rightarrow -\int_a^b f(x) dx \leq \int_a^b |f(x)| dx \quad \text{--- (3)}$$

From (2) and (3), we get:

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

Prove the inequality:

$$1 \leq \int_0^1 e^{x^2} dx \leq e$$

Here $f(x) = e^{x^2}$ which is an increasing function in $[0, 1]$

$$0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq 1$$

$$1 \leq e^{x^2} \leq e$$

$$\int_0^1 1 dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e dx$$

$$[x]_0^1 \leq \int_0^1 e^{x^2} dx \leq e [x]_0^1$$

$$1 \leq \int_0^1 e^{x^2} dx \leq e$$

Hence the result.

Primitive of a function:-

A function 'F' defined on $[a, b]$ is said to be a primitive of a function 'f' defined on $[a, b]$ if $F(x) = f(x)$ for all $x \in [a, b]$

Statement: If f is integrable on $[a, b]$ and F is the primitive of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Proof: As F is primitive of f on $[a, b]$

$$\Rightarrow F'(x) = f(x) \text{ for all } x \in [a, b]$$

Consider a partition $P = [a = x_0, x_1, \dots, x_n = b]$

Applying Lagrange's mean value theorem to primitive F on each sub-interval $[x_{i-1}, x_i]$ where $1 \leq i \leq n$

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(\xi_i) \text{ where } \xi_i \in (x_{i-1}, x_i)$$

$$\begin{aligned} F(x_i) - F(x_{i-1}) &= F'(\xi_i) (x_i - x_{i-1}) \\ &= f(\xi_i) (x_i - x_{i-1}) \end{aligned}$$

$$[\because F'(x) = f(x)]$$

$$\sum_{i=1}^n f(\xi_i) (x_i - x_{i-1}) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})]$$

$$= \sum_{i=1}^n f(\xi_i) \delta x_i = F(b) - F(a)$$

$$\delta x_i = (x_i - x_{i-1})$$

-QED

$$L(f, P) \leq \sum_{i=1}^n F(\xi_i) \delta x_i \leq U(f, P)$$

$$L(f, P) \leq F(b) - F(a) \leq U(f, P)$$

$$\int_a^b f(x) dx \leq F(b) - F(a) \leq \int_a^b f(x) dx \quad \text{--- (2)}$$

But F is integrable, therefore

$$\int_a^b f(x) dx \leq \int_a^b f(x) dx = \int_a^b f(x) dx \quad \text{--- (3)}$$

From (2) and (3) we get

$$\begin{aligned} \int_a^b f(x) dx &\leq F(b) - F(a) \leq \int_a^b f(x) dx \\ \int_a^b f(x) dx &= F(b) - F(a) \end{aligned}$$

Ans

Ex.

Evaluate the integral $\int_{-1}^1 ([x] - x) dx$

Sol.

Here $f(x) = [x] - x$ which has only finite points of discontinuity namely -1 or 1 and f is bounded in $[-1, 1]$

Therefore f is Riemann integrable on $[-1, 1]$

$$-1 \leq x \leq 0 \quad [x] = -1$$

$$f(x) = -1 - x$$

For $0 \leq x \leq 1$ $[x] = 0$

$$F(x) = 0 - x = -x$$

$$x=1 \quad [x] = 1 \Rightarrow F(x) = 1 - 1 = 0$$

$$\begin{aligned} \int_{-1}^1 F(x) dx &= \int_{-1}^0 F(x) dx + \int_0^1 F(x) dx \\ &= \int_{-1}^0 (-1-x) dx + \int_0^1 -x dx \end{aligned}$$

$$\begin{aligned} &= \left[\frac{(-1-x)^2}{-2} \right]_{-1}^0 - \left[\frac{x^2}{2} \right]_0^1 \\ &= \left(\frac{-1}{2} - 0 \right) - \frac{1}{2} = -1 \quad \underline{\text{Ans}} \end{aligned}$$

Chapter - 2nd

Improper Integrals and their Convergence

Improper Integral:-

The integral $\int_a^b f(x) dx$ is called an Improper integral if either 'a' or 'b' both or 'a' and 'b' are infinite or the integrand 'f' becomes infinite (unbounded) on the interval $[a, b]$

Types of Improper Integrals:-

Improper Integral of First Kind:-

The integral $\int_a^b f(x) dx$ is said to be improper integral of first kind if either 'a' or 'b' or both 'a' and 'b' are infinite but the integrand 'f' is bounded.

Improper Integral of Second Kind:-

The integral $\int_a^b f(x) dx$ is said to be improper integral of second kind if 'f' has a finite number of points of infinite discontinuity in $[a, b]$

For exp. $\int_0^1 \frac{1}{x(1-x)} dx = \int_0^1 \frac{1}{x} dx$

Improper Integral of Mixed kind
or third kind:-

An infinite interval having points of infinite discontinuity of integrand is then such type of integral is said to be an improper integral of mixed kind or third kind.

$$\int_0^{\infty} \frac{1}{x} dx, \quad \int_{-\infty}^0 \frac{1}{1-x} dx \text{ are improper}$$

integrals of mixed kind.

Ex. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

To examine the convergence of this improper integral, we must select a convenient number 'a' such that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$\lim_{t \rightarrow -\infty} \int_t^0 \frac{dx}{1+x^2} + \lim_{t \rightarrow \infty} \int_0^t \frac{dx}{1+x^2}$$

$$\lim_{t \rightarrow -\infty} [\tan^{-1}x]_t^0 + \lim_{t \rightarrow \infty} [\tan^{-1}x]_0^t$$

$$= \lim [\tan^{-1} 0 - \tan^{-1} t] + \lim [\tan^{-1} t - \tan^{-1} 0]$$

$$= -\tan^{-1}(-\infty) + \tan^{-1}(\infty)$$

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi$$

(9)

Test the following improper integrals for convergence.

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\epsilon \rightarrow 0^+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x}}$$

$$\lim_{\epsilon \rightarrow 0^+} \left[\frac{2(1-x)^{1/2}}{-1} \right]_0^{1-\epsilon}$$

$$= \lim_{\epsilon \rightarrow 0^+} [-2(\epsilon)^{1/2} + 2] = 2$$

Hence the given improper integral converges to 2.

(i) A The Improper integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if $n < 1$

(ii) The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ is convergent if only if $n < 1$

Proof: The given integral $\int_a^b \frac{dx}{(x-a)^n}$ is

a proper integral for $n \leq 0$ and therefore convergent. If $n > 0$ then it is an improper integral and a is the only point of infinite discontinuity.

Case 1. If $n \neq 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b \frac{dx}{(x-a)^n}$$

$$0 < \epsilon < b-a$$

$$\lim_{\epsilon \rightarrow 0^+} \left[\frac{(x-a)^{-n+1}}{-n+1} \right]_{a+\epsilon}^b$$

$$= \lim_{\epsilon \rightarrow 0^+} \frac{1}{1-n} \left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\epsilon^{n-1}} \right]$$

$$= \begin{cases} \frac{1}{(1-n)(b-a)^{n-1}} \text{ finite } n < 1 \\ \infty \text{ if } n > 1 \end{cases}$$

$\int_a^b \frac{dx}{(x-a)^n}$ converges if $0 \leq n < 1$ and diverges for $n \geq 1$

Case II: If $n = 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a}$$

$$\lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\epsilon \rightarrow 0^+} [\log|x-a|]_{a+\epsilon}^b$$

$$= \lim_{\epsilon \rightarrow 0^+} [\log(b-a) - \log \epsilon]$$

$$= \lim_{\epsilon \rightarrow 0^+} \left(\log \frac{b-a}{\epsilon} \right) = \infty$$

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n = 1$

Hence $\int_a^b \frac{dx}{(x-a)^n}$ is convergent for $n < 1$ and divergent for $n \geq 1$.

Show that the integral $\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$ exists if and only if $n < m+1$

Sol Let $I = \int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$

$$\text{Here } f(x) = \frac{\sin^m x}{x^n} = \left(\frac{\sin x}{x} \right)^m \frac{1}{x^{n-m}}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \right)^m \frac{1}{x^{n-m}}$$

$$= \begin{cases} 0 & \text{if } n-m < 0 \\ 1 & \text{if } n=m \\ 0 & \text{if } n-m > 0 \end{cases}$$

Therefore the given integral I is a proper integral and hence convergent for $n \leq m$ and '0' is the only point of infinite discontinuity of f in $[0, \frac{\pi}{2}]$ if $n > m$

$$f(u) > 0 \text{ in } (0, \frac{\pi}{2}]$$

Take $g(u) = \frac{1}{u^{n-m}}$ when $n > m$

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{g(u)} = \lim_{u \rightarrow 0^+} \left(\frac{\sin u}{u} \right)^m = 1 \neq 0, \infty$$

$\therefore$ The integrals $\int_0^{\pi/2} f(u) du$ and $\int_0^{\pi/2} g(u) du$ converge or diverge together

$$\text{But the integral } \int_0^{\pi/2} g(u) du = \int_0^{\pi/2} \frac{1}{u^{n-m}} du$$

$n > m$ is convergent if and only if $n-m < 1$

$\therefore \int_0^{\pi/2} f(u) du$ is convergent if and only if $m < n < m+1$

Also $\int_0^{\pi/2} f(u) du$ is convergent for $n \leq m$ therefore the given integral

is convergent if and only if $n < m+1$

Ex. Discuss the convergence of the Beta function or

Show that $\int_0^1 x^{m-1} (1-x)^{n-1} dx$

exists if and only if m, n are both positive.

$$\text{Let } I = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

The integral I is proper if $m > 1$ and $n > 1$ and so it is convergent for $m > 1$ and $n > 1$ clearly 0 and 1 are the points of infinite discontinuity if $m < 1$ and $n < 1$ resp. For $m < 1$ and $n < 1$, take a number $\frac{1}{2}$ b/w 0 and 1 so that we can write

$$I = \int_0^{1/2} x^{m-1} (1-x)^{n-1} dx + \int_{1/2}^1 x^{m-1} (1-x)^{n-1} dx$$

$$I = I_1 + I_2 \text{ (say)} \quad \text{--- (1)}$$

To discuss the convergence of

$$I_1 = \int_0^{1/2} x^{m-1} (1-x)^{n-1} dx \text{ at } x=0 \text{ when } m < 1$$

Here $f(u) = x^{m-1} (1-u)^{n-1} = \frac{(1-u)^{n-1}}{x^{1-m}}$

Take $g(u) = \frac{1}{x^{1-m}}$ so that $\lim_{x \rightarrow \infty} \frac{f(u)}{g(u)} = (1-u)^{n-1}$

which is finite and non-zero

$\therefore I_1$ and $\int_0^{1/2} g(u) du = \int_0^{1/2} \frac{1}{x^{1-m}} du$

converge or diverge together.

