Notes of Dynamics.

B.Sc – Vth Sem (Third Year).

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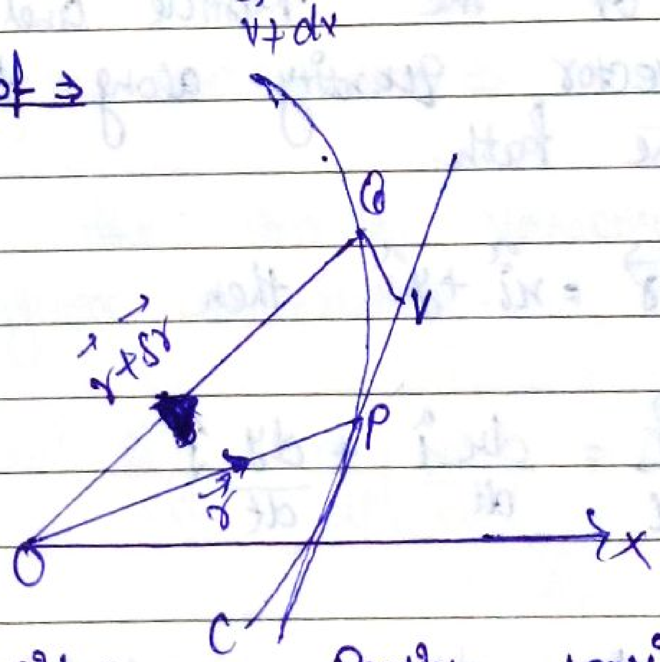
CHAPTER ⇒ 1

Motion along a Plane Curve

Velocity Along a Plane Curve ⇒ The velocity of a Particle moving along a curve is the rate change of its displacement with respect to time.

Proof ⇒



Consider a Particle moving along a curve C. let P and Q be its Position at times t and $t+\delta t$ respectively with respect to O as origin of vectors.

let $\vec{OP} = \vec{r}$

and $\vec{OQ} = \vec{r} + \delta\vec{r}$

$\vec{PQ} = \vec{OQ} - \vec{OP} = \vec{\delta r}$

Thus the displacement in time δt is δr. Suppose v and $v + dv$ are the velocity of particle at P and Q respectively. Then the velocity at P is given by $\vec{u} = \lim_{\delta t \to 0} \frac{\delta r}{\delta t}$

$\vec{v} = \frac{d\vec{r}}{dt}$ is along the tangent to the path of the particle and hence it is vector quantity along the tangent to the path. The velocity

if $\vec{r} = x\hat{i} + y\hat{j}$ then

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

Here $\frac{dx}{dt}, \frac{dy}{dt}$ are called the resolved parts of velocity u along the axes of x and y respectively.

Speed of particle at $P = |\vec{V}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Resultant velocity ⇒

$$V = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

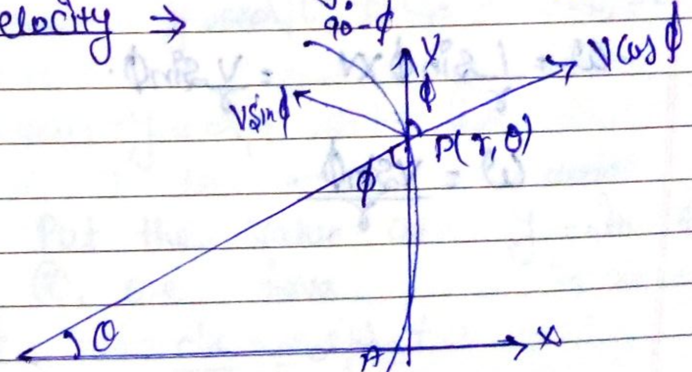
Resultant acceleration

$$\Rightarrow \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

Angular acceleration ⇒ The rate chang of the angular velocity is called angular acceleration.

$$\frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

Relation b/w Angular velocity and linear velocity ⇒



Let the linear velocity of the Point $P(r,\theta)$ at any time t be v which is along the tangent at P. Let ϕ be the angle which the tangent at P makes with the radius vector OP. Then

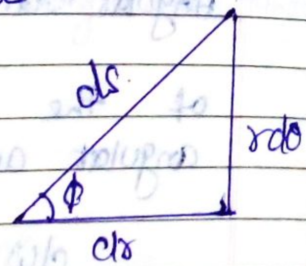
$$\omega = \frac{d\theta}{dt} = \frac{d\theta}{ds}\cdot\frac{ds}{dt} \quad -①$$

Now

$$\frac{d\theta}{ds} = \frac{1}{r}\cdot r\frac{d\theta}{ds}$$

and

$$\sin\phi = \frac{r\,d\theta}{ds}$$



$$\therefore \frac{d\theta}{ds} = \frac{1}{r}\sin\phi.$$

also $\frac{ds}{dt} = v$.

$\therefore$ from ①, we have

$$\omega = \frac{1}{r}\sin\phi \times v = \frac{v}{r}\sin\phi.$$

$$\omega = \frac{v\sin\phi}{r}.$$

Question ⇒ 1. A Particle is moving with a Constant velocity Parallel to axis of y and a velocity Proportional to y, Parallel to the axis of x. Prove that it will describe a Parabola.

Sol^n ⇒ Here $\frac{dy}{dt}$ = Constant

and $\frac{dx}{dt} \propto y$

$$\therefore \frac{dy}{dt} = \alpha_1 \quad --- ①$$

$$\frac{dx}{dt} = \alpha_2 y \quad --- ②$$

where α_1 and α_2 are Constants.

Integration equ^n ①, we get

$$y = \alpha_1 t + c_1 \quad [\because c_1 = 0.$$

$$y = \alpha_1 t \quad --- ③.$$

Put the value of y in equ^n ②, we have

$$\frac{dx}{dt} = \alpha_1\alpha_2 t$$

$\alpha_1\alpha_2 = \alpha$

Then $\frac{dx}{dt} = \alpha t$ - (4).

Integration equ^n (4). we get

$x = \alpha t^2 + C_2$

$(\because C_2 = 0$, when $t=0)$

$x = \frac{\alpha t^2}{2}$ ... (5)

Now, we shall eliminate t b/w (3) and (5) equ^n

from (3), we get $t = \frac{y}{\alpha_1}$

$x = \frac{\alpha \cdot y^2}{2\alpha_1^2}$

$y^2 = lx$ (where l is a constant)

Thus the path of the particle is a Parabola.

Question No => 2

A Particle start from the origin and the components of its velocity parallel to the axes of co-ordinates at time t and $2t+3$ and $4t$. find the Path.

Sol^n =>

$\frac{dx}{dt} = 2t + 3$ - (1)

and

$\frac{dy}{dt} = 4t$ - (2).

Integration (1) & (2), we get

$x = t^2 + 3t + C_1$

$y = 2t^2 + C_2$

Initially at $t=0$, we get

$x=0, \quad y=0$ - 3

$C_1 = C_2 = 0$

$x = t^2 + 3t$ - (3)

$y = 2t^2$ - (4)

Elimnating t from (3) & (4), we get

$t^2 = \frac{y}{2}$ in (3) equ^n, we get

$$\frac{x-y}{3} = 3t$$

Squaring both side.

$$x^2 + \frac{y^2}{4} - xy = 9t^2 = 9\left(\frac{y}{2}\right).$$

$$4x^2 + y^2 - 4xy - 18y = 0.$$

Radial acceleration ⇒

The resolved Part of the acceleration at P along the radius vector $\vec{OP}$ in the sense of r increasing is called radial acceleration.

Transerve acceleration ⇒

The resolved Part of the acceleration at P along a line through P but right angle to radius vector $\vec{OP}$ and in the sense in which θ increase is called the transerve acceleration.

Question No. 2

A Point P moves with a constant angular velocity about O, where O is the Pole of the equiangular spiral $r = ae^{\theta}$. Obtain radial and transerve acceleration of P

Sol^n ⇒ The equation of the curve

$$r = ae^{\theta} \quad -①$$

Since angular velocity is constant

$$\frac{d\theta}{dt} = \text{constant} = \omega (\text{say}). \quad -②$$

Diff equ^n ① w r to time, we get

$$\frac{dr}{dt} = ae^{\theta} \cdot \left(\frac{d\theta}{dt}\right)$$

$$\frac{dr}{dt} = r\omega \quad \cdots ③$$

Diff equ^n ③ w r to time, we get

$$\frac{d^2r}{dt^2} = \omega\frac{dr}{dt} = \omega(\omega r) = \omega^2 r$$

∴ Radial acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$= \omega^2 r - r\omega^2 = 0$$

Transerve acceleration ⇒

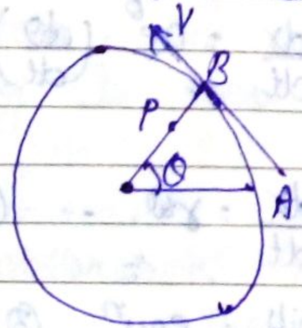
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\omega\right)$$

$$= \frac{1}{r}(2r\omega)\cdot\frac{dr}{dt} = 2\omega(r\omega) = 2\omega^2 r.$$

Question No => 4

An Insects Crawls at a Constant rate u along the spoke of a cart wheel of radius a and the cart is moving with velocity v. find the acceleration along the Perpendicular to the spoke.

Sol^n => Let OA be the spoke and Initially the insect be at O.



Let OB be the new Position of the spoke OA after time t and the Insect reaches at point P during this Interval. Since the Insect crawls along the spoke OA with constant velocity u, so distance $OP = ut$.

let $\angle AOB = \theta$.