But $\int_0^{1/2} \frac{1}{x^{1-m}} du$ converges at

$x=0$ iff $1-m < 1 \quad m > 0$

$\therefore I_1$ Converges at 0 iff $0 < m < 1$

To discuss the convergence $I_2 = \int_{1/2}^1 \frac{x^{m-1}}{(1-x)^{n-1}} dx$ at $x=1$ when $n < 1$

Here $f(u) = x^{m-1} (1-u)^{n-1} = \frac{x^{m-1}}{(1-u)^{1-n}}$

Take $g(u) = \frac{1}{(1-u)^{1-n}}$ so that

$\lim_{x \rightarrow \infty} \frac{f(u)}{g(u)} = \lim_{x \rightarrow \infty} x^{m-1} = 1$

which is finite and non-zero

$\therefore I_2$ and $\int_{1/2}^1 g(u) du$ converge or diverge together.

But $\int_{1/2}^1 g(u) du = \int_{1/2}^1 \frac{1}{(1-u)^{1-n}} du$

converges at $x=1$ iff $1-n < 1 \quad n > 0$

$\therefore I_2$ Converges at 1 iff $0 < n < 1$ therefore from (1) the integral I is convergent if and only if $0 < m < 1$ and $0 < n < 1$. Also it is a proper integral for $m > 1$ and $n > 1$

Hence I is convergent iff m, n are both positive.

Thm. The improper integral $\int_a^{\infty} \frac{dx}{x^n}$ converges if and only if $n > 1$ and diverges for $n \leq 1$

Proof.

$\int_a^{\infty} \frac{dx}{x^n} = \lim_{t \rightarrow \infty} \int_a^t \frac{dx}{x^n}$

$= \left[\frac{x^{-n+1}}{-n+1} \right]_a^t$ if $n \neq 1$

$$\lim_{n \rightarrow \infty} \left[\frac{1+n}{1-n} - \frac{a^{1+n}}{1-n} \right] \text{ if } n \neq 1$$

$$= \begin{cases} -\frac{a^{1+n}}{1-n} & \text{which is finite if } n > 1 \\ \infty & \text{if } n < 1 \end{cases}$$

Also when $n = 1$

$$\int_a^\infty \frac{1}{x^n} dx = \int_a^\infty \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_a^t \frac{1}{x} dx$$

$$= \lim_{t \rightarrow \infty} [\log x]_a^t = \lim_{t \rightarrow \infty} [\log t - \log a]$$

$$= \log \infty - \log a = \infty$$

Hence, $\int_a^\infty \frac{dx}{x^n}$ converges if and only if $n > 1$ and diverges for $n \leq 1$

Ex 4. Let $I = \int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$

$$= \int_0^1 \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx + \int_1^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$$

$$I = I_1 + I_2$$

the first integral I_1 is a proper integral and hence convergent.

$$\text{Let } f(x) = \frac{x \tan^{-1} x}{(1+x^4)^{1/3}}, \quad \frac{x \tan^{-1} x}{x^{1/3} (x^4+1)^{1/3}}$$

$$= \frac{\tan^{-1} x}{x^{2/3} (x^4+1)^{1/3}}, \quad \frac{\tan^{-1} x}{x^{2/3} (1+\frac{1}{x^4})^{1/3}}$$

Take $g(x) = \frac{1}{x^{1/3}}$ so that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{(1+\frac{1}{x^4})^{1/3}}$$

$\tan^{-1} \infty = \frac{\pi}{2}$ which is finite and non-zero.

$\therefore$ By comparison test, the integrals $\int_0^\infty f dx$ and $\int_0^\infty g dx$ converge or diverge together.

But the integral $\int_1^\infty g dx = \int_1^\infty \frac{1}{x^{1/3}} dx$ is

$$I_2 = \int_1^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx \text{ diverges.}$$

Hence by (i), the integral

$$I = \int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx \text{ is divergent.}$$

Discuss the convergence of
Gamma function or
Show that the integral $\int_0^{\infty} x^{n-1} e^{-x} dx$
is convergent if $n > 0$

Sol. Let $I = \int_0^{\infty} x^{n-1} e^{-x} dx$
 $= \int_0^1 x^{n-1} e^{-x} dx + \int_1^{\infty} x^{n-1} e^{-x} dx$

$I = I_1 + I_2$

$f(x) > x^{n-1} e^{-x} > \frac{e^{-x}}{x^{1-n}} \quad \text{--- (1)}$

The integrand f has an infinite discontinuity at $x=0$ in $[0,1]$ if $n < 1$ and I_1 is proper integral and hence convergent for $n > 1$

To test the convergence of I_2 at ∞ when $n < 1$

Take $g(x) > \frac{1}{x^{1-n}}$ so that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$

$\lim_{x \rightarrow \infty} e^{-x} = 0$ which is finite and non-zero.

But the integral $\int_1^{\infty} g(x) dx = \int_1^{\infty} \frac{1}{x^{1-n}} dx$ is

convergent if and only if $1-n < 1$, $n > 0$
 Therefore by comparison test, the integral $\int_1^{\infty} f(x) dx$ converges for $0 < n < 1$
 Also it is a proper integral for $n > 1$, Hence I_2 is convergent for all $n > 0$

To test the convergence

of I_2 :-

Take $g(x) = \frac{1}{x^2}$ so that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$

$\lim_{x \rightarrow \infty} \frac{x^{n+1}}{e^x} = 0$ for all n

Now $\int_1^{\infty} g(x) dx = \int_1^{\infty} \frac{1}{x^2} dx$ is conv.
 $n > 2 > 1$

By comparison test, the integral I_2 is also convergent for all n .

Hence by (1) we conclude that the given integral I is convergent if and only if $n > 0$

Abel's Test for convergence:-

If $\int_a^\infty f(x) dx$ is convergent at ∞ and g is bounded and monotonic in $[a, \infty)$ then $\int_a^\infty f(x)g(x) dx$ is convergent at ∞ .

Proof:-

As g is monotonic and bounded on $[a, \infty)$ therefore it is integrable on $[a, t]$ for all $t \geq a$.

Also f is integrable on $[a, t]$ therefore by second mean value theorem.

$$\int_{t_1}^{t_2} f(x)g(x) dx = g(t_1) \int_{t_1}^{\xi} f(x) dx + g(t_2) \int_{\xi}^{t_2} f(x) dx$$

when $a < t_1 \leq \xi \leq t_2$

$$\Rightarrow \left| \int_{t_1}^{t_2} f(x)g(x) dx \right| \leq |g(t_1)| \left| \int_{t_1}^{\xi} f(x) dx \right| + |g(t_2)| \left| \int_{\xi}^{t_2} f(x) dx \right|$$

when $a < t_1 \leq \xi \leq t_2$

Since $g(x)$ is bounded in $[a, \infty)$ therefore there exists a positive number k such that

$$|g(x)| \leq k \quad \text{for all } x \geq a$$

Since (1) holds for all $x \geq a$ therefore it holds for t_1 and t_2 also.

$$|g(t_1)| \leq k \quad \text{and} \quad |g(t_2)| \leq k$$

Also it is given that $\int_a^\infty f(x) dx$ is convergent therefore for a given $M > 0$ there exists a number t_0 such that

$$\left| \int_{t_1}^{t_2} f(x) dx \right| < \frac{M}{2k} \quad \text{for all } t_1, t_2 \geq t_0 \quad \text{--- (2)}$$

Select the number t_1 and t_2 in (1) such that both are $\geq t_0$ and number ξ which lies b/w t_1 and t_2 also $\geq t_0$.

Hence from (1) we obtain

$$\left| \int_{t_1}^{\xi} f(x) dx \right| < \frac{M}{2k} \quad \text{and} \quad \left| \int_{\xi}^{t_2} f(x) dx \right| < \frac{M}{2k} \quad \text{--- (3)}$$

using (2), (3) and (3) in (1) we get

$$\left| \int_{t_1}^{t_2} f(x)g(x) dx \right| < k \frac{M}{2k} + k \frac{M}{2k} = M$$

$$\therefore \left| \int_{t_1}^{t_2} f(x)g(x) dx \right| < M \quad \text{for all } t_1, t_2 \geq t_0$$

Hence by Cauchy's test

$\int_a^\infty f(x)g(x) dx$ is convergent at ∞

Ex.

Examine the convergency of $\int \sin u^2 du$

$$\text{Let } I = \int_0^{\infty} \sin u^2 du$$

$$= \int_0^1 \sin u^2 du + \int_1^{\infty} \sin u^2 du$$

$$I = I_1 + I_2 \quad \text{--- (1)}$$

the integral $I_1 = \int_0^1 \sin u^2 du$ is ?

proper integral and hence it is convergent.