Taking O as pole and OA as initial line, the Polar co-ordinates of P are (r, θ) where $r = ut$

$\therefore$ Radial velocity of the Insect $= \frac{dr}{dt} = u$

Transerve velocity of the Insect $= a\frac{d\theta}{dt} = v$.

$$\therefore \frac{dr}{dt} = u \quad \text{-(1) and} \quad \frac{d\theta}{dt} = \frac{v}{a} = \text{constant} \quad \text{-(2)}$$

Diff. equ^n (1) and (2) w.r.to time, we have

$$\ddot{r} = 0, \quad \ddot{\theta} = 0.$$

Radial acceleration of the Insect along the spoke

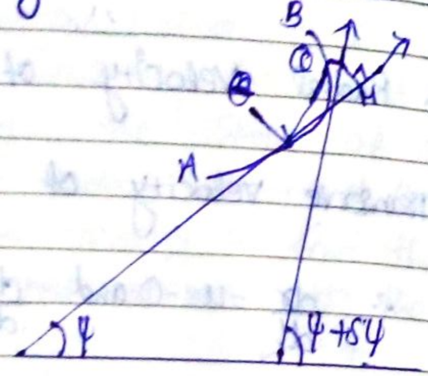
$$= \ddot{r} - r\dot{\theta}^2 = 0 - r\frac{v^2}{a^2} = -ut\left(\frac{v^2}{a^2}\right).$$

Transerve acceleration of the Insect Perpendicular to the spoke

$$= 2\dot{r}\dot{\theta} + r\ddot{\theta} = 2u\frac{v}{a} + 0 = \frac{2uv}{a}.$$

State ⇒ To find the tangential and normal components of velocity of a particle moving along a plane curve.

Proof ⇒



Let A be a fixed point on the plane curve AB of a moving particle. Let P and Q be the positions of the particle at time t and $t+\delta t$ respectively such that arc $AP = S$ and the arc $AQ = S + \delta S$.

$\therefore$ Arc $PQ = \delta S$.

Let ψ and $\psi + \delta\psi$ be the angles that the tangents at P and Q makes with a fixed line OX (x-axis).

Draw QM perpendicular to OP and join the chord PQ.

Let $\angle QPM = \theta$ so that when $Q \to P$, $\theta \to 0$.

$\therefore$ Tangential velocity at P,

$$= \lim_{\delta t \to 0} \frac{\text{displacement along the tangent}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{PM}{\delta t} = \lim_{\delta t \to 0} \frac{(\text{Chord } PQ)\cos\theta}{\delta t}$$

$$\Rightarrow \lim_{\delta t \to 0} \left(\frac{\text{Chord } PQ}{\text{arc } PQ}\right) \cdot \frac{(\text{arc } PQ)\cos\theta}{\delta t}$$

$$\Rightarrow \lim_{\delta t \to 0} \left(\frac{\text{Chord } PQ}{\text{arc } PQ}\right) \cdot \frac{\delta S}{\delta t} \cdot \cos\theta$$

$$\Rightarrow 1 \cdot \frac{ds}{dt} \cdot 1 = \frac{ds}{dt}.$$

Hence tangential velocity $= \frac{ds}{dt}$.

Again, Normal velocity at $P = \lim_{\delta t \to 0} \frac{\text{dis. along N}}{\delta t}$

$$\lim_{\delta t \to 0} \frac{QM}{\delta t} = \lim_{\delta t \to 0} \frac{(\text{Chord } PQ)\sin\theta}{\delta t}.$$

$$\lim_{\delta t \to 0} \frac{\text{Chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t} \sin\theta$$

$$\lim_{\delta t \to 0} 1 \cdot \frac{ds}{dt} \cdot 0 = 0.$$

Hence, Normal velocity $= 0$.

Question No ⇒ 5

A particle describe the cycloide $S = 4a\sin\psi$ with uniform velocity v. find its acceleration at any point

Sol ⇒ The equation of cycloide is

$$S = 4a\sin\psi \quad -(1)$$

Since the particle moves with uniform speed v.

∴ Tangential velocity acceleration, $\frac{dv}{dt} = 0$

Diff of eqn (1) w.r to ψ, we have

$$\frac{ds}{d\psi} = 4a\cos\psi, \text{ i.e. } \rho = 4a\cos\psi.$$

also,

Normal acceleration

$$\frac{v^2}{\rho} = \frac{v^2}{4a\cos\psi} = \frac{v^2}{4a\sqrt{1-\sin^2\psi}}$$

$$= \frac{u^2}{\sqrt{16a^2 - 16a^2\sin^2\psi}} = \frac{v^2}{\sqrt{16a^2 - s^2}} \quad (\because s = 4a\sin\psi)$$

∴ Resultant acceleration

$$= \sqrt{(\text{Tangential acceleration})^2 + (\text{Normal accel})^2}$$

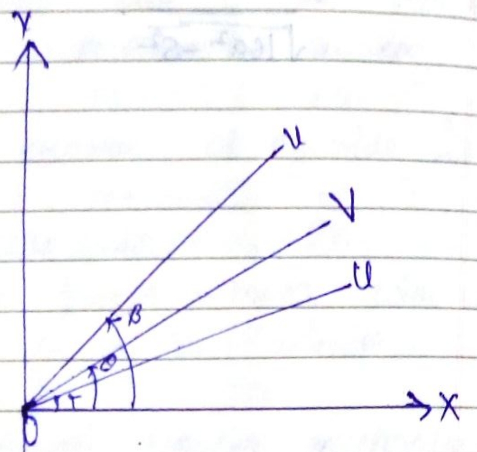
$$= \frac{v^2}{\sqrt{16a^2 - s^2}}.$$

CHAPTER ⇒ 2.

Relative Motion

Statement ⇒ Expression for the magnitude and dir^n of relative velocity.

Proof ⇒



Take OX and OY as the axes of reference. Consider two particles P and Q moving in a plane with velocity u and v and making angle α and β with the axis respectively. Let V be the velocity of P relative to Q making an angle θ with the x-axis.

Actual velocity of P along OX = $u\cos\alpha$.
Actual velocity of P along OY = $u\sin\alpha$.
Reserved velocity of Q along OX = $-v\cos\beta$.
Reserved velocity of Q along OY = $-v\sin\beta$.

Now, the velocity of P relative to Q is the resultant of the velocity of P and the reserved velocity of Q.

Taking resolved parts along x-axis and y-axis.

$$V\cos\theta = u\cos\alpha - v\cos\beta. \quad -(1)$$

and $$V\sin\theta = u\sin\alpha - v\sin\beta. \quad -(2)$$

Squaring and adding (1) & (2) eq^n, we get

$$V^2(\cos^2\theta + \sin^2\theta) = (u\cos\alpha - v\cos\beta)^2 + (u\sin\alpha - v\sin\beta)^2$$

$$V^2 = u^2\cos^2\alpha + v^2\cos^2\beta - 2uv\cos\alpha\cdot\cos\beta + u^2\sin^2\alpha + v^2\cos^2\beta - 2uv\cos\alpha\cdot\sin\beta.$$

$$V^2 = u^2 + v^2 - 2uv(\cos\alpha\cos\beta + \sin\alpha\cdot\sin\beta)$$

$$V^2 = u^2 + v^2 - 2uv\cos(\alpha-\beta).$$

$$V = \sqrt{u^2 + v^2 - 2uv\cos(\alpha-\beta)} \quad ---(3).$$

Dividing (2) by (1), we get

$$\frac{V\sin\theta}{V\cos\theta} = \frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - u\cos\beta}.$$

$$\tan\theta = \frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta}.$$

Relative acceleration ⇒

Let P and Q be the two points moving with velocities u and v respectively. If V is the relative velocity of P with respect to Q, then.

$$V = u - v.$$

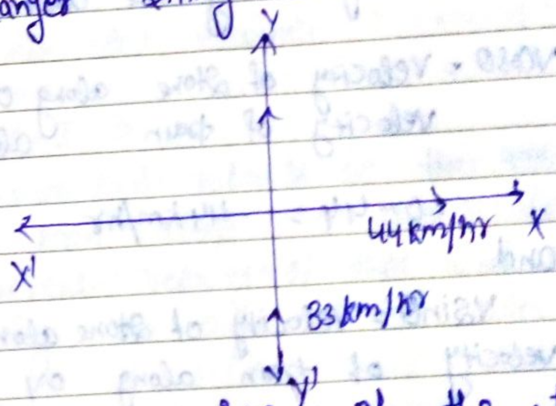
Diff w.r. to time

$$\frac{dV}{dt} = \frac{du}{dt} - \frac{dv}{dt}$$

∴ Acceleration of Point P relative to Q is obtained by compounding the actual acceleration of P with reversed acceleration of Q.

Question No ⇒ 1 A Train is moving at a speed of 44 km/hr. A stone strikes it at right angles with a speed of 33 km/hr. Find the magnitude and the direction of the velocity of the stone with which it appears to strike the passanger sitting in the train.

Sol^n ⇒



Let the velocities of the train and the stone be represent along X'OX and Y'OY respectively.

Now it is req. to find the velocity of stone relative to the train.

Now, actual velocity of the train along OX = 44 km/hr.

actual velocity of train along OY = 0.

Reversed velocity of train along OX = -44 km/hr.

Reversed velocity of train along OY = 0

Actual velocity of the Stone along OY = 33

Actual velocity of Stone along OX = 0

∴ Resolving along the axes

$V\cos\theta$ = Velocity of Stone along OX + Reversed velocity of train along OX.

$= 0 - 44 = -44$ km/hr.

and

$V\sin\theta$ = Velocity of Stone along OY + Reversed velocity of train along OY

$= 33 + 0 = 33$ km/hr

Squaring and adding ① and ②, we have

$$V^2 = (-44)^2 + (33)^2$$

$$= (11\times4)^2 + (11\times3)^2$$

$$= 121 \times 25$$

$$V = 11\times5 = 55 \text{ km/hr.}$$

Dividing (2) by (1), we have

$$\tan\theta = \frac{33}{-44} = \frac{-3}{4}$$

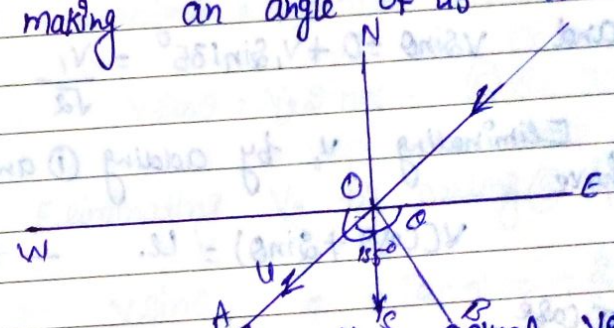
$$\theta = \pi - \tan^{-1}\frac{3}{4}.$$

Question No → 2

To a Person travelling due east, the wind appears to come from the north east, but when he doubles his speed, it appears to come from a direction $\tan^{-1}\frac{2}{3}$ north to east. Find the direction of wind.

Solⁿ ⇒

Case-I ⇒ let u be the velocity of the man along OE and V_1 be the apperent velocity of the wind along OA making an angle of 45° with OS.



let u be the actual velocity of the wind making an angle θ with the South of east. Then u is the resultant of actual velocity of the man and relative velocity of the wind.

Actual velocity of wind along OE ...

actual velocity of man along OE = u

apparent velocity of wind along OE = $V_1 \cos 135^\circ$

actual velocity of man along OE = $V\sin\theta$

Actual velocity of man along OS = 0.

Apparent velocity of wind along OS = $V_1 \sin$ ...

Resolving velocities along OE and along OS, we have

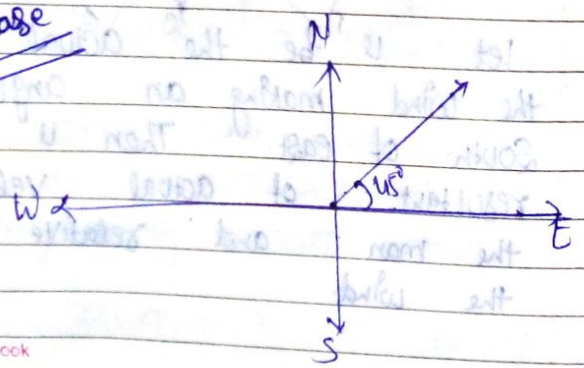
$$V\cos\theta = u + V_1 \cos 135^\circ = u - \frac{V_1}{\sqrt{2}} \quad -(1)$$

and $$V\sin\theta = 0 + V_1 \sin 135^\circ = \frac{V_1}{\sqrt{2}} \quad -(2)$$

Eliminating v_1 by adding (1) and (2), we have

$$V(\cos\theta + \sin\theta) = u. \quad -(3)$$

II case



In this case, velocity of man along OE = 2u. Let V_2 be the apparent velocity of wind along OA', making an angle α with OW such that

$$\tan\alpha = \frac{2}{3}.$$

Resolving velocities along OE and along OS, we get

$$V\cos\theta = 2u + V_2\cos(180^\circ - \alpha).$$

$$= 2u - V_2\cos\alpha.$$

i.e, $$V\cos\theta - 2u = -V_2\cos\alpha. \quad \cdots (4)$$

and

$$V\sin\theta = V_2\sin(180 - \alpha) + 0$$

$$V\sin\theta = V_2\sin\alpha \quad \cdots (5)$$

Eliminating V_2 by adding (5) by (4), we

$$\frac{V\sin\theta}{V\cos\theta - 2u} = -\tan\alpha = -\frac{2}{3}.$$

$$\Rightarrow V(3\sin\theta + 2\cos\theta) = 4u \quad -(6)$$

Dividing (3) by (6), we get

$$\frac{\cos\theta + \sin\theta}{3\sin\theta + 2\cos\theta} = \frac{1}{4}$$

$$4\cos\theta + 4\sin\theta = 3\sin\theta + 2\cos\theta$$

$$\sin\theta = -2\cos\theta$$

$$\tan\theta = -2 \Rightarrow -\tan\theta = 2$$

$$\tan(\pi - \theta) = 2 \Rightarrow \pi - \theta = \tan^{-1} 2$$

$$\theta = \pi - \tan^{-1} 2$$

CHAPTER ⇒ 3

SIMPLE HARMONIC Motion

S.H.M ⇒ A particle is said to execute Simple Harmonic motion (S.H.M) if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.

⇒ To OBTain the expression for veloc[ity] and Position of a particle executing Simple Harmonic motion ⇒

Statement ⇒ A particle moves in a straight line such that its acceleration is always directed toward a fixed point in the line and is proportional to the distance of the particle from the fixed point. discuss the motion.

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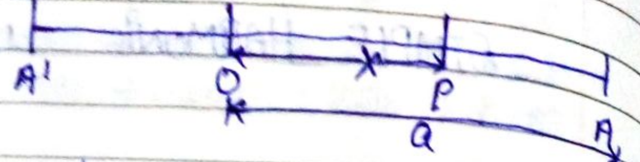
Proof ⇒



let O be the fixed Point in the line A'OA and P be the position of the particle moving with velocity u in the direction OA at time t and at a distance at x from O.

By the defination of S.H.M., the acceleration of the particle at P is Proportional to x and is directed towards O.

$$\therefore\ u\frac{dv}{dx} = -\mu x \quad \text{--- (1).}$$

Integration equ^n (1) w.r to x, we have

$$\frac{v^2}{2} = \frac{-\mu x^2}{2} + c \quad \text{--- (2).}$$

$\therefore$ At A, $u=0$, $x=a$.

$$0 = \frac{-\mu a^2}{2} + c \Rightarrow c = \frac{\mu a^2}{2}.$$

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Then from (2), we get

$$\frac{u^2}{2} = \frac{-\mu x^2}{2} + \frac{\mu a^2}{2}$$

$$u^2 = \mu(a^2 - x^2).$$

$$v = \pm\sqrt{\mu(a^2-x^2)}$$

$$\frac{dx}{dt} = +\sqrt{\mu(a^2-x^2)}$$

I case when time is measured from the fixed point ⇒

$$\frac{dx}{dt} = \sqrt{\mu}\cdot\sqrt{a^2-x^2}.$$

or $$\int\frac{1}{\sqrt{a^2-x^2}}\,dx = \int\sqrt{\mu}\,dt.$$

$$\sin^{-1}\frac{x}{a} = \sqrt{\mu}t + c_1 \quad \text{- (3).}$$

At $t=0$, $x=0$.

$$\sin^{-1}(0) = 0 + c_1 \Rightarrow c_1 = 0.$$

from equ^n (3), we have.

$$\sin^{-1}\frac{x}{a} = \sqrt{\mu}t + 0.$$

$$\frac{x}{a} = \sin\sqrt{\mu}t$$

$$x = a\sin\sqrt{\mu}t.$$

II case

When the time is measured from the instant when particle is at A ⇒

$$\frac{dx}{dt} = -\sqrt{\mu}\cdot\sqrt{a^2-x^2}$$

$$\frac{-dx}{\sqrt{a^2-x^2}} = \sqrt{\mu}dt.$$

Integration

$$\cos^{-1}\frac{x}{a} = \sqrt{\mu}t + C_2$$

at $x=a$, $t=0$ then $C_2=0$.

$$\cos^{-1}\frac{x}{a} = \sqrt{\mu}t.$$

$$x = a\cos\sqrt{\mu}t..$$

Maximum velocity ⇒ $\sqrt{\mu}\cdot\sqrt{a^2-x^2}$.

Minimum velocity = $\sqrt{\mu}\cdot a$.

Amplitude of S.H.M ⇒

The maximum distance between the centre of attraction O and the position of instantaneous rest A or A' is called Amplitude.

Frequency ⇒

frequency of a simple harmonic motion is the number of complete oscillations made in one sec.

Time of oscillations ⇒

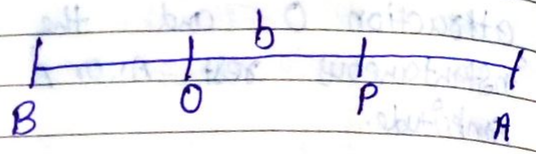
$$T = \frac{\sqrt{\mu}}{2\pi}.$$

Question No ⇒ 1 ⇒

A particle is performing S.H.M of period T about a centre O and it passes through the position P(OP=b) with velocity u in the dirⁿ OP. Prove that the time elapses before it returns to P is

$$\frac{T}{\pi}\tan^{-1}\frac{vt}{2\pi b}.$$

Sol^n ⇒ Time period $= T = \frac{2\pi}{\sqrt{\mu}}$ $\sqrt{\mu} = \frac{2\pi}{T}$.



Time period from P to A and back = T[illegible]
the time from A to p.
for $x = OP = b$, let t_1 be the time from A to p.

$$b = a\cos\sqrt{\mu}t_1 \qquad -②$$

also $u^2 = \mu(a^2 - b^2)$

$$= \mu(a^2 - a^2\cos^2\sqrt{\mu}t_1)$$

$$= \mu a^2\sin^2\sqrt{\mu}t_1$$

$$v = \mu a\sin\sqrt{\mu}t_1 \qquad -③$$

Dividing ③ by ②, we have

$$\frac{v}{b} = \sqrt{\mu}\tan\sqrt{\mu}t_1$$

or $\tan\sqrt{\mu}t_1 = \frac{v}{\sqrt{\mu}b}$

$$t_1 = \frac{1}{\sqrt{\mu}}\tan^{-1}\frac{v}{\sqrt{\mu}\cdot b}$$

$$t_1 = \frac{T}{2\pi}\tan^{-1}\frac{vT}{2\pi b}.$$

∴ Time from P to A and Back $= 2t_1$

$$= \frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi b}.$$

Question No⇒2

The maximum velocity of a body moving with S.H.M. is 2 unit/sec. and its period is $\frac{1}{5}$ sec what is the amplitude?

Sol^n ⇒ Time period $T = \frac{1}{5}$ sec.

$$\frac{2\pi}{\sqrt{\mu}} = \frac{1}{5} \qquad = \sqrt{\mu} = 10\pi.$$

maximum velocity $= \sqrt{\mu}\cdot a$.

$$2 = 10\pi\cdot a$$

$$a = \frac{2}{5\cdot 10\pi} = \frac{1}{5\pi}\ \text{unit/sec}^2.$$

CHAPTER ⇒ 4th.