To test the convergency of $I_2 = \int_1^{\infty} \sin u^2 du$ at ∞ :-

$$\text{Let } f(u) = 2u \sin u^2 \quad \text{and } g(u) = \frac{1}{2u}$$

$$\left| \int^+ f du \right| = \left| \int^+ 2u \sin u^2 du \right|$$

$$= \left| \left[-\cos u^2 \right]^+ \right| = |0 - (-1)|$$

$$\leq 1 + 1 = 2$$

$\therefore \int^+ f du$ is bounded for all $t \geq 1$

Also $g(u) = \frac{1}{2u} \rightarrow 0$ as $u \rightarrow \infty$. Thus $g(u)$ is bounded and monotonically decreasing function tending to 0 as $u \rightarrow \infty$.

$\therefore$ By Dirichlet's test, $\int fg du$
 $= \int_0^{\infty} \sin u^2 du$ is convergent

Hence from (1) $\int_0^{\infty} \sin u^2 du$ is convergent.

Frulloni's thm:-

If $f(u)$ is a continuous function on $[0, \infty)$ having singularities at 0 and ∞ only, then $\int_0^{\infty} \frac{f(u) - f(v)}{n} du$ is equal to

i. 0 if $\int_0^{\infty} \frac{f(u)}{n} du$ converges at 0 and ∞

ii. $F_1 \log \left(\frac{a}{b} \right)$ if $\int_0^{\infty} \frac{f(u)}{n} du$ converges

only at 0.

iii. $F_0 \log \left(\frac{1}{a} \right)$ if $\int_0^{\infty} \frac{f(u)}{n} du$ converges

only at ∞

iv. $(F_0 - F_1) \log \left(\frac{1}{a} \right)$ if $\int_0^{\infty} \frac{f(u)}{n} du$

neither converges at 0 nor at ∞ when

$$\lim_{u \rightarrow 0} f(u) = F_0 \quad \text{and}$$

$$\lim_{u \rightarrow \infty} f(u) = F_1$$

Exb. If $a > 0, b > 0$ prove that

$$\int_0^{\infty} \frac{e^{-ay} - e^{-by}}{y} dy = \log \frac{b}{a}$$

Let $I = \int_0^{\infty} \frac{e^{-ay} - e^{-by}}{y} dy$

Take $F(y) = e^{-y}$

Since $\int_0^{\infty} \frac{e^{-y}}{y} dy$ is convergent at ∞

and $\lim_{y \rightarrow 0} e^{-y} = 1$

$\therefore$ By Frullani's integral case (iii) we have

$$\begin{aligned} \int_0^{\infty} \frac{e^{-ay} - e^{-by}}{y} dy &= f(0) \log \left| \frac{f(b)}{f(a)} \right| \\ &= 1 \log \frac{b}{a} = \log \frac{b}{a} \end{aligned}$$

Integrals as a Function of a Parameter

Show that $\int_0^1 \frac{x^a - 1}{\log x} dx = \log(a+1)$

Sol.

Let $F(x) = \int_0^1 \frac{x^y - 1}{\log x} dy$ $a > -1$

Here $F(x, y) = x^y - 1$ and $g(y) = \frac{1}{\log x}$

where x is parameter. The function $F(x, y)$ and $g(y) = \frac{1}{\log x}$ where x is a parameter.

thus function $F(x, y)$ and $\frac{\partial}{\partial x} (x^y - 1)$

$x^y \log x$ are continuous in $[0, 1]$ and $a > -1$ and $g(y) = \frac{1}{\log x}$ is

bounded and integrable.

therefore $F'(x) = \frac{d}{dx} F(x)$ exists and

is given by

$$F'(x) = \int_0^1 \frac{\partial}{\partial x} (F(x, y) g(y)) dy$$

But $\frac{\partial}{\partial x} F(x, y) = \frac{\partial}{\partial x} (x^y - 1)$
 $= x^y \log x$

$$F'(x) = \int_0^1 \frac{x^y \log x}{\log x} dy = \int_0^1 x^y dy$$

$$F(x) = \left[\frac{x^2+1}{2+1} \right]^{-1} = \frac{1}{1+x^2} \quad \text{--- (1)}$$

Integrating (1) w.r.t x we have

$$F(x) = \log(1+x^2) + C$$

$$F(0) = \log(1+0) = C$$

$$F(0) = \int_0^0 0 dx = 0$$

$$C = 0$$

$$F(x) = \log(1+x^2)$$

Exmp. Evaluate $\int_0^{\pi/2} \log(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) d\theta$

Sol. Let $F(\alpha, \beta) = \int_0^{\pi/2} \log(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) d\theta$

Here $g(\theta) = 1$ and $F(\theta, \alpha, \beta)$

$$= \log(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta)$$

This function is a function of two parameters α and β and we can diff. it with respect to any of the parameters, α or β

Diff (2) w.r.t parameter α , we get

$$\frac{\partial}{\partial \alpha} (F(\alpha, \beta)) = \frac{\alpha \cos^2 \theta}{\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta}$$

$$= \frac{\alpha}{\alpha^2 + \beta^2 \tan^2 \theta}$$

$$B \frac{d}{d\alpha} F(\alpha, \beta) = F'(\alpha, \beta)$$

$$= \int_0^{\pi/2} \frac{\partial}{\partial \alpha} F(\theta, \alpha, \beta) g(\theta) d\theta$$

$$= \int_0^{\pi/2} \frac{\partial}{\partial \alpha} F(\theta, \alpha, \beta) d\theta \quad [\because g(\theta) = 1]$$

$$= \int_0^{\pi/2} \frac{\alpha}{\alpha^2 + \beta^2 \tan^2 \theta} d\theta$$

but $\tan \theta = t$
 $\sec^2 \theta d\theta = dt$

$$(1 + \tan^2 \theta) d\theta = dt$$

$$d\theta = \frac{dt}{1+t^2}$$

When $\theta = 0, t = 0$, and $\theta = \pi/2, t = \infty$

$$F'(\alpha, \beta) = \alpha \int_0^{\infty} \frac{dt}{(\alpha^2 + \beta^2 t^2)(1+t^2)}$$

$$\frac{\alpha}{\beta^2 \alpha^2} \int_0^{\infty} \left[\frac{\beta^2}{\alpha^2 + \beta^2 t^2} - \frac{1}{1+t^2} \right] dt$$

$$\frac{2x}{\beta^2 - x^2} \left[\frac{\beta}{\alpha} \tan^{-1} \frac{\beta + x}{\alpha} \right]_0^{\infty}$$

$$\frac{2x}{\beta^2 - x^2} \left[\frac{\beta}{\alpha} \frac{\pi}{2} \right]$$

$$F(0, \alpha, \beta) = \frac{2x}{\beta^2 - x^2} \frac{\beta - x}{\alpha} \frac{\pi}{2}$$

$$= \frac{\pi}{\beta + \alpha}$$

Integrating w.r.t x we get

$$F(0, \alpha, \beta) = \int \frac{\pi}{\alpha + \beta} dx + C$$

$$\pi \log(\alpha + \beta) + C \quad \text{--- (3)}$$

Taking $\beta = x \tan u$

$$F(0, \alpha, x) = \int_0^{\pi/2} \log x^2 dx$$

$$= 2 \log x \left[0 \right]_0^{\pi/2}$$

$$\pi \log x$$

From (3) we have

$$F(0, \alpha, x) = \pi \log(\alpha + x) + C$$

$$\pi \log x = \pi \log(\alpha + x) + C$$

$$C = \pi [\log x - \log(\alpha + x)]$$

$$\pi \log \left(\frac{x}{\alpha} \right) = \pi \log \frac{1}{\alpha}$$

$$= -\pi \log \alpha$$

From (3) we get

$$F(0, \alpha, \beta) = \pi [\log(\alpha + \beta) - \log \alpha]$$

$$\pi \log \left(\frac{\alpha + \beta}{\alpha} \right)$$

Exp. show that $\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 u + b^2 \cos^2 u)^2}$

$$= \frac{\pi(a^2 + b^2)}{4a^2 b^3}$$

Let $I = \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 u + b^2 \cos^2 u}$

$$= \int_0^{\pi/2} \frac{\sec^2 u \, du}{a^2 \tan^2 u + b^2}$$

(Divide the num. and den. by $\cos^2 u$)

but $\tan u = t$
 $\sec^2 u \, du = dt$

when $u = 0, t = 0$ when $u = \pi/2, t = \infty$

$$I = \int_0^{\infty} \frac{dt}{a^2 t^2 + b^2}$$

$$= \frac{1}{a^2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}$$

$$\frac{1}{a^2} \cdot \frac{1}{\frac{b}{a}} \left[\tan^{-1} \frac{1}{\frac{b}{a}} \right]_0^{\frac{\pi}{2}}$$

$$\frac{1}{ab} \left[\tan^{-1} \frac{a}{b} \right]_0^{\frac{\pi}{2}}$$

$$\frac{1}{ab} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{2ab}$$

$$\text{Thus } I = \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab} \quad (1)$$

Diff both sides of (1) w.r.t. a we get

$$\frac{dI}{da} = \int_0^{\pi/2} \frac{d}{da} \left(\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \right) dx$$

$$= \frac{d}{da} \left(\frac{\pi}{2ab} \right)$$

$$= - \int_0^{\pi/2} \frac{2a \sin^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} dx$$

$$= - \frac{\pi}{2a^2 b}$$

$$\therefore \int_0^{\pi/2} \frac{\sin^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} dx = \frac{\pi}{4a^2 b} \quad (2)$$

Similarly diff both side of (1) w.r.t. b

$$\frac{dI}{db} = - \int_0^{\pi/2} \frac{2b \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} dx$$

$$= - \frac{\pi}{2a^2 b}$$

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} dx = \frac{\pi}{4a^2 b} \quad (3)$$

Adding (2) and (3) we get

$$\int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} dx = \frac{\pi}{4a^2 b} + \frac{\pi}{4a^2 b}$$

$$\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi}{4a^2 b} (b^2 + a^2)$$

Metric and Metric Space:-

Let X be

a non-empty set. A metric (or distance function) on X is a real function $d: X \times X \rightarrow \mathbb{R}$ (set of reals) which satisfies the following axioms:

- i. $d(x, y) \geq 0$ for all $x, y \in X$
- ii. $d(x, y) = 0 \Leftrightarrow x = y$ when $x, y \in X$
- iii. $d(x, y) = d(y, x)$ for all $x, y \in X$
- iv. $d(x, y) \leq d(x, z) + d(z, y)$ for

all $x, y, z \in X$. If d is a metric for X , then the ordered pair (X, d) is called a metric space and $d(x, y)$ is called the distance b/w x and y .

Ex. Discrete Metric space:- Let X be an arbitrary non-empty set and $d: X \times X \rightarrow \mathbb{R}$ be a function such that $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

show that (X, d) is a metric space.

Sol.

we have $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

i.

Clearly $d(x, y) \geq 0$ for all $x, y \in X$

ii.

If $x = y$ then $d(x, y) = 0$ (given) and if $d(x, y) = 0$ then $x = y$

For if $x \neq y$ then $d(x, y) = 1 \neq 0$

Hence $d(x, y) = 0 \Leftrightarrow x = y$

iii.

If $x = y$ then $d(x, y) = 0 = d(y, x)$

and if $x \neq y$ then $d(x, y) = 1 = d(y, x)$

Hence $d(x, y) = d(y, x)$ for all

$x, y \in X$

iv.

Let $x, y, z \in X$

The following cases arise:-

a)

If $x = y = z$ then $d(x, y) = 0$
 $d(x, z) = 0$ and $d(z, y) = 0$

$\therefore d(x, y) = d(x, z) + d(z, y)$

b)

If $x = y \neq z$ then
 $d(x, y) = 0$, $d(x, z) = 1$
and $d(z, y) = 1$

$$\therefore d(x, y) = d(x, z) + d(z, y)$$

(b) If $x \neq y = z$, then
 $d(x, y) = 1$, $d(x, z) = 1$ and $d(z, y) = 0$

$$d(x, y) = d(x, z) + d(z, y)$$

(c) If $x = z \neq y$ then
 $d(x, y) = 1$, $d(x, z) = 0$
 and $d(z, y) = 1$

$$d(x, y) = d(x, z) + d(z, y)$$

(e) If $x \neq z \neq y$ then $d(x, y) = d(x, z)$
 $d(z, y) = 1$

$$d(x, y) \leq d(x, z) + d(z, y)$$

Thus in all five cases, we have

$$d(x, y) \leq d(x, z) + d(z, y) \text{ for}$$

$$x, y, z \in X$$

Hence d is a metric for X ,
 known as a discrete metric
 and (X, d) is a discrete metric
 space.

Let d be a function of $\mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$
 such that

(a) $d(x_1, x_2), (y_1, y_2) = \max$

$\{ |x_1 - y_1|, |x_2 - y_2| \}$ where
 $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ then show
 that d is a metric of $\mathbb{R}^2$

Proof. Let $x = (x_1, x_2)$ $y = (y_1, y_2)$
 $z = (z_1, z_2)$ be arbitrary elements
 of $\mathbb{R}^2$

Also let $d(x, y) = \max \{ |x_1 - y_1|, |x_2 - y_2| \}$

(i) $d(x, y) \geq 0$ since modulus of any
 quantity is non-negative.

(ii) Now $d(x, y) = 0$
 $\Rightarrow \max \{ |x_1 - y_1|, |x_2 - y_2| \} = 0$

$$\Rightarrow |x_1 - y_1| = 0 = |x_2 - y_2|$$

$$\Rightarrow x_1 = y_1, x_2 = y_2$$

$$\Rightarrow (x_1, x_2) = (y_1, y_2)$$

$$\Rightarrow x = y$$

$$\therefore d(x, y) > 0 \text{ if } x \neq y$$

(i): we have $d(x, y) = \max \{ |x_1 - y_1|, |x_2 - y_2| \}$

$$= \max \{ |y_1 - x_1|, |y_2 - x_2| \}$$

$$[\because |a-b| = |b-a|]$$

$$d(y, x)$$

(iv) let $x = (x_1, x_2)$ $y = (y_1, y_2)$ $z = (z_1, z_2)$
 $\in \mathbb{R}^2$

we have

$$|x_1 - z_1| \leq \max \{ |x_1 - y_1|, |x_2 - y_2| \}$$

$$= d(x, y)$$

$$|z_1 - y_1| \leq \max \{ |z_1 - y_1|, |z_2 - y_2| \}$$

$$= d(z, y)$$

$$|x_1 - z_1| + |z_1 - y_1| \leq d(x, y) + d(z, y)$$

$$\text{But } |x_1 - y_1| = |x_1 - z_1 + z_1 - y_1|$$

$$\leq |x_1 - z_1| + |z_1 - y_1|$$

$$\text{So } |x_1 - y_1| \leq d(x, y) + d(z, y) \quad \text{--- (1)}$$

$$\text{Similarly } |x_2 - y_2| \leq d(x, y) + d(z, y) \quad \text{--- (2)}$$

From (1) and (2) we have

$$\max \{ |x_1 - y_1|, |x_2 - y_2| \} \leq d(x, y) + d(z, y)$$

$$d(x, y) \leq d(x, z) + d(z, y)$$

Hence d is a metric for $\mathbb{R}^2$

Bounded Sequence:-

A sequence $\langle x_n \rangle$ of real numbers is said to be bounded sequence if there exists a real number $k > 0$ such that $|x_n| \leq k$ for all $n \in \mathbb{N}$

Bounded Function:-

A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be bounded if there exists a real number $k > 0$ such that $|f(x)| \leq k$ for all $x \in [a, b]$

Ex.

Let X be the set of all real valued bounded function defined on $[a, b]$ and let d be a function such that

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)| \text{ for all } f, g \in X$$

show that (X, d) is a metric space.

For $f, g \in X$ we have

$$\sup_{x \in [a, b]} |f(x) - g(x)| \leq \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |g(x)|$$

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)| < \infty$$

so $d(f, g)$ is well defined.

Also (i) $d(f, g) \geq 0$

since $\sup_{x \in [a, b]} |f(x) - g(x)| \geq 0$

ii, $d(f, g) = 0 \Leftrightarrow \sup_{x \in [a, b]} |f(x) - g(x)| = 0$

$$\Leftrightarrow |f(x) - g(x)| = 0 \text{ for all } x \in [a, b]$$

$$\Leftrightarrow f(x) = g(x) \text{ for all } x \in [a, b] \Leftrightarrow f = g$$

iii, $d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|$

$$= \sup_{x \in [a, b]} |g(x) - f(x)| = d(g, f)$$

iv, let $f, g, h \in X$ then $d(f, g)$

$$= \sup_{x \in [a, b]} |f(x) - g(x)|$$

$$= \sup_{x \in [a, b]} |f(x) - h(x) + h(x) - g(x)|$$

$$\leq \sup_{x \in [a, b]} |f(x) - h(x)| + \sup_{x \in [a, b]} |h(x) - g(x)|$$

$$= d(f, h) + d(h, g)$$

$$d(f, g) \leq d(f, h) + d(h, g)$$

This d is a metric for X , known as supremum or uniform metric.

Semi-Metric Space:-

A semi metric space (X, d) consists of a non-empty set X and a function $d: X \times X \rightarrow \mathbb{R}$ called semi metric such that

(i), $d(x, y) \geq 0$ for all $x, y \in X$

(ii), if $x \neq y$ then $d(x, y) > 0$ for all $x, y \in X$,

(iii), $d(x, y) = d(y, x)$, for all $x, y \in X$

(v), $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Diameter of a Subset:

Let (X, d) be a metric space and let A be any non-empty subset of X , then the diameter of A , denoted by $\delta(A)$, is defined by

$$\delta(A) = \sup \{ d(x, y) : x, y \in A \}$$

For exp. if d is usual metric for $\mathbb{R}$ such that $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ and if $A = [1, 3]$ and $B = [3, 6]$ then $\delta(A) = 2$ and $\delta(B) = 3$

Distance b/w two Subsets:-

Let A and B be two subset for a metric space (X, d) . then distance b/w A and B is defined as

$$d(A, B) = \inf \{ d(x, y) : x \in A, y \in B \}$$

For exp. if d is a usual metric for $\mathbb{R}$ such that $d(x, y) = |x - y|$

and if $A = [1, 2]$ $B = [2, 4]$ then $d(A, B) = 0$

Again if $A = [1, 4]$ and $B = [6, 9]$ then

$$d(A, B) = \inf \{ d(x, y) : x \in A, y \in B \}$$

* Bounded And Unbounded metric spaces

Let (X, d) be a metric space, then it is said to be bounded if there exists a real number $k > 0$ such that

$$d(x, y) \leq k \text{ for all } x, y \in X$$

or $\delta(X) \leq k$

A metric space which is not bounded is said to be unbounded metric space.