ELASTIC STRINGS

Hooke's Law ⇒

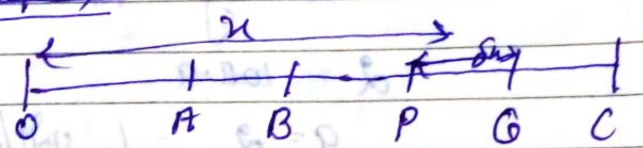
Tension in an elastic string is proportional to the extension produced beyond its natural length is called Hooke's law.

$$T = \frac{\lambda \cdot x}{l}.$$

Statement ⇒

Prove that the work done against the tension in stretching a light elastic string is equal to the product of its extension and the mean of the initial and final tension.

Proof ⇒



Let $OA = a$ be the natural length of the string whose one end is fixed at O.

let the string be stretched such that B and C be the two position of the free end A of the string during extension. let $OB = b$ and $OC = c$.

let λ be the modulus of elasticity of the string. Then by Hooke's law

$$\text{Tension at } B = \frac{\lambda(b-a)}{a}.$$

$$\text{Tension at } C = \frac{\lambda(c-a)}{a}.$$

let P be any position of the free end of the string during this extension. suppose that the free end of the string is slightly stretched from P to Q so that $PQ = \delta x$ where $OP = x$ and $OQ = x + \delta x$.

$$\text{Tension at } P = \frac{\lambda(x-a)}{a}.$$

The work done against tension in stretching the string from P to Q

$$(\text{Tension at } P)\delta x = \frac{\lambda(x-a)}{a}\cdot\delta x.$$

$\therefore$ Work done from B to C

$$\int_b^c \frac{\lambda}{a}(x-a)\,dx = \frac{\lambda}{2a}\Big[(x-a)^2\Big]_b^c$$

$$= \frac{\lambda}{2a}\Big[(c-a)^2 - (b-a)^2\Big].$$

$$= \frac{\lambda}{2a}\Big[(c-a-b+a)(c-a+b-a)\Big]$$

$$= \frac{\lambda}{2a}\Big[(c-b)\cdot(c-a+b-a)$$

$$= \frac{c-b}{2}\Big[\frac{\lambda}{a}(c-a) + \frac{\lambda}{a}(b-a)\Big].$$

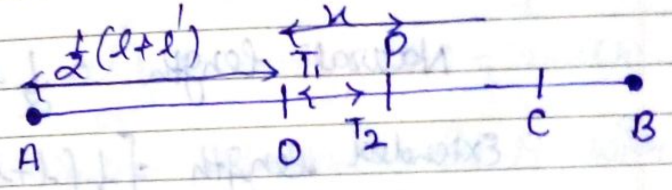
$$= (c-b)\cdot\frac{1}{2}\Big[\text{Tension at } C + \text{Tension at } B\Big].$$

Question No $\Rightarrow$ 1

A particle is attached to the middle point of a uniform elastic string which is stretched b/w two points A and B on a smooth horizontal table. If the particle is pulled to some point C b/w A and B and then librated, show that the motion is simple harmonic and find the period of motion. What is the amplitude?

Sol^n $\Rightarrow$

Let the elastic string of natural length l be stretched to the length $l+l'$ so that $AB = l+l'$ where A and B are the point where the ends are fixed.



Let λ be the modulus of elasticity, O be the middle point so that

$$AO = OB = \frac{1}{2}(l+l').$$

Let P be the position of particle at time t, such that $OP = x$.

(i) Tension T_1 of the portion AP of the string in the dirⁿ PA ⇒

Here the natural length $= \frac{1}{2}(l+l')$

Extended length $= AP = \frac{1}{2}(l+l') + x$.

Extension ⇒ $\left[\frac{1}{2}(l+l') + x\right] - \frac{1}{2}l$

$= \frac{1}{2}l' + x$.

(ii) Tension T_2 at PB ⇒

Natural length $= \frac{1}{2}l$.

Extended length $= \left[\frac{1}{2}(l+l') - x\right]$

∴ Extension $= \left[\frac{1}{2}(l+l') - x\right] - \frac{1}{2}l$

$= \frac{1}{2}l' - x$.

By Hooke's Law ⇒

$$T_1 = \lambda \frac{\frac{1}{2}l' + x}{\frac{1}{2}l} = \lambda\left(\frac{l' + 2x}{l}\right).$$

Similarly

$$T_2 = \frac{\lambda(l' - 2x)}{l}.$$

By the eqⁿ of motion for the mass m is

$$m\frac{d^2x}{dt^2} = T_2 - T_1 = \lambda\cdot\frac{l' - 2x}{l} - \lambda\frac{l' + 2x}{l}$$

$$= \frac{-4\lambda}{l}\cdot x.$$

$$\frac{d^2x}{dt^2} = \frac{-4\lambda}{lm}\cdot x = -\mu x.$$

which represent S.H.M with centre O and $\mu = \frac{4\lambda}{lm}$.

∴ Period of motion $T = \frac{2\pi}{\sqrt{\mu}}$

$$= \frac{2\pi}{\sqrt{\frac{4\lambda}{lm}}} = \pi\sqrt{\frac{lm}{\lambda}}.$$

Amplitude = OC.

Question No => 2

If t_1 and t_2 be the [illegible] corresponding to two diff. attached weights and C_1, C_2 the Statical ex[illegible] due to these weights, Prove th[illegible]

$$g = \frac{4\pi^2 (C_1 - C_2)}{t_1^2 - t_2^2}.$$

Sol^n => We Know that Periodic time

$$= 2\pi \sqrt{\frac{e}{g}},$$

Here, when time Period = T_1

$$e = C_1$$

When time Period = T_2

$$e = C_2.$$

$$t_1 = 2\pi \sqrt{\frac{C_1}{g}} \quad , \quad t_2 = 2\pi \sqrt{\frac{C_2}{g}}$$

$$t_1^2 - t_2^2 = \frac{4\pi^2}{g}(C_1 - C_2).$$

$$\boxed{g = \frac{4\pi^2 (C_1 - C_2)}{t_1^2 - t_2^2}}.$$

5th CHAPTER

Newton's Law of motions

Mass ⇒

Mass of a body is t Quantity of matter it contains.

Unit's of mass ⇒

M.K.S. System ⇒ Kilogram (kg)
C.G.S System ⇒ gramme (g)
F.P.S System ⇒ Pound (Pon)

Momentum ⇒

momentum of a body is the quantity of motion possessed by a body.

momentum = mass × velocity.

Impressed force ⇒

The external force acting upon a body are called the impressed force.

Newton first Law ⇒

Every body continues in its state of rest, or of uniform motion in a straight line unless it is compelled by impressed force to change that state is called Newton's first law.

2nd law ⇒

$P = mf$.

3rd Law ⇒

To every rea^n equal and opposite re^n.

Weight ⇒

weight of a body is the force with which the body attracted towards the centre of the earth.

$W = m.g$.

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Question No ⇒ 1

Find the uniform force will move 1 kg mass from rest 1 meter in 1 second.

Sol^n ⇒ Mass = 1 kg.

Initial Velocity $u = 0$.

displacement $S = 1$ m.

Time = 1 sec.

using relation.

$$S = ut + \frac{1}{2}at^2$$

$$1 = 0.1 + \frac{1}{2}.a \times 1$$

$$a = 2 m/sec^2$$

If P be the Impressed force then

$$P = ma.$$

$$P = 2 \times 1 = 2 \text{ Newtons.}$$

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Question No ⇒ 2.

A Train whose mass is 20 tons moves at the rate of 60 mph after steam off, it is brought to rest by breaks in 500 yds. find the retardation force, assuming it to be uniform.

Sol^n ⇒ Initial velocity = 60 M.P.h = $\frac{60 \times 22}{15}$ = 88 ft/sec

final velocity = 0.

Distance travelled = 500 yd = 1500 ft.

using $v^2 - u^2 = 2fs$

$$0 - (88)^2 = 2f \times 1500 \Rightarrow f = -\frac{968}{375} ft/sec^2$$

∴ Retardation (r) = $\frac{968}{375}$ ft/sec^2

Given mass = 20 tons = 20 × 2240 lbs

∴ Retarding force = mr

$$= \left(20 \times 2240 \times \frac{968}{375}\right) \text{Poundals}$$

$= \left(20 \times 2240 \times \frac{968}{375} \times \frac{1}{32}\right)$ lbs wt.

$= \left(20 \times 2240 \times \frac{968}{375} \times \frac{1}{32} \times \frac{1}{2240}\right)$ tons wt

$= 1\frac{46}{75}$ tons wt.

When the horizontal plane is moving vertically upwards with acceleration

$R = m(g+f).$

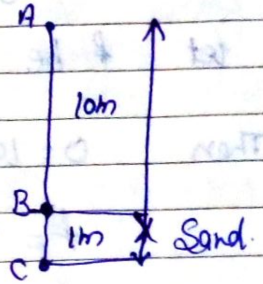
When the horizontal Plane is moving vertically downwards with acceleration

$R' = m(g-f).$

Question No→3

A mass of 10kg falls from a distance of 10m from rest and is then brought to rest after penetrating through 1m in sand. find the average force exerted by the sand on it.

Solⁿ⇒ (i) Motion from rest to the ground:



let v be the velocity with which the mass strikes the ground after falling. 10m from A to B.

Here Initial velocity, $u=0$. $h=10m$,

$g = 9.8 m/sec^2$.

using $v^2 = u^2 + 2gh$

$v^2 = 0 + 2 \times 9.8 \times 10 = 196$

$v = \sqrt{196} = 14 m/sec.$

(ii) motion from ground to sand ⇒

when the body penetrates 1m into the sand, the velocity becomes zero.

Initial velocity, $u = 14m/sec$.

final velocity, $v=0$ and $S=$ 1m

let f be the acceleration

Then $0 = 196 + 2f \times 1$

$\therefore f = -98 m/sec^2$

let R Newton be the average thrust of the sand on the m acting upwards

Then effective force downwards $= (10g - R)$

$10g - R = 10f$

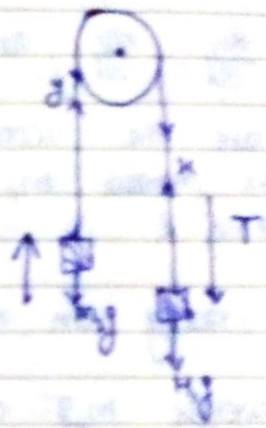
$R = 10(g - f)$

$R = 10 \times 9.8 + 10 \times 98$

$= 1078$ Newtons $\Rightarrow \frac{1078}{9.8} = 110$ kg wt

⇒ Two masses m_1 and m_2 $(m_1 > m_2)$ are suspended by a light inextensible and flexible string which passes over a smooth fixed and light pulley. To find the motion of the system, the tension in the string and the pressure on the pulley

Sol^n ⇒



let l be the length of the whole string on both sides of the pulley and x, y be the depths of masses m_1 and m_2 respectively below the pulley at any time t

Then $x + y = l$ ①

Diff w.r.t. time, we have

$\frac{dx}{dt} + \frac{dy}{dt} = 0$

$\frac{dx}{dt} = \frac{-dy}{dt}$

again D. w. r to time, we have

$$\frac{d^2x}{dt^2} = \frac{-d^2y}{dt^2}.$$

But $\frac{dx}{dt}$, $\frac{dy}{dt}$ are the velocity $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ are the acceleration components of the masses m_1 and m_2 respectively at time t.

Force acting on mass m_1 are:

(i) weight $m_1 g$ downwards.

(ii) Tension T upwards.

Equation of motion of mass m_1 in downward dirn is

$$m_1 f = m_1 g - T. \quad -(3)$$

Similarly, Equn of motion of mass m_2 in upward dirn is

$$m_2 f = T - m_2 g. \quad -(4)$$

adding (3) and (4), we have

$$(m_1 + m_2) f = (m_1 - m_2) g$$

$$f = \frac{m_1 - m_2}{m_1 + m_2} \cdot g \quad -(5).$$

Put the value of f in (3) or (4), we get

$$T = \frac{2m_1 m_2}{m_1 + m_2} \cdot g.$$

Pressure on the Pulley:

$$R = 2T = \frac{4m_1 m_2}{m_1 + m_2} \cdot g$$

Question No→4

A mass of 4kg is drawn across a rough horizontal table by means of a string passing over a smooth pulley fixed at an edge of the table and supports a mass of 2kg at the other end. If the coefficient of friction be 5/16, find the acceleration of the masses.

Sol^n ⇒ let $m_1 = 2kg$, $m_2 = 4kg$

$\mu = 5/16$

using

$$f = \frac{m_1 - \mu m_2}{m_1 + m_2}$$

$$= \frac{2 - \frac{4 \times 5}{16}}{2 + 4} = \frac{1}{8} g \, m/s^2$$

6th CHAPTER

Work, Power and Energy

Work ⇒

A force acting on a body is said to do work, when its point of application moves through some distance along the line of action of the force.

$$W = F \times S$$

Unit of Work ⇒

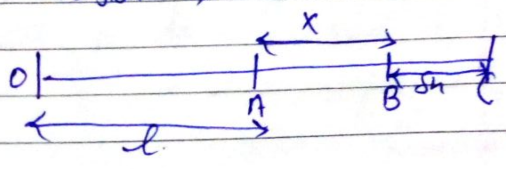
In F.P.S System = foot - Poundal.

In C.G.S System = Erg.

In M.K.S. System = Joule.

⇒ State → To prove that the work done in stretching an elastic string is equal to the produced of the extension and the mean of Initial and final tension ⇒

Proof ⇒



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let one end of the elastic string with modulus λ be fixed. let OA be its natural length and AB = x be the extension produced. Then by Hooke's law

$$T = \lambda \cdot \frac{x}{l}$$

Assuming this to be constant for a further small extension BC = δx, small work done is

$$\delta W = T \cdot \delta x = \frac{\lambda}{l} \cdot x \cdot \delta x$$

$\therefore$ W, the total work done in stretching the elastic string from its natural length $l+x$ is given by

$$W = \int_0^x \lambda \cdot \frac{x}{l}\, dx = \frac{1}{2} \frac{\lambda x^2}{l} \quad (1)$$

Now, from (1), work done for an extension $x_1 = \frac{1}{2} \frac{\lambda x_1^2}{l}$

Work done for extension $x_2 = \frac{1}{2} \frac{\lambda x_2^2}{l}$

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Hence, the work done in increasing the extension from x_1 to x_2.

$$= \frac{1}{2} \frac{\lambda x_2^2}{l} - \frac{1}{2} \frac{\lambda x_1^2}{l}$$

$$= \frac{\lambda}{2l} (x_2 + x_1)(x_2 - x_1)$$

$$\Rightarrow (x_2 - x_1) \cdot \frac{1}{2} \left(\frac{\lambda x_1}{l} + \frac{\lambda x_2}{l} \right).$$

Here extension $= x_2 - x_1$

T_1, Initial tension $= \frac{\lambda x_1}{l}$.

T_2, final tension $= \frac{\lambda x_2}{l}$.

Hence the result.

Question $\Rightarrow$ 1.

Show that the work done in stretching an elastic string of natural length l and modulus λ from tension T_1 to tension T_2 is $\frac{1}{2\lambda}(T_2^2 - T_1^2)$.

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Sol^n ⇒ let l_1 and l_2 be the extension when the tension are T_1 and T_2 respectively. Then

$$T_1 = \frac{\lambda . l_1}{l} \quad \text{and} \quad T_2 = \frac{\lambda l_2}{l}$$

Mean of tensions $= \frac{1}{2}[T_2 + T_1]$

∴ Work done in stretching the string

= Extension × mean of Tensions

$$= (l_2 - l_1) \cdot \frac{1}{2}(T_2 + T_1)$$

$$= \frac{1}{2}(T_2 + T_1) \cdot \left(\frac{lT_2}{\lambda} - \frac{lT_1}{\lambda}\right].$$

$$= \frac{l}{2\lambda}[T_2^2 - T_1^2].$$

Power ⇒ The rate of doing work is called Power.

Units of Power ⇒

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In F.P.S. system ⇒ one foot Poundal/sec.

In C.G.S. system ⇒ one erg/sec.

In. S.I. system ⇒ one watt pr Joule/sec.

One. H.P = 746 watts approx.

__Question No⇒2__

An engine of horse - Power H, draws a train of mass M tons up an Incline 1 in n against a resistance of m lbs wt. per ton. Show that maximum speed of the train is

$$\frac{550\, Hn}{M(2240 + mn)} \text{ ft/sec.}$$

Sol^n ⇒ Mass of the train = M tons.

Resistance R = M m lbs wt.

Gravitational resistance = 2240 M sin α. lbs. wt

$$= M \times 2240 \times \frac{1}{n} = \frac{2240 M}{n} \text{ lbs. wt.}$$

Since the train is moving at its

Maximum speed.

∴ force exerted by engine = Total resistance
$= R + G$.

$= Mm + \frac{2240M}{n} = \frac{M}{n}(mn + 2240)$ lbs

let v ft/sec. be the maximum speed

$H.P. = \frac{F.v}{550}$

$H = \frac{Mv}{550n}(mn + 2240)$

$\Rightarrow v = \frac{550Hn}{M(2240 + mn)}$ ft/sec.

Energy ⇒
The capacity of doing work is called energy.

Type of energy ⇒
It is two type –

(i) Kinetic Energy ⇒ $\frac{1}{2}Mv^2$

(ii) Potential Energy. = Mgh.

Question No ⇒ 3
A stone weighing 3kg has fallen through 10 meters. What is the K.E. Now and what force will stop it in 2 meters?

Sol^n ⇒
let u be the velocity acquired by the stone in falling through 10m.

$u^2 = 2gh = 2 \times 9.8 \times 10 = 196$.

$v = 14m/sec$.

$K.E = \frac{1}{2}mv^2 = \frac{1}{2} \times 3 \times 196 = 294J$.

let P kg. wt. force be req. to stop the stone in 2 meters.

Then the resultant force which reduce it to rest in 2m.
$(P - 3)$ kg wt.

Work done $= -(P-3)2g$ J

By principle of work and energy

work done = Change in K.E.

$$-(P-3)2g = 0 - \frac{1}{2}(8)(196).$$

$$(P-3) = \frac{294}{2\times 9.8} = 15$$

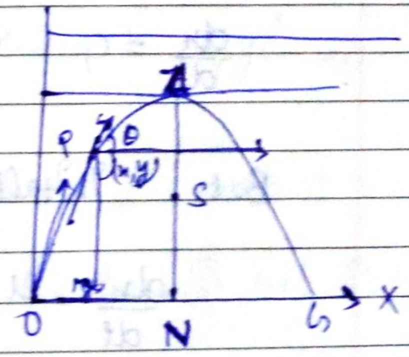
$P = 18$ kg. wt.

CHAPTER ⇒ 8th

PROJECTILES

State ⇒ A particle of mass 'm' is projected in a vertical plane through the point of projection with velocity 'u' in a dir^n making an angle α with the horizontal, to find its motion and path described.

Proof ⇒



let O be the the point of projection as the origin and the horizontal and vertical line OX and OY through O as x- axis and y-axis respectively.

let $P(x,y)$ be the position of particle at time t.