For exp. let $X = \mathbb{R}$

$$d(x, y) = |x - y| \text{ for } x, y \in \mathbb{R}$$

This metric space (X, d) is unbounded since the diameter of $\mathbb{R}$ is infinite

A discrete metric space (X, d) when

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is bounded since $d(x, y) \leq 1$ for all $x, y \in X$

Ex 16 Any metric space (X, d) bounded
 A mtr. can be converted into a
 bounded metric space (X, d^*)
 where $d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$

Sol. i) $d^*(x, y) \geq 0$ since $d(x, y) \geq 0$

ii) $d^*(x, y) = 0 \iff \frac{d(x, y)}{1 + d(x, y)} = 0$

$\iff d(x, y) = 0 \iff x = y$
 $(\because d \text{ is metric})$

iii) $d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, x)}{1 + d(y, x)}$
 $= d^*(y, x)$

iv) Let $x, y, z \in X$

Consider $d^*(x, z) + d^*(z, y)$
 $= \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)}$

using $\frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|} \geq \frac{|a + b|}{1 + |a + b|}$

$d^*(x, z) + d^*(z, y) \geq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)}$

$d(x, y) \leq d(x, z) + d(z, y)$

$(\because d \text{ is metric})$

$\frac{1}{d(x, z) + d(z, y)} \leq \frac{1}{d(x, y)}$

$\frac{1}{d(x, z) + d(z, y)} + 1 \leq \frac{1}{d(x, y)} + 1$

$\frac{1 + d(x, z) + d(z, y)}{d(x, z) + d(z, y)} \leq \frac{1 + d(x, y)}{d(x, y)}$

$\frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)}$

$d^*(x, y) \leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)}$

$\leq d^*(x, z) + d^*(z, y)$

Thus d^* is a metric on X

Also $0 \leq d^*(x, y) < 1$

Hence (X, d^*) is a bounded metric space.

* Open sphere (or open Ball) :-

Let (X, d) be a metric space. Also let $a \in X$ and $r > 0$ be any real number. Then the set $\{x \in X : d(x, a) < r\}$ is called an open sphere (or open ball). The point a is called the centre of the sphere and r is the radius of the sphere. It is denoted by $S_r(a)$ or $S(a, r)$ or $B_r(a)$ or $B(a, r)$.

$$S_r(a) = \{x \in X : d(x, a) < r\}$$

closed sphere :-

Let (X, d) be a metric space. Also let $a \in X$ and $r > 0$ be any real number. Then the set $\{x \in X : d(x, a) \leq r\}$ is called a closed sphere with centre a and radius r . It is denoted by

$$S_r[a] \text{ or } S[a, r] \text{ or } B_r[a] \text{ or } B[a, r]$$

Thus

$$S_r[a] = \{x \in X : d(x, a) \leq r\}$$

Ex.

If d be the usual metric $d(x, y) = |x - y|$ for $x, y \in [0, 1]$ describe $S_{1/8}(0)$ $S(0, \frac{1}{8})$

$$S_{1/8}(0) = \{x \in [0, 1] : |x - 0| < \frac{1}{8}\}$$

$$= \{x \in [0, 1] : 0 - \frac{1}{8} < x < 0 + \frac{1}{8}\}$$

$$= \{x \in [0, 1] : -\frac{1}{8} < x < \frac{1}{8}\}$$

$$= \left(0, \frac{1}{8}\right)$$

* Interior point And Neighbourhood of a point :-

Let A be any subset of a metric space (X, d) . Then a point $a \in A$ is called interior point of A if a is the centre of an open sphere contained in A , then exist a real number $r > 0$ such that $S_r(a) \subseteq A$. In this A is said to be neighbourhood of point a .

Interior of a Set :-

The set of all interior point of A is called the interior of A .

5.4

The set of all interior points of A is called the interior of A and is denoted by A° .

Thus $A^\circ = \{x: x \text{ is an interior point of } A\}$

$$A^\circ = \bigcup \{S_r(x): S_r(x) \subseteq A\}$$

Thm. If A and B are subsets of metric space (X, d) , then

$$i, (A \cap B)^\circ = A^\circ \cap B^\circ$$

Proof:- i let $x \in (A \cap B)^\circ$

$\Rightarrow x$ is an interior point of $A \cap B$

$\Rightarrow$ there exists an open sphere $S_r(x)$ such that $S_r(x) \subseteq A \cap B$

$$S_r(x) \subseteq A \cap B \subseteq A, \quad S_r(x) \subseteq A \cap B \subseteq B$$

$$S_r(x) \subseteq A, \quad S_r(x) \subseteq B$$

$$x \in A^\circ, \quad x \in B^\circ$$

$$x \in A^\circ \cap B^\circ$$

$$(A \cap B)^\circ \subseteq A^\circ \cap B^\circ \quad - (1)$$

Conversely suppose

$$x \in (A^\circ \cap B^\circ) \Rightarrow x \in A^\circ \text{ and } x \in B^\circ$$

then exists open spheres $S_{r_1}(x), S_{r_2}(x)$ such that

$$S_{r_1}(x) \subseteq A, \quad S_{r_2}(x) \subseteq B$$

$$\text{Choose } r = \min\{r_1, r_2\}$$

$$S_{r_1}(x) \subseteq S_r(x) \subseteq A, \quad S_{r_2}(x) \subseteq S_r(x) \subseteq B$$

$$\Rightarrow S_r(x) \subseteq A, \quad S_r(x) \subseteq B$$

$$S_r(x) \subseteq A \cap B$$

$$x \in (A \cap B)^\circ$$

$$A^\circ \cap B^\circ \subseteq (A \cap B)^\circ \quad - (2)$$

Combining (1) and (2) we get

$$(A \cap B)^\circ = A^\circ \cap B^\circ$$

Ex

Show that the open interval $(a, b) \subseteq \mathbb{R}$ is an open set.

Let $x \in (a, b)$ be arbitrary. then we can find $\gamma > 0$ such that

$$(x - \gamma, x + \gamma) \subseteq (a, b)$$

$$S_\gamma(x) \subseteq (a, b)$$

thus (a, b) is nbhd of all $x \in (a, b)$ points and so is an open set.

Thm In a metric space, the intersection of a finite number of open sets is open.

Proof: - Let (X, d) be a metric space and let $\{G_i, i=1, 2, 3, \dots, n\}$ be a finite collection of open subsets of X .

Let $G = \bigcap_{i=1}^n G_i$ we shall now prove that G is an open set.

If each $G_i = \emptyset$ then $G = \emptyset$ and so G is open.

Let $G \neq \emptyset$ and if $x \in G$ then $x \in \bigcap_{i=1}^n G_i \Rightarrow x \in G_i$ for each $i, 1 \leq i \leq n$.

Since each G_i is open, then exists $\delta_i > 0$ such that $S_{\delta_i}(x) \subset G_i$ for each $i, 1 \leq i \leq n$.

Let $\delta = \min(\delta_1, \delta_2, \dots, \delta_n)$.

Then $S_\delta(x) \subset S_{\delta_1}(x)$

$S_\delta(x) \subset G_i$ for each $i, 1 \leq i \leq n$

$S_\delta(x) \subset \bigcap_{i=1}^n G_i = G$

Since $S_\delta(x)$ is an open sphere centred on x and contained in G therefore G is neighbourhood of x .

But x is an arbitrary point of G and hence G is open.

Thm

Let A be a subset of metric space (X, d) , then A° is the union of all open set contained in A .

Proof

Let $\{G_\alpha\}_{\alpha \in A}$ be an arbitrary collection of open sets contained in A .

Let $G = \bigcup_{\alpha \in A} G_\alpha$

We have to prove that $A^\circ = G$.
As arbitrary union of open sets is open, so G is open.

Also $G_\alpha \subset A$ for each $\alpha \in A$
 $\bigcup_{\alpha \in A} G_\alpha \subset A \Rightarrow G \subset A$

Thus G is an open set contained in A .
But A° is the largest open set contained in A .

So $G \subset A^\circ$ — ①
 Now, let $x \in A^\circ$. then x is an interior point of A . So there exists $\delta > 0$ such that

$$S_\delta(x) \subseteq A$$

As every open sphere is an open set so $S_\delta(x)$ is an open set contained in A .

But G is the union of all open sets contained in A . So

$$S_\delta(x) \subseteq G \quad x \in G \quad - ②$$

$$A^\circ \subseteq G$$

From ① and ②, we have

$$A^\circ = G.$$

* Adherent point:-

Let (X, d) be a metric space and let A be a subset of X . A point $x \in X$ is called an adherent point of A if every open sphere centred on x contains a point of A if $S_\delta(x)$ is an open sphere centred on x , then $S_\delta(x) \cap A \neq \emptyset$ for every $\delta > 0$.

Limit point:-

A point $x \in X$ is said to be limit point of subset A of X , if for every real number $\delta > 0$, there exists an open sphere $S_\delta(x)$ such that

$$A \cap S_\delta(x) - \{x\} \neq \emptyset$$

Isolated point:-

Let (X, d) be a metric space and A be a subset of X . A point $x \in X$ is said to be isolated point of a subset A if $x \in A$ but x is not a limit point of A .

Derived Set:-

Let A be a subset of metric space (X, d) . Then the set of all limit points of A is called derived set of A . It is denoted by $d(A)$ or A' .

Thm

Every derived set is a closed set.

Proof:-

Let A be a subset of a metric space (X, d) and let

$d(A)$ be its derived set. $\forall \epsilon$
shall prove that $d(A)$ is closed.

Let x be any limit point of $d(A)$.
Now, we have to prove that
 $x \in d(A)$

For all $n > 0$ we have
 $S_n(x) \cap d(A) - \{x\} \neq \emptyset$

[$\because x$ is limit point
of $d(A)$]

Let $a \in S_n(x) \cap d(A)$ such that $a \neq x$

$\Rightarrow a \in S_n(x)$ and $a \in d(A)$

Let $n_1 = n - d(a, n) > 0$ then we
claim that

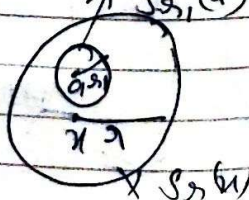
$$S_{n_1}(a) \subset S_n(x)$$

Let $y \in S_{n_1}(a) \Rightarrow d(a, y) < n_1$

$$d(y, n) \leq d(y, a) + d(a, n)$$

$$< n_1 + d(a, n)$$

$$< n \Rightarrow y \in S_n(x)$$



$$y \in S_n(x)$$

$$S_{n_1}(a) \subset S_n(x)$$

$$S_{n_1}(a) \cap A - \{a\} \subset S_n(x) \cap A - \{a\}$$

$$\text{But } S_{n_1}(a) \cap A - \{a\} \neq \emptyset$$

$$S_n(x) \cap A - \{a\} \neq \emptyset$$

$$x \in d(A)$$

Hence $d(A)$ is closed.