During the motion,

$$m\frac{d^2x}{dt^2} = 0 \Rightarrow \frac{d^2x}{dt^2} = 0 \quad -(1)$$

and $$m\frac{d^2y}{dt^2} = -mg \text{ or } \frac{d^2y}{dt^2} = -g \quad -(2)$$

Integration (1) and (2), we get

$$\frac{dx}{dt} = c_1 \quad \text{and} \quad \frac{dy}{dt} = -gt + c_2$$

But initially, when $t=0$

$$\frac{dx}{dt} = u\cos\alpha = c_1$$

$$\frac{dy}{dt} = u\sin\alpha = c_2.$$

$$\frac{dx}{dt} = u\cos\alpha \quad -(3)$$

and $$\frac{dy}{dt} = u\sin\alpha - gt \quad -(4)$$

Integration again (3) and (4), we get

$$x = u\cos\alpha t + c_3$$

$$y = u\sin\alpha t - \frac{1}{2}gt^2 + c_4$$

Applying Initial conditions

$t=0$, $x=0$ and $y=0$, $c_3=0$ and $c_4=0$.

$$y = u\sin\alpha t - \frac{1}{2}gt^2 \quad -(6)$$

$$x = u\cos\alpha t \quad -(5)$$

Eliminating (t) from (5) and (6), we

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}.$$

Time of flight $\Rightarrow$ The time of flight is the time that elapses from the instant of projection till the particle reaches the horizontal plane through the point of projection.

$$T = \frac{2u\sin\alpha}{g}$$

Horizontal range ⇒ Horizontal range is the distance b/w the point of projection and point where the particle strike the horizontal plane.

$$= \frac{u^2\sin 2\alpha}{g}$$

Maximum Horizontal Range ⇒ $\frac{u^2}{g}$.

Greatest height ⇒

$$H = \frac{u^2\sin^2\alpha}{2g}$$

Question No ⇒ 1 A particle is projected with a velocity 49m/sec. In a dir^n making an angle of 45° with the horizontal. find (i) time of flight (ii) horizontal range (iii) greatest height.

Sol^n ⇒

$u = 49 m/sec$. $\alpha = 45°$

(i) Time of flight $= \frac{2u\sin\alpha}{g}$

$$= \frac{2 \times 49 \sin 45°}{9.8}$$

$$= \frac{2 \times 49 \times 1}{9.8 \times \sqrt{2}} = 5\sqrt{2} \text{ sec}$$

(ii) Horizontal range $= \frac{u^2\sin 2\alpha}{g}$

$$= \frac{(49)^2 \cdot \sin 90}{9.8} = \frac{49 \times 49 \times 1}{9.8} = 245m.$$

(iii) Greatest height $= \frac{u^2 \sin^2\alpha}{2g}$

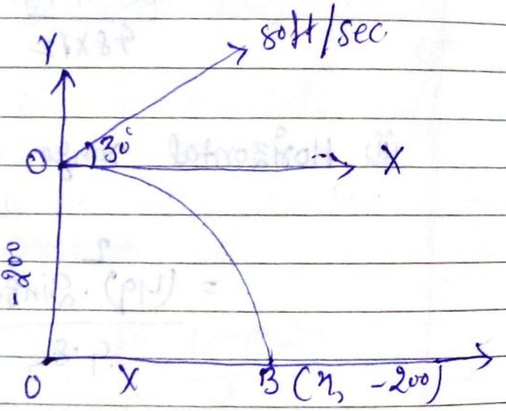
$= \frac{(49)^2 \cdot \sin^2 45^\circ}{2 \times 9.8}$

$= \frac{49 \times 49 \times 1}{2 \times 9.8 \times 2} = 61.25\,m.$

Question No ⇒ 2

A ball is thrown from the top of a tower 200ft height with velocity of 80ft/sec at any elevation of 30° above the horizon. find the horizontal distance from the foot of the tower of the point where it hits the ground.

Soln ⇒



let the point of projection O be taken as origin and B be the point where stone strike the ground.

$u = 80ft/sec. \quad \alpha = 30°$

Co-ordinate of B are $(x, -200)$.

using $y = u\sin\alpha t - \frac{1}{2}gt^2$

$= -200 = 80\sin 30° t - 16t^2$

$= 2t^2 - 5t - 25 = 0.$

$(t-5)(2t+5) = 0.$

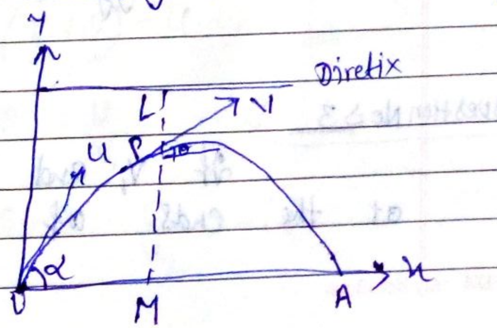
$t = 5.$

Horizontal distance $= u\cos\alpha t$

$= 80 \cdot \cos 30° (5)$

$= 200\sqrt{3}$ ft.

→ Velocity At any point of the trajectory ⇒



We have,

$$v\cos\theta = u\cos\alpha \quad \text{--- (1)}$$

and $v\sin\theta = u\sin\alpha - gt$ --- (2)

Dividing (2) by (1), we get

$$\tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha}$$

which gives dir^n of velocity

again sq and Adding (1) & (2), we get

$$v^2 = u^2\cos^2\alpha + (u\sin\alpha - gt)^2$$

$$= u^2 - 2ugt\sin\alpha + g^2t^2$$

$$= u^2 - 2g\left(u\sin\alpha . t - \frac{1}{2}gt^2\right).$$

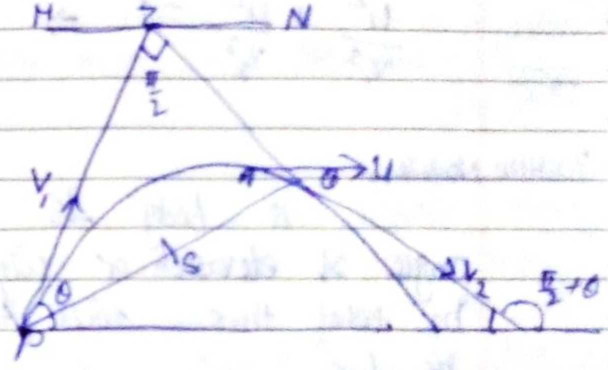
$$= u^2 - 2gy \quad (\because \text{ co-ordinate of } P)$$

Question No ⇒ 3

If v_1 and v_2 be the velocity at the ends of a focal chord Projectile's path and 'u' the horizontal component of the velocity, show that

$$\frac{1}{v_1^2} + \frac{1}{v_2^2} = \frac{1}{u^2}$$

Sol^n ⇒



let A be the vertex, S be the focus and MZN the directrix of the path of projectile. let PQ be the focal chord of the parabola.

Resolving horizontally, we have

$$v_1\cos\theta = u.$$

and $v_2\cos\left(\frac{\pi}{2}+\theta\right) = u.$

$$\therefore \cos\theta = \frac{u}{v_1} \quad \text{-- (1)}$$

and $-\sin\theta = \frac{u}{v_2}$ -- (2)

Squaring ① and ②, and adding, we have

$$\sin^2\theta + \cos^2\theta = \frac{u^2}{v_1^2} + \frac{u^2}{v_2^2}$$

$$\frac{u^2}{v_1^2} + \frac{u^2}{v_2^2} = 1 \Rightarrow \frac{1}{v_1^2} + \frac{1}{v_2^2} = \frac{1}{u^2}$$

Question

A body is projected at an angle of elevation α with a velocity 9.8 m/sec. In what time will the dir^n of motion be $\frac{\alpha}{2}$?

Sol^n ⇒ Here $u = 9.8$ m/sec

$$\theta = \frac{\alpha}{2}$$

Putting these value in

$$\tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha}$$, we have

$$\tan\frac{\alpha}{2} = \frac{9.8\sin\alpha - 9.8t}{9.8\cos\alpha}$$

$$\therefore \frac{t}{\cos\alpha} = \tan\alpha - \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\alpha\cos\frac{\alpha}{2}}$$

Hence $t = \tan\frac{\alpha}{2}$.

⇒ Maximum range down the plane $= \frac{u^2}{g(1-\sin\alpha)}$

Question No 25

Show that for a given velocity of projection the maximum range down a plane of inclination α is greater than up the plane in the ratio $\frac{1+\sin\alpha}{1-\sin\alpha}$.

Sol^n ⇒

let u be the velocity of projection. The maximum range up the inclined plane $= \frac{u^2}{g(1+\sin\alpha)}$ —①

and maximum range down the plane

$$= \frac{u^2}{g(1-\sin\alpha)}$$

$\therefore$ from (1) and (2), we get

$$= \frac{1+\sin r}{1-\sin r}.$$

9th CHAPTER
Central Orbits

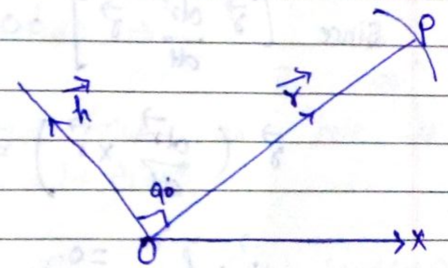
⇒ **Central force ⇒**

A force whose line of action always passes through a fixed point is called a central force.

⇒ **Central orbit ⇒**

The path described by a particle moving in a plane under the action of force, which is always directed towards a fixed point in the plane is called the central orbit.

⇒ **A central orbit is always a plane curve**



let O be the centre of central force. Take it as origin of vectors. let OX be a fixed straight line through O.

The acceleration vector $\frac{d^2\vec{r}}{dt^2}$ is parallel to $\vec{OP}$.

$$\frac{d^2 r}{dt^2} \times \vec{OP} = \vec{0}$$

$$\frac{d^2\vec{r}}{dt^2} \times \vec{r} = \vec{0}$$

$$\frac{d^2\vec{r}}{dt^2} \times \vec{r} + \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = \vec{0}$$

or $\frac{d}{dt}\left(\frac{d\vec{r}}{dt} \times \vec{r}\right) = \vec{0}$

$$\frac{d\vec{r}}{dt} \times \vec{r} = \text{constant vector } \vec{h} \text{ (say)}.$$

Since $\left[\vec{r}\ \frac{d\vec{r}}{dt}\ \vec{r}\right] = 0$

$$\vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \vec{r}\right) = 0.$$

$$\vec{r} \cdot \vec{h} = 0.$$

$\Rightarrow$ $\vec{r}$ is perpendicular to a constant vector $\vec{h}$.