Let A subset A is a closed set
if and only if $\bar{A} = A$

Proof: If A is closed then $d(A) \subset A$

$$\bar{A} = A \cup d(A) \supset A$$

Conversely If $\bar{A} = A$ then

$$A \cup d(A) \supset A$$

$$d(A) \subset A \Rightarrow A \text{ is closed}$$

Chapters - 6th
Completeness in metric space.

DATE:
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* Sequence in metric spaces:- A sequence $\langle a_n \rangle$ in a metric space (X, d) is a function of $\mathbb{N}$ into X . Thus $f: \mathbb{N} \rightarrow X$ be a function such that $f(n) = a_n$ for all $n \in \mathbb{N}$. Then $\langle a_n \rangle$ is called a sequence in (X, d) whose n th term is a_n .

* Convergence in a metric spaces:-

A Sequence $\langle a_n \rangle$ in a metric space (X, d) is said to converge to a point $a \in X$ if the distance $d(a_n, a)$ tends to 0 as $n \rightarrow \infty$ for given $\epsilon > 0$. There exist a positive integer n_0 such that $d(a_n, a) < \epsilon$ for all $n \geq n_0$.

Thm Every Convergent sequence in a metric space has a ^{unique} limit.

Proof Suppose a Sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in metric space (X, d) converges to two limits say a and b such that $a \neq b$.

Then $d(a_n, a) \rightarrow 0$ as $n \rightarrow \infty$
 $d(a_n, b) \rightarrow 0$ as $n \rightarrow \infty$

For any $\epsilon > 0$ there exist positive integers N_1 and N_2 such that

$$d(a_n, a) < \frac{\epsilon}{2} \text{ for all } n \geq N_1$$

$$d(a_n, b) < \frac{\epsilon}{2} \text{ for all } n \geq N_2$$

Take $N = \max(N_1, N_2)$ then

$$0 \leq d(a, b) \leq d(a, a_n) + d(a_n, b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } n \geq N$$

Since $\epsilon > 0$ is arbitrary $\Rightarrow d(a, b) = 0$

$$\therefore a = b$$

Thm Every convergent sequence in a metric space is a Cauchy but converse need not be true.

Proof Let (X, d) be a metric space and let $\langle a_n \rangle$ be a convergent sequence in X . Suppose that $\lim_{n \rightarrow \infty} a_n = a$.

then for given $\epsilon > 0$ there exists a positive integer n_0 such that $d(a_m, a_n) < \frac{\epsilon}{2}$ for all $n, m \geq n_0$

$$\text{therefore } d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$d(a_n, a_m) < \epsilon \text{ for all } n, m \geq n_0$$

Hence $\langle a_n \rangle$ is a Cauchy Sequence

But converse of this thm is not true, Cauchy Sequence need not be convergent in an metric space.

For Ex. let us consider the space $\mathbb{Q}$ of rational numbers with usual metric.

$$d(x, y) = |x - y| \text{ then the sequence } \langle 1.4, 1.41, 1.411, 1.4142, \dots \rangle$$

of finite decimals converging to $\sqrt{2}$ is a Cauchy Sequence in $\mathbb{Q}$ but it does not converge in $\mathbb{Q}$. Since $\sqrt{2} \notin \mathbb{Q}$

Ex.

Show that the space $[0, 1]$ with usual metric is not complete.

Sol.

Consider the sequence $\langle \frac{1}{n} \rangle$ then the sequence $\langle \frac{1}{n} \rangle$ is Cauchy sequence in $[0, 1]$ but does not converge in $[0, 1]$ as $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$ and $0 \notin [0, 1]^n$

Hence $[0, 1]$ is not complete.

Thm

The usual metric space $(\mathbb{R}, d)$ is complete.

Proof:-

Let $\langle a_n \rangle$ be a Cauchy sequence in $(\mathbb{R}, d)$ then $\langle a_n \rangle$ is bounded sequence in $\mathbb{R}$.

Now by Bolzano Weierstrass thm of real number $\langle a_n \rangle$ must have a limit point say a .

We have to prove that $\langle a_n \rangle$ converges to a . Let $\epsilon > 0$ be given.

Since $\langle a_n \rangle$ is a Cauchy sequence, so there exists positive integer n_0 such that

$$d(q_n, q_m) < \varepsilon \text{ for all}$$

$$n, m \geq n_0$$

$$|q_n - q_m| < \frac{\varepsilon}{3} \text{ for all}$$

$$n, m \geq n_0$$

Also as a limit point of $\{q_n\}$ so there exists positive integer $k > n_0$ such that

$$|q_k - q| < \frac{\varepsilon}{3}$$

Since $k > n_0$ so by (1) we have

$$|q_k - q_m| < \frac{\varepsilon}{3}$$

$$|q_n - q| = |q_n - q_m + q_m - q_k + q_k - q|$$

$$\leq |q_n - q_m| + |q_m - q_k| + |q_k - q|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}, \text{ for all } n \geq n_0$$

$$|q_n - q| < \varepsilon \text{ for all } n \geq n_0$$

$$d(q_n, q) < \varepsilon \text{ for all } n \geq n_0$$

Thus $\{q_n\}$ is a Cauchy sequence

in R is convergent.

Hence R is a complete metric space.

Ex 4

Let d be the usual metric on R^2 and $F: (R^2, d) \rightarrow (R^2, d)$

be a function such that then

$$\text{sh. } F(x) = \frac{x}{3} \text{ for all } x = (x_1, x_2) \in R^2$$

Then show that F is a contraction on R^2

$$\text{let } x = (x_1, x_2), y = (y_1, y_2) \in R^2$$

$$d(Fx, Fy) = d\left(\frac{x}{3}, \frac{y}{3}\right)$$

$$d\left\{\frac{1}{3}(x_1, x_2), \frac{1}{3}(y_1, y_2)\right\}$$

$$= \sqrt{\left(\frac{x_1}{3} - \frac{y_1}{3}\right)^2 + \left(\frac{x_2}{3} - \frac{y_2}{3}\right)^2}$$

$$= \frac{1}{3} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 \right]^{1/2}$$

$$= \frac{1}{3} d(x, y)$$

$$\text{Thus } d(Fx, Fy) = \alpha \cdot d(x, y)$$

$$\text{where } \alpha = \frac{1}{3} < 1$$

Hence F is a contraction on R^2

Fixed point:-

Let X be a non-empty set $T: X \rightarrow X$ be mapping.
A point $x \in X$ is said to be a fixed point of T if $T(x) = x$

Ex Let $T: X \rightarrow X$ be an identity map on X .

Since $T(x) = x$ for all $x \in X$, therefore every point of X is a fixed point of T .

Continuous Function:-

Let (X, d) and (Y, d^*) be two metric space and let $F: X \rightarrow Y$ be a mapping of X into Y . Then F is said to be continuous at a point $a \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, a) < \delta \Rightarrow d^*(F(x), F(a)) < \epsilon$$

where $x \in X$ and $F(a) \in Y$

Ex

Let k be a given point of the metric space (X, d) and (R, d^*) be the usual metric space. Then show that the function $F_k: (X, d) \rightarrow (R, d^*)$ such that $F_k(x) = d(x, k)$ $x \in X$ is cont.

Sol

Let $a \in X$, $\epsilon > 0$ be given. Now, choose $\delta = \epsilon$. Then $d(x, a) < \delta$

Consider

$$|F_k(x) - F_k(a)| = |d(x, k) - d(a, k)|$$

$$\leq d(x, a) < \epsilon = \delta$$

Hence F_k is continuous on X .

thm.

Let (X, d) and (Y, d^*) be two metric spaces and let f, g be two cont. functions of X into Y .
then the set $\{x \in X : f(x) = g(x)\}$ is a closed subset of X .

Proof:

$$\text{let } A = \{x \in X : f(x) = g(x)\}$$

we shall prove that A is a closed subset of X .

for this it is sufficient to prove that its complement $X - A = A^c$ is open.

If $A^c = \emptyset$ then obviously A is closed.

so let $A^c \neq \emptyset$ Take $a \in A^c$ then

$$a \in A^c \Rightarrow a \notin A$$

$$f(a) \neq g(a)$$

set $\Rightarrow$

$$d^*(f(a), g(a)) > 0$$

Since f and g are cont. thus for $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x, a) < \delta$$

$$\Rightarrow d^*(f(x), f(a)) < \frac{\epsilon}{3}$$

$$d^*(g(a), g(a)) < \frac{\epsilon}{3}$$

$$d^*(f(a), g(a)) \leq d^*(f(a), f(u)) + d^*(f(u), g(u)) + d^*(g(u), g(a))$$

$$< \frac{\epsilon}{3} + d^*(f(u), g(u)) + \frac{\epsilon}{3} \quad \text{--- (1)}$$

$$\epsilon < d^*(f(u), g(u)) + 2\frac{\epsilon}{3}$$

$$d^*(f(u), g(u)) > \frac{\epsilon}{3} > 0$$

$$f(u) \neq g(u)$$

thus we have

$$d(u, a) < \delta \quad f(u) \neq g(u)$$

$$u \in S(a, \delta) \Rightarrow u \in A^c$$

$$S(a, \delta) \subseteq A^c$$

This shows that A^c is the nb. of a . Since a is arbitrary. so A^c is the neighbourhood of all of its point and therefore A^c is open.

Hence, A is closed.

thm

Let (X, d) and (Y, d^*) be metric spaces and f be a function of X into Y . then f is continuous if and only if $f^{-1}(U)$ is open

PAGE :
in X whenever G is open in Y .

Proof: Let f be continuous on (X, d) and let G be any open subset of Y .

To prove that $f^{-1}(G)$ is open in X .

If $f^{-1}(G) = \emptyset$ it is clearly open.

So let $f^{-1}(G) \neq \emptyset$

Let $a \in f^{-1}(G) \Rightarrow f(a) \in G$

Since G is open, there exists an open sphere $S(f(a), \epsilon)$ such that

$$S(f(a), \epsilon) \subseteq G \quad \text{--- (1)}$$

Also as f is cont. at a there exists an open sphere $S(a, \delta)$ such that

$$f(S(a, \delta)) \subseteq S(f(a), \epsilon) \quad \text{--- (2)}$$

From (1) and (2)

$$f(S(a, \delta)) \subseteq G$$

$$S(a, \delta) \subseteq f^{-1}(G)$$

therefore $f^{-1}(G)$ is open.

Conversely, Suppose that the inverse image under f of every open set in Y is an open set in X .

we have to prove that f is continuous.

Let $a \in X$

For any $\epsilon > 0$, consider the open sphere $S(f(a), \epsilon)$

By assumption $f^{-1}(S(f(a), \epsilon))$ is open.

Since $a \in f^{-1}(S(f(a), \epsilon))$ there exists an open sphere $S(a, \delta) \subseteq f^{-1}(S(f(a), \epsilon))$

$$f(S(a, \delta)) \subseteq S(f(a), \epsilon)$$

this shows that f is continuous at $x = a$

Since a was an arbitrarily chosen point, it follows that f is cont. at every point of X and hence f is continuous.

Thm

Let (X, d) and (Y, d') be two metric spaces and let $f: X \rightarrow Y$. Then f is continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

Proof

Let f be continuous and $A \subseteq X$.

we have to prove that

$$f(\bar{A}) \subseteq \overline{f(A)} \quad \bar{A} \subseteq f^{-1}[\overline{f(A)}] \text{ for all } A \subseteq X$$

Now, $\overline{f(A)}$ is a closed subset of Y [$\because$ closure of a set is always closed]

and f is continuous

$\therefore f^{-1}[\overline{f(A)}]$ is a closed subset of X .

consequently,

$$\overline{f^{-1}[\overline{f(A)}]} = f^{-1}[\overline{f(A)}] \quad - (1)$$

$\because B$ is closed iff $\bar{B} = B$

$$\text{Now } f(A) \subseteq \overline{f(A)} \\ A \subseteq f^{-1}[\overline{f(A)}]$$

$$\bar{A} \subseteq \overline{f^{-1}[\overline{f(A)}]}$$

$$\bar{A} \subseteq f^{-1}[\overline{f(A)}]$$

$$f(\bar{A}) \subseteq \overline{f(A)} \quad \text{By (1)}$$

Conversely

let $f(\bar{A}) \subseteq \overline{f(A)}$ for every subset A of X .

then we have to prove that f is continuous.

let B be any closed subset of Y .

$$\bar{B} = B \Rightarrow f^{-1}(\bar{B}) = f^{-1}(B) \subseteq X$$

In given hypothesis taking

$A = f^{-1}(B)$ we have

$$f[\overline{f^{-1}(B)}] \subseteq \overline{f[f^{-1}(B)]}$$

$$f[\overline{f^{-1}(B)}] \subseteq \bar{B} = B$$

$$\bar{f^{-1}(B)} \subseteq f^{-1}(B) \quad - (2)$$

$$f^{-1}(B) \subseteq \overline{f^{-1}(B)} \quad - (3)$$

from ② and ③, we get

$$F^{-1}(B) = F^{-1}(B)$$

②) $F^{-1}(B)$ is closed whenever B is closed.

Hence F is continuous.

Exmp. Show that the function $F: [0,1] \rightarrow \mathbb{R}$ such that $F(x) = x^2$ is uniformly cont.

$$\text{let } x_1, x_2 \in [0,1]$$

let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2} > 0$ then

$$|x_1 - x_2| < \delta$$

$$|F(x_1) - F(x_2)| = |x_1^2 - x_2^2|$$

$$= |(x_1 - x_2)(x_1 + x_2)|$$

$$\leq |x_1| + |x_2| |x_1 - x_2|$$

$$\leq 2|x_1 - x_2| < 2\delta = \varepsilon$$

$$d(x_1, x_2) < \delta \Rightarrow d^*(F(x_1), F(x_2)) < \varepsilon$$

which show that f is uniformly continuous.

Prove that an isometry is a uniformly continuous function.

Sol. let $x, y \in X$ be arbitrary.
let $\varepsilon > 0$ be given. choose $\delta > 0$ such that $\delta \leq \varepsilon$

$$d(x, y) < \delta$$

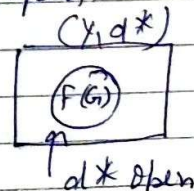
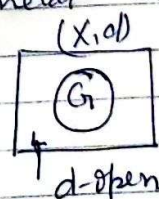
$$d^*(F(x), F(y)) = d(x, y) < \delta \leq \varepsilon$$

$$\text{thus } d(x, y) < \delta \Rightarrow d^*(F(x), F(y)) < \varepsilon$$

Hence f is uniformly cont.

* Some Mappings:-

(i) Open Mapping:- The mapping $F: X \rightarrow Y$ is said to be an open mapping if $F(G)$ is d^* open whenever G is d -open.



(ii) closed Mapping:- The mapping $F: X \rightarrow Y$ is said to be a closed mapping if $F(H)$ is closed mapping if $F(H)$ is

* closed whenever H is σ -closed

(iii) Bicontinuous Mapping:- The mapping $F: X \rightarrow Y$ is said to be bicontinuous if F is open and continuous.

(iv) Homeomorphism:- The mapping $F: (X \rightarrow Y)$ is said to be homeomorphism if
 i. F is one-one, onto
 ii. F is bicontinuous.

Cover:- i. Let A be a subset of a metric space (X, d) then a collection $\{G_\alpha : \alpha \in I\}$ of subsets of X is said to be a cover of A if $A \subseteq \bigcup_{\alpha \in I} G_\alpha$

ii. If each member of a cover is open, then it is said to be an open cover.

iii. A Subcover C^* of a cover C is said to be a finite subcover if C^* has finite number of members.

* Compact set and compact metric space:-

A subset A of a metric space (X, d) is said to be compact if every open cover of A has finite subcover.

Ex

Prove that the usual metric space $(\mathbb{R}, d)$ is not compact.

Sol.

Let $\mathcal{D} = \{G_n : n \in \mathbb{N}\}$ where $G_n = (-n, n)$

Clearly, $\mathcal{D}$ is a σ -open cover of $\mathbb{R}$
 If $\{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$ be any finite sub-collection of $\mathcal{D}$,

Let $n_0 = \max \{n_1, n_2, \dots, n_k\}$

Then $n_0 \in G_{n_i} \ (i=1, 2, \dots, k)$

Since $n_0 \in \mathbb{R}$, we see that no finite sub-collection of $\mathcal{O}$ can cover $\mathbb{R}$.

Hence $(\mathbb{R}, d)$ is not compact.

Bolzano Weierstrass Property (BWP)

A metric space (X, d) is said to have Bolzano Weierstrass property if every infinite subset of X has a limit point.

Sequentially compact metric space:-

A metric space (X, d) is said to be sequentially compact metric space if every sequence of points of the space contains a convergent subsequence.

* A metric space is sequentially compact iff it has the BWP.
in $\mathbb{R}$.

A metric space is sequentially compact iff every infinite subset has a limit point.

Proof:- Let the metric space (X, d) is sequentially compact and let A be any infinite subset of X .

Let $\langle a_n \rangle$ be any sequence of distinct points of A .

Since X is sequentially compact, this sequence has a convergent subsequence $\{a_{n_i}\}$ whose limit is a .

$\{a_{n_i}\} \subset \{a_n\} \subset A$ and $\langle a_{n_i} \rangle \rightarrow a$

$\therefore a$ is a limit point of A .

Hence X has BWP.

Conversely:-

Let X have BWP. Let $\langle a_n \rangle$ be any sequence of points of X .

Let A be the range of the sequence $\langle a_n \rangle$.

If A is finite, then there is at least one point say a_{n_0} , which occurs infinitely many times in the sequence $\langle a_n \rangle$. It follows that $\langle a_{n_0}, a_{n_0}, \dots \rangle$ is a subsequence of $\langle a_n \rangle$ which converges to $a_{n_0} \in X$.

If A is infinite, then by hypothesis, A has at least one limit point, say a .

Then it is easy to extract from $\langle a_n \rangle$, a subsequence which converges to a .

Hence X is sequentially compact.

Thm Every compact (sequentially compact) metric space is complete.

Proof:-

Let (X, d) be a compact metric space. Then X is sequentially compact.

Let $\langle a_n \rangle$ be a Cauchy sequence in X . Then $\langle a_n \rangle$ has convergent subsequence $\langle a_{n_k} \rangle$

converging to some $a \in X$. We shall prove that $\langle a_n \rangle \rightarrow a$.

Let $\epsilon > 0$ be given.

Since $\langle a_n \rangle$ is a Cauchy sequence, then exists $n_0 \in \mathbb{N}$ such that

$$d(a_{m_i}, a_{n_i}) < \frac{\epsilon}{2} \text{ for all } m_i, n_i > n_0 \quad (1)$$

Again $\langle a_{n_k} \rangle \rightarrow a$ then for this exists $k_0 \in \mathbb{N}$ such that

$$d(a_{n_k}, a) < \frac{\epsilon}{2} \text{ for all } k > k_0 \quad (2)$$

Choose $k_1 > k_0$ such that

$n_{k_1} > n_0$ then by (1) and (2) we have

$$d(a_n, a_{n_{k_1}}) < \frac{\epsilon}{2} \text{ for all } n > n_{k_1}$$

$$d(a_{n_{k_1}}, a) < \frac{\epsilon}{2}$$

$$d(a_n, a) \leq d(a_n, a_{n_{k_1}}) + d(a_{n_{k_1}}, a) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$d(a_n, a) < \epsilon \text{ for all } n > n_{k_1}$$

$$\lim_{n \rightarrow \infty} a_n = a$$

Hence (X, d) is complete.