⇒ **Areal velocity ⇒** When a particle moves along a plane curve, the rate of change of the sectorial area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle.

⇒ **ELLIPTIC orbit ⇒** A particle moves in an ellipse under a force which is always directed towards its focus; to find

(i) the law of force.

(ii) the velocity at any point of its path.

(iii) the periodic time.

Sol^n ⇒

(i) Law of force ⇒ The eq^n of ellipse with focus as pole is

$$\frac{l}{r} = 1 + e\cos\theta,$$

Taking $u = \frac{1}{r}$. $\Rightarrow u = \frac{1 + e\cos\theta}{l}$ —(1).

Diff. eq^n (1). w.r. to θ, we get

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta.$$

$$\frac{d^2u}{d\theta^2} = \frac{-e}{l}\cos\theta \quad -②.$$

The diff. equation of central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \quad -③.$$

Substituting the values from ① and ② in ③, we have

$$\frac{F}{h^2u^2} = \frac{-e\cos\theta}{l} + \frac{1+e\cos\theta}{l}$$

$$\frac{F}{h^2u^2} = \frac{1}{l} \Rightarrow F = \frac{h^2u^2}{l} \rightarrow \frac{\lambda}{r^2}$$

$$\left(\because \lambda = \frac{h^2}{l}\right)$$

(ii) velocity $\Rightarrow$

The velocity of central orbit is given by

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right].$$

$$= h^2\left[\left(\frac{-e\sin\theta}{l}\right)^2 + \left(\frac{1+e\cos\theta}{l}\right)^2\right].$$

$$= \frac{\lambda}{l}\left[1+e^2+2e\cos\theta\right].$$

$$= \lambda\left[2\left(\frac{1+e\cos\theta}{l}\right) - \frac{1-e^2}{l}\right]$$

$$= \lambda\left[2u - \frac{1-e^2}{l}\right]$$

$$= \lambda\left[2u - \frac{\frac{b^2}{a^2}}{\frac{b^2}{a}}\right]$$

$$u^2 = \lambda\left[\frac{2}{r} - \frac{1}{a}\right].$$

(iii) Periodic time $\Rightarrow$

In a central orbit, the rate of description of the sectional area is constant and is equal to $\frac{h}{2}$.

$$\frac{h}{2} \times T = \pi ab.$$

$$T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\lambda l}}$$

$$T = \frac{2\pi ab}{\sqrt{\frac{\lambda b^2}{a}}}.$$

Hence $T = \dfrac{2\pi a^{3/2}}{\sqrt{\lambda}}$.

Question No ⇒ 1 ⇒

A particle describes the equiangular spiral $r = ae^{\theta\cot\alpha}$ under a force to the pole. Find the law of force.

Solⁿ ⇒ The given eqⁿ of curve is

$$r = ae^{\theta\cot\alpha}$$

$$\frac{1}{u} = ae^{\theta\cot\alpha}$$

$$1 = aue^{\theta\cot\alpha} \quad \text{---(1)}$$

Diff (1) w.r. to θ, we get

$$0 = ae^{\theta\cot\alpha}\frac{du}{d\theta} + au\cot\alpha\cdot e^{\theta\cot\alpha}$$

$$0 = ae^{\theta\cot\alpha}\left[\frac{du}{d\theta} + au\cot\alpha\right].$$

$$\therefore \frac{du}{d\theta} + u\cot\alpha = 0. \quad (\because ae^{\theta\cot\alpha} \neq 0)$$

$$\frac{du}{d\theta} = -u\cot\alpha. \quad \text{---(2)}$$

Diff. eqⁿ (2) w.r. to θ, we get

$$\frac{d^2u}{d\theta^2} = -\frac{du}{d\theta}\cot\alpha.$$

or $$\frac{d^2u}{d\theta^2} = (u\cot\alpha)\cdot\cot\alpha.$$

$$\frac{d^2u}{d\theta^2} = u\cot^2\alpha.$$

The diff. eqⁿ of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

$$u\cot^2\alpha + u = \frac{F}{h^2u^2}.$$

$$u\operatorname{cosec}^2\alpha = \frac{F}{h^2u^2}$$

$$F = h^2u^3\operatorname{cosec}^2\alpha.$$

$$F = \frac{h^2\operatorname{cosec}^2\alpha}{r^3}$$

$$F \propto \frac{1}{r^3}$$

⇒ **Apse** ⇒

An apse is a point on a central orbit at which the radius vector drawn from the centre of force to this point has a maximum or minimum value.

Apsidal distance ⇒

The distance b/w an apse and the central of force is called the apsidal distance.

Apsidal angle ⇒

The distance b/w two consecutive apsidal distances is called an apsidal angle.

⇒ **Statement** ⇒

At an apse, the particle moves at right angle to the radius vector i.e at an apse the radius vector is perpendicular to the tangent.

Proof ⇒ From the definition of an apse, the apsidal distance r is maximum or minimum.

$\therefore u = \frac{1}{r}$

$\Rightarrow \frac{du}{d\theta} = 0.$

For any curve, we have $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

$\frac{1}{p^2} = u^2$

$\frac{1}{p^2} = \frac{1}{r^2}$

$p = r.$

$r\sin\phi = r.$

$\sin\phi = 1.$

$\phi = 90^\circ.$

Hence proved.

Question No ⇒ 2

A particle moves under with a central acceleration $\frac{\lambda}{(\text{distance})^3}$. find the path and distinguish cases.

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Sol^n ⇒ let F be the central acceleration.

Then

$$F = \frac{\lambda}{r^3} = \lambda u^3.$$

The diff. eq^n of the Path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{F}{u^2}.$$

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\lambda u^3}{u^2}.$$

$$\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\lambda u}{h^2}$$

or $$\frac{d^2u}{d\theta^2} = \frac{\lambda u}{h^2} - u.$$

or $$\frac{d^2u}{d\theta^2} = \left(\frac{\lambda}{h^2} - 1\right) u.$$

Case ⇒ 1

When $\frac{\lambda}{h^2} > 1$. i.e. $\frac{\lambda}{h^2} - 1 > 0$

let $\frac{\lambda}{h^2} - 1 = n^2$ (say).

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∴ Eq^n ① become $\frac{d^2u}{d\theta^2} = n^2 u$.

Auxillary eq^n is $D^2 - n^2 = 0$

$$D^2 = n^2 \quad D = \pm n.$$

Thus general Sol^n is

$$u = c_1 e^{n\theta} + c_2 e^{-n\theta}$$

$$u = A\cosh n\theta + B\sinh n\theta.$$

Case ⇒ II

When $\frac{\lambda}{h^2} = 1$. i.e. $\lambda = h^2$.

In this case, equation ① become $\frac{d^2u}{d\theta^2} = 0$.

$$\frac{du}{d\theta} = c_1 \quad \text{i.e.} \quad du = c_1 d\theta.$$

again Integration

$$u = c_1\theta + c_2.$$

The general Sol^n is

$$u = c_1\theta + c_2$$

$$u = c_1(\theta - k)$$

Case ⇒ III

When $\frac{\lambda}{h^2} < 1$ i.e. $\frac{\lambda}{h^2} - 1 < 0$.

let $\frac{\lambda}{h^2} - 1 = -n^2$ (say).

In this case, eqn ① becomes

$\frac{d^2u}{d\theta^2} = -n^2u$, where $n^2 > 0$

$$\frac{d^2u}{d\theta^2} + n^2u = 0.$$

The Auxiliary eqn is $D^2 + n^2 = 0$

$D^2 = -n^2$ i.e. $D = \pm in$

Thus the soln of ② is $u = c_1 \cos(n\theta + c_2)$ ⇒

$$u = c_1 \cos n(\theta - \alpha).$$

The apse is given by

$$\theta = \alpha, \quad u = c_1.$$

10th CHAPTER

KEPLER'S LAW OF PLANETARY Motion.

These are three law's of Plantary motion:-

(i) The squares of the periodic times of the vectors planets are proportional to the cubes of the major axes of their orbits.

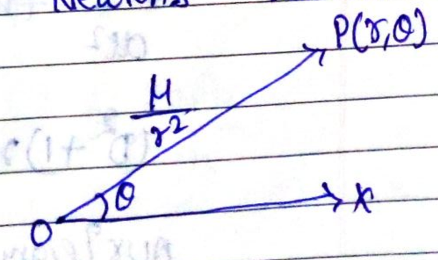
(ii) Each planet describes an ellipse having the sun as its focus.

(iii) The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.

Statements ⇒

To establish the equivalence of kepler's law of Planetary motion and Newton's law of gravitation.