Page:
An Important Remark: -

Let (X, d) be a metric space.
then the following statements
are equivalent.

- (a) X is compact
- (b) X is countably compact
- (c) X has BWP
- (d) X is sequentially compact
- (e) X is complete and totally bounded
- (f) Every countable collection of closed subsets of X having finite intersection property has a non-empty intersection.

Thm Prove that continuous image
of a compact metric space is
compact.

or

Let F be a continuous mapping
of a compact metric space X
into a metric space Y
then $F(X)$ is compact.

Proof: Let F be a continuous
mapping of a compact metric

space X into a metric space
 Y .

Let $\{H_\alpha\}_{\alpha \in I}$ be an open cover
of $F(X)$.

Since F is continuous, $F^{-1}\{H_\alpha\}$
is an open set in X .

It follows that the collection
 $\{F^{-1}(H_\alpha)\}_{\alpha \in I}$ is an open cover of
 X .

Since X is compact, there exist
finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n$

$$\text{such that } X = \bigcup_{i=1}^n F^{-1}(H_{\alpha_i}) = F^{-1}\left(\bigcup_{i=1}^n H_{\alpha_i}\right)$$

$$\text{It follows that } F(X) = F\left(F^{-1}\left(\bigcup_{i=1}^n H_{\alpha_i}\right)\right) \subset \bigcup_{i=1}^n H_{\alpha_i}$$

Hence $F(X)$ is compact.

Chapter 9-4
Connectedness in Metric Spaces

Separated Sets:-

Let (X, d) be a metric space. Two non-empty A and B of X said to be separated if

$$A \cap \bar{B} = \emptyset \text{ and } \bar{A} \cap B = \emptyset$$

$$(A \cap \bar{B}) \cup (\bar{A} \cap B) = \emptyset$$

Thus A and B are separated if and only if A and B are disjoint and neither of them contains limit point of the other.

$$\text{Let } A \cap B \neq \emptyset$$

$$A \cap \bar{B} = \emptyset \Rightarrow A \cap (B \cup d(B)) = \emptyset$$

$$= (A \cap B) \cup (A \cap d(B)) = \emptyset$$

$$\Rightarrow \emptyset \cup (A \cap d(B)) = \emptyset$$

$$\Rightarrow A \cap d(B) = \emptyset$$

$\Rightarrow A$ contains no limit of B .

Similarly, B contains no limit point of A

thm

If E is connected subset of a metric space (X, d) such that $E \subset A \cup B$, where A and B are separated sets in X then either $E \subset A$ or $E \subset B$.

Proof.

Since A and B are separated

$$\therefore A \cap \bar{B} = \emptyset \quad \bar{A} \cap B = \emptyset$$

$$\text{Now, } E \subset A \cup B \Rightarrow E = E \cap (A \cup B) \\ = (E \cap A) \cup (E \cap B) \quad \text{--- (1)}$$

we claim that atleast one of the sets $E \cap A$ and $E \cap B$ is empty.

Let if possible $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$

$$\text{then } (E \cap A) \cap (E \cap B) \subset (E \cap A) \cap (\bar{E} \cap B)$$

$$[\because A \cap B \subset \bar{A} \cap \bar{B}]$$

$$= (E \cap \bar{E}) \cap (A \cap B)$$

$$= (E \cap \bar{E}) \cap \emptyset = \emptyset$$

Similarly,

$$(\bar{E} \cap A) \cap (E \cap B) \neq \emptyset$$

Hence $E \cap A$ and $E \cap B$ are separated sets.

Thus E has been expressed as the union of two non-empty separated sets and consequently E is disconnected. But this is a contradiction.

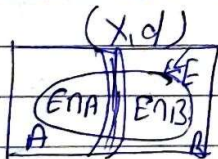
Hence at least one of the set $E \cap A$ and $E \cap B$ is empty.

If $E \cap A = \emptyset$ then from ① we get

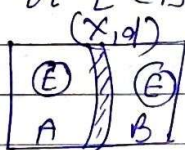
$$E = \emptyset \cup (E \cap B) = E \cap B \Rightarrow E \subset B$$

Again if $E \cap B = \emptyset$ then $E \subset A$

Hence either $E \subset A$ or $E \subset B$



(E is separated)



(E is connected)

Thm.

Let (X, d) be a metric space and let E be a connected subset of X . If F is a subset of X such that $E \subset F \subset \bar{E}$, then F is connected, in particular $\bar{E}$ is connected.

Closure of connected subset in (X, d) is also connected.

and

Closure of a connected subset in (X, d) is also connected.

Proof.

Let F is not connected. then there exists non-empty separated sets whose union is F . then exists A and B such that $F = A \cup B$ and $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$

Since $F \subset E$ we have $E \subset A \cup B$

It follows that either $E \subset A$ or $E \subset B$

$$\text{Let } E \subset A \Rightarrow \bar{E} \subset \bar{A} \Rightarrow \bar{E} \cap B \subset \bar{A} \cap B = \emptyset$$

$$\Rightarrow \bar{E} \cap B = \emptyset \quad \text{--- (1)}$$

Again $F = A \cup B$ and $F \subset E$

$$\Rightarrow B \subset F \subset E$$

$$\bar{B} \cap B = B \quad \text{--- (2)}$$

From ① and ② we conclude that $B = \emptyset$ which is a contradiction, since B is

non-empty.
Hence F is connected.

thm the real line in a usual metric space $(\mathbb{R}, d)$ is connected.

A
The set of real numbers with usual metric is a connected space.

Proof:

Let if possible $(\mathbb{R}, d)$ is disconnected.

Then there exist non-empty & closed subsets A and B of $\mathbb{R}$ such that

$$\mathbb{R} = A \cup B \text{ and } A \cap B = \emptyset$$

Let $x \in A$ and $y \in B$ then $x \neq y$

$\Rightarrow$ either $x < y$ or $x > y$

Let $x < y$. Since $x, y \in \mathbb{R}$ and $\mathbb{R}$ is interval, we have

$$[x, y] \subset \mathbb{R} = A \cup B$$

Let z

$$\text{Sub } [A \cap (x, y)]$$

then clearly $x < z < y$
Consequently $z \in \mathbb{R}$

But A is closed in $\mathbb{R}$, so it has l.u.b. say $z < y$ and $z \in A$. Moreover the definition of z shows that

$$z + \epsilon \in B \text{ for each } \epsilon > 0 \text{ for which } z + \epsilon < y$$

$\Rightarrow$ Every neighbourhood of z contains at least one point B other than z .

So z is a limit point of B

But B is closed. Therefore $z \in B$
 $\Rightarrow z \in A \cap B$

this is contradiction since $A \cap B = \emptyset$

Hence $(\mathbb{R}, d)$ is a connected space

thm

A continuous image of a connected space is connected.
OR

Let (X, d) and (Y, d^*) be two metric space and F be a continuous mapping of X onto Y . If X is connected, then Y is also connected.

Proof: Let $F: X \rightarrow Y$ be a cont. mapping of a connected space X into an arbitrary metric space Y . We shall prove that $F(X)$ is connected as a subspace of Y .

Let us suppose $F(X)$ is disconnected. Then there exist set A and B both open in Y such that

$$A \cap F(X) \neq \emptyset \quad B \cap F(X) \neq \emptyset$$

$$(A \cap F(X)) \cap (B \cap F(X)) = \emptyset$$

$$F(X) = (A \cap F(X)) \cup (B \cap F(X))$$

It follows that $\emptyset = F^{-1}(\emptyset)$

$$F^{-1}[(A \cap F(X)) \cap (B \cap F(X))]$$

$$= F^{-1}[(A \cap B) \cap F(X)]$$

$$= F^{-1}(A) \cap F^{-1}(B) \cap F^{-1}(F(X))$$

$$= F^{-1}(A) \cap F^{-1}(B) \cap X$$

$$= F^{-1}(A) \cap F^{-1}(B)$$

$$X = F^{-1}(F(X))$$

$$= F^{-1}[(A \cap F(X)) \cup (B \cap F(X))]$$

$$= F^{-1}[(A \cup B) \cap F(X)] \quad \text{From (1)}$$

$$F^{-1}(A \cup B) \cap F^{-1}(F(X))$$

$$= F^{-1}(A) \cup F^{-1}(B) \cap X$$

$$= F^{-1}(A) \cup F^{-1}(B)$$

Since F is continuous and A and B are open in Y both intersecting $F(X)$. It follows that

$F^{-1}(A)$ and $F^{-1}(B)$ are both non-empty open subsets of X .

thus we have shown that X is the union of two disjoint open subset of X and consequently X is disconnected, which is contradiction.

Hence $F(X)$ must be connected.

Totally Disconnected Spaces: -

A metric space (X, d) is said to be totally disconnected if for each pair of distinct points

$x, y \in X$ there exists a disconnection $A \cup B$ of X such that $x \in A$ and $y \in B$.

Ex Show that every discrete metric space is totally disconnected.

Sol. For any $x, y \in X$ such that $x \neq y$ we have.

$A = \{x\}$ and $B = X - \{x\}$ are both non-empty open disjoint sets whose union is X such that $x \in A$ and $y \in B$.

Hence $A \cup B$ is a disconnection of X with $x \in A$ and $y \in B$.

It follows that (X, d) is totally disconnected.

Locally Connected Spaces: -

A metric space (X, d) is said to be locally connected at $x \in X$ if every open neighbourhood of x contains a connected open neighbourhood of x .

This space (X, d) is said to be locally connected if it is

locally connected at each of its points.

Show that every discrete metric space (X, d) is locally connected.

Sol. For any $x \in X$, the set $\{x\}$ is connected neighbourhood of x . Also every neighbourhood of x must