Proof ⇒



Take, Sun the fixed particle in Space and its centre as the origin. Let $P(r,\theta)$ be the position of the Planet at time t.

$$F = \frac{\mu}{r^2}$$

The diff. equ^n of the central orbit

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^2}{h^2u^2}$$

$$\frac{d^2u}{d\theta^2} - \frac{\mu}{h^2} + u = 0$$

$$\frac{d^2}{d\theta^2}\left[u - \frac{\mu}{h^2}\right] + \left[u - \frac{\mu}{h^2}\right] = 0$$

$$\frac{d^2x}{dt^2} + x = 0, \quad \text{where } x = u - \frac{\mu}{h^2}$$

$$(D^2+1)x = 0.$$

Auxiliary equ^n is $D^2+1=0$

$D^2 = -1$ i.e. $= \pm i$

$$x = C_1\cos(\theta - C_2)$$

Putting $x = u - \frac{\mu}{h^2}$, we get

$$u - \frac{\mu}{h^2} = C_1\cos(\theta - C_2)$$

$$\frac{1}{r} - \frac{\mu}{h^2} = C_1\cos(\theta - C_2)$$

Dividing both side by $\frac{\mu}{h^2}$, we get

$$\frac{h^2/\mu}{r} = 1 + \frac{C_1h^2}{\mu}\cos(\theta - C_2)$$

This is kepler's first law.

$$T = \frac{\text{Area of the ellipse}}{\text{Rate of description of the sectional}}$$

$$= \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}.$$

$$= \frac{2\pi ab}{\sqrt{\mu \cdot L}} = \frac{2\pi ab}{\sqrt{\mu \frac{b^2}{a}}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$\therefore T^2 = \frac{4\pi^2 a^3}{\mu} = \frac{\pi^2}{2\mu}(2a)^3.$$

$\Rightarrow T^2 \propto (2a)^3$, which verifies kepler's third law.

from kepler's second law, the areal velocity is constant i.e. $\frac{h}{2}$ constant

$$\frac{1}{2}(r^2\dot{\theta}) = \text{constant}$$

$$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0. \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$\frac{l}{r} = 1 + e\cos\theta$, where $e \leq 1$

$$ul = 1 + e\cos\theta.$$

$$u = \frac{1}{l}(1 + e\cos\theta).$$

Diff w r to θ, we have

$\frac{du}{d\theta} = \frac{1}{l}(-e\sin\theta]$ and $\frac{d^2u}{d\theta^2} = \frac{-e}{l}\cos\theta$

$$F = \mu h^2 \left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= h^2u^2 \left[\frac{1+e\cos\theta}{L} - \frac{e\cos\theta}{L}\right].$$

$$= h^2u^2\left[\frac{1}{l}\right] = \frac{h^2}{r^2}\left[\frac{1}{l}\right]. \quad -(2)$$

$$= \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}.$$

Acc. to kepler's third law

$$T^2 \propto (2a)^3$$

$$\frac{T^2}{a^3} = \text{constant}$$

$$\frac{4\pi^2 a^2 b^2}{h^2 a^3} = \text{constant}.$$

$$\frac{4\pi^2}{h^2}\left(\frac{b^2}{a}\right) = \text{constant}.$$

$\frac{4\pi^2}{h^2}(l) =$ constant $\left(\frac{b^2}{a} = l \cdot \text{latus rectum}\right)$

$\frac{l}{h^2}$ = constant for all planets.

$\frac{h^2}{l} = \mu$.

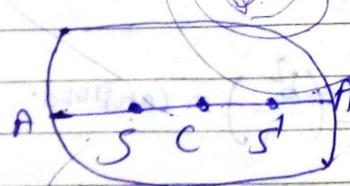
from ②, we get

$$F = \frac{1}{r^2}\left(\frac{h^2}{l}\right) = \frac{1}{r^2}[\mu]$$

$$\therefore \boxed{mF = \frac{m\mu}{r^2}}.$$

Question ⇒ If v_1 and v_2 are the linear velocities of a planet when it respectively nearest and farthest from the sun. Prove that $(1+e)v_2 = (1-e)v_1$

Sol^n ⇒



let the sun be at the focus S. v_1 is the velocity of the planet when it is at A, where A is the nearest point from focus S.

$$\therefore SA = CA - CS = a - ae = a(1-e).$$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\therefore v_1^2 = \mu\left(\frac{2}{AS} - \frac{1}{a}\right) = \mu\left[\frac{2}{a(1-e)} - \frac{1}{a}\right]$$

$$= \frac{\mu(1+e)}{a(1-e)} \quad \text{--- ①}.$$

$$SA' = SC + CA' = ae + a = a(1+e)$$

$$\therefore v_2^2 = \frac{\mu(1-e)}{a(1+e)} \quad \text{--- ②}.$$

Dividing ① by ②, we get

$$\frac{v_1^2}{v_2^2} = \frac{(1+e)}{(1-e)} \times \frac{(1+e)}{(1-e)} = \frac{(1+e)^2}{(1-e)^2}$$

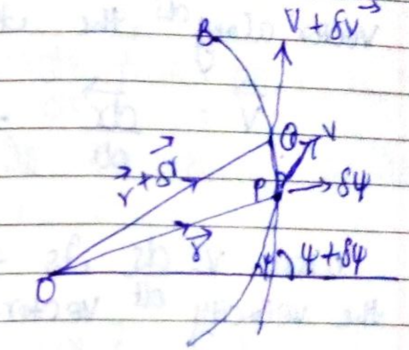
$$\frac{v_1}{v_2} = \frac{1+e}{1-e}.$$

$v_1(1-e) = v_2(1+e)$.

11th CHAPTER

Motion of a Particle in 3 Dimension

⇒ Velocity and Acceleration of a Particle Along A Curve ⇒



Prove ⇒ Consider a Particle moving along a curve AB, Not necessarily a plane curve. let P and Q be the position of the particle at the instant t and $t+\delta t$ respectively. let $\vec{OP} = \vec{r}$ and $\vec{OQ} = \vec{r} + \delta\vec{r}$

$$\therefore \vec{PQ} = \delta\vec{r}$$

The velocity of the Particle at time t is defined as

$$\vec{V} = \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{d\vec{r}}{dt}$$

if arc $PQ = \delta S$, then

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \hat{t}\,\frac{ds}{dt}.$$

Where $\hat{t} = \frac{d\vec{r}}{ds}$ is the unit tangent vector along the tangent at P.

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \hat{t}\cdot v$$

where $v = \frac{ds}{dt}$ is the magnitude of the velocity vector $\vec{V}$.

$$\vec{f} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}(v\hat{t})$$

$$= \hat{t}\,\frac{dv}{dt} + v\,\frac{d\hat{t}}{dt}.$$

$$\Rightarrow \frac{dv}{dt}\cdot\hat{t} + v\frac{d\hat{t}}{ds}\cdot\frac{ds}{dt}$$

$$= \frac{dv}{dt}\cdot\hat{t} + v^2\frac{d\hat{t}}{ds}$$

$\frac{d\hat{t}}{ds}$ is the rate change of tangent vector and acts in the

(Direction) (time vect)

dir^n Perpendicular to $\hat{t}$ at P.

$$\frac{d\hat{t}}{ds} = \hat{n}\left|\frac{d\vec{t}}{ds}\right|$$

$$= \hat{n}\cdot k \quad \left(\because k = \left|\frac{d\vec{t}}{ds}\right|\right)$$

$$f = \frac{dv}{dt}\cdot\hat{t} + v^2 k\hat{n}$$

if ρ is the radius of curvature, then $\rho = \frac{1}{k}$

$$\vec{f} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}.$$

Question No$\Rightarrow$ 1

A particle moves on a smooth sphere under no force except the pressure of the surface. Show that its path is given by the equation $\cot\theta = \cot\beta\cdot\cos\phi$, where θ and ϕ are its angular co-ordinates.

Sol^n $\Rightarrow$ At time t, let P be the position of the particle of mass m and θ and ϕ be it's angular co-ordinates. Let a be the radius sphere which is constant.

let R be the rea^n of the surface of P.

$$a\left(\frac{d\theta}{dt}\right)^2 + a\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 = \frac{R}{m} \quad (1)$$

$$a\frac{d^2\theta}{dt^2} - a\cos\theta\cdot\sin\theta\left(\frac{d\phi}{dt}\right)^2 = 0.$$

$$\frac{d^2\theta}{dt^2} = \cos\theta\cdot\sin\theta\left(\frac{d\phi}{dt}\right)^2 \quad (2)$$

$$\frac{1}{\sin\theta}\cdot\frac{d}{dt}\left(\sin^2\theta\cdot\frac{d\phi}{dt}\right) = 0 \quad (3)$$

Integration eq^n (3), we get

$$\sin^2\theta\,\frac{d\phi}{dt} = C.$$

$$\frac{d\phi}{dt} = \frac{C}{\sin^2\theta}$$

Substituting the value of $\frac{d\phi}{dt}$ from (4) in (2), we get

$$\frac{d^2\theta}{dt^2} = \sin\theta\cdot\cos\theta\cdot\frac{C^2}{\sin^4\theta} = \frac{C^2\cos\theta}{\sin^3\theta}$$

multiplying both side by $2\frac{d\theta}{dt}$, we get

$$2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} = \frac{2C^2\cos\theta}{\sin^3\theta}\,\frac{d\theta}{dt}$$

$$\int 2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}\,dt = 2C^2\int\frac{\cos\theta}{\sin^3\theta}\,d\theta$$

or $$\left(\frac{d\theta}{dt}\right)^2 = -\frac{C^2}{\sin^2\theta} + C_1$$

Initially let $\theta = \beta$, $\frac{d\theta}{dt} = 0$

$$C_1 = \frac{C^2}{\sin^2\beta}.$$

$$\left(\frac{d\theta}{dt}\right)^2 = C^2\left(\frac{1}{\sin^2\beta} - \frac{1}{\sin^2\theta}\right).$$

Taking square roots $\frac{d\theta}{dt} =$

$$\frac{C}{\sin\beta\,\sin\theta}\left(\sin^2\theta - \sin^2\beta\right)^{1/2}.$$

$$\frac{d\phi}{dt}\cdot\frac{dt}{d\theta} = \frac{d\phi}{d\theta} = \frac{\sin\beta}{\sin\theta\sqrt{\sin^2\theta - \sin^2\beta}}$$

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$$\Rightarrow \frac{\operatorname{cosec}\theta \sin\beta \cdot \operatorname{cosec}\beta \operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\beta - \operatorname{cosec}^2\theta}}$$

$$= \frac{\operatorname{cosec}^2\theta}{\sqrt{\cot^2\beta - \cot^2\theta}}$$

Integration

$$\phi = \cos^{-1}\left(\frac{\cot\theta}{\cot\beta}\right) + C_2$$

$$\phi = \cos^{-1}\left(\frac{\cot\theta}{\cot\beta}\right) + 0.$$

$$\cos\phi = \frac{\cot\theta}{\cot\beta}.$$

Hence Prove.