

NOTES

Subject - Linear Algebra

Semester - VI

Vector spaces and Subspaces.

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Date :

Internal and External Binary operation
Internal Binary operation:- If $\forall u, v \in V$ and $u+v$ is unique

External Binary operation:- If for each $u \in V$ and $a \in F$, $a \cdot u \in V$ and $a \cdot u$ is unique

Vector space:- A vector space over the field F , is a non empty set V together with two operations $+$ and $\cdot$ such that for each $u, v \in V$ there is a unique element $u+v \in V$ and for each $a \in F$ and $u \in V$ there is a unique element $a \cdot u \in V$

- (1) $u+(u+v) = (u+u)+v$
- (2) $u+v = v+u$
- (3) $\exists$ an element $0 \in V$ st $u+0 = 0+u = u \forall u \in V$
- (4) For each $u \in V$, $\exists$ a unique element $-u \in V$

$$u+(-u) = 0 = -u+u$$

Abelian group:- Let $\langle V, + \rangle$ be an abelian group and $\langle F, +, \cdot \rangle$ be a field. Define scalar multiplication st $\forall a \in F, u \in V, au \in V$ via a

- Vector space over F
- ① $a(u+v) = au + av$
 - ② $(a+b)u = au + bu$
 - ③ $a(bu) = (ab)u$
 - ④ $1u = u \quad \forall u \in V$

N vector $\rightarrow$ An ordered n -tuple of real numbers.

Column vector - An ordered array of real numbers can also be written in column vector.

Plane vector $\rightarrow$ A plane vector is an ordered pair of real numbers. The set of plane vector is denoted by $\mathbb{R}^2$ which is a Cartesian product $\mathbb{R} \times \mathbb{R}$.

Space vector - A space vector is an ordered pair triple of real numbers. The set of all plane vectors is denoted by $\mathbb{R}^3$ which is a Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Ex 7 Show that the set $\mathcal{C}(\mathbb{R}) = \{a + b\alpha\}$ where α is a vector space over $\mathbb{Q}$ with the composition

$$(a + b\alpha) + (c + d\alpha) = (a+c) + (b+d)\alpha$$

$$a(a + b\alpha) = aa + ba\alpha$$

closed and have all functional number

The given set is $\mathcal{C}(\mathbb{R}) = \{a + b\alpha\} : a, b \in \mathbb{Q}$.

Properties of addition.

$$(a + b\alpha) + (c + d\alpha) = (a+c) + (b+d)\alpha \in \mathcal{C}(\mathbb{R})$$

Associativity

$$(a + b\alpha) + (c + d\alpha) + (e + f\alpha) = (a+c+e) + (b+d+f)\alpha$$

$$= a + c + e + (b + d + f)\alpha$$

$$(a + b\alpha) + (c + d\alpha) + (e + f\alpha) = (a+c+e) + (b+d+f)\alpha$$

$$= a + c + e + (b + d + f)\alpha$$

$$= (a + b\alpha) + (c + d\alpha) + (e + f\alpha)$$

Commutativity

$$(a + b\alpha) + (c + d\alpha) = (c + a) + (d + b)\alpha = (c + a) + (d + b)\alpha$$

$$= (c + a) + (d + b)\alpha = (c + a) + (d + b)\alpha$$

Existence of identity.

$$0 + 0\alpha = 0 \in \mathcal{C}(\mathbb{R}) \quad \forall (a + b\alpha) \in \mathcal{C}(\mathbb{R})$$

$$= a + b\alpha$$

$$(0 + 0\alpha) + (a + b\alpha) = a + b\alpha$$

(iv)

Existence of inverse

For each $a + b\beta \in V$ $\exists$ $(c + d\beta) \in V$.

$$(a + b\beta) + (-a - b\beta) = 0 + 0\beta = 0$$

$$(-a - b\beta) + (a + b\beta) = 0 + 0\beta = 0.$$

II

Properties of multiplication.

If $a + b\beta \in V$ and $\alpha \in \mathbb{Q}$.

$$\alpha(a + b\beta) = \alpha a + \alpha b\beta \in V.$$

$$u = a + b\beta, v = c + d\beta,$$

$$d(u + v) = d(a + c + b\beta + d\beta)$$

$$= d(a + c + (b + d)\beta) = d(a + c) + d(b + d)\beta$$

$$= (da + dc) + (db + dd)\beta = du + dv$$

(v)

$$(a + b\beta)u = (a + b\beta)(c + d\beta)$$

$$= (a + b\beta)a + (a + b\beta)b\beta.$$

$$= (\alpha a + \alpha b\beta) + (\beta a + \beta b\beta)$$

$$= \alpha u + \beta u$$

(vi)

$$(\alpha\beta)u = (\alpha\beta)(c + d\beta)$$

$$= \alpha\beta a + \alpha\beta b\beta$$

$$= \alpha(\beta a + \beta b\beta) = \alpha(\beta u)$$

(vii)

$$1 \cdot u = 1 \cdot (a + b\beta)$$

$$= a + b\beta = u$$

$\mathbb{Q}\beta$ is a vector space over $\mathbb{Q}$

Q3.

If H is a subfield of a field F , then show that F can be considered as a vector space over H .

Ans 3

Let us take the addition in F as the internal binary operation. Thus F being a field, forms an abelian group w.r to addition.

The multiplication of the elements of the field F as an external binary operation over H .

$u \in H, v \in F \Rightarrow uv \in F$.

Some properties of scalar multiplication

(i)

$$a(uv) = (au)v \quad \forall a \in H, u \in F, v \in F$$

(ii)

$$(ab)u = a(bu) \quad \forall a, b \in H, u \in F$$

(iii)

$$1u = u \quad \forall u \in H, u \in F.$$

(iv)

$$F$$
 is a vector space over H as H is a subfield of F .

Some general properties of vector space

(i)

$$0u = 0 \quad \forall u \in F$$

$$a0 = a(0 + 0) = a0 + a0.$$

$$0 + a0 = a0 + a0.$$

$$0 = a0.$$

$$a0 = 0.$$

② $0 \cdot u = 0 \quad \forall u \in V.$

$0 + 0 = 0.$

$(0+0)u = 0 \cdot u.$

$0u + 0u = 0u$

$0u + 0u = 0u + 0$

$0u = 0.$

③ $(-1)u = -u \quad \forall u \in V.$

$(-1)u + u = (-1+1)u.$

$(-1)u + u = (-1+1)u.$

$(-1)u + u = 0u$

$(-1)u + u = 0$

$-1u = -u.$

④ $a(-u) = -(au) = (-a)u \quad \forall a \in F, u \in V$

$au + (-au) = au + a(-u)$

$0 = au + a(-u)$

$0 = au + a(-u)$

$a(-u)$ is additive inverse of au .

$a(-u) = -au.$

⑤ Let $au = 0$ and $a \neq 0$

Now $a \neq 0 \Rightarrow a^{-1} \text{ exists.}$

$a^{-1}(au) = a^{-1}0.$

$(a^{-1}a)u = 0$

$1 \cdot u = 0$

$u = 0$

Also if $au = 0$ and $u \neq 0$; we have to show that $a = 0.$

Let $a \neq 0 \quad a^{-1} \text{ exists.}$

$a^{-1}(au) = (a^{-1}a)u = 0$

$1u = 0$

$u = 0.$

which is a contradiction $\therefore a$ must be equal to 0.

Thus $au = 0$ and $u \neq 0 \Rightarrow a = 0.$

⑦

$au = bu$

$au - bu = 0$

$(a-b)u = 0$

$u \neq 0$

$a-b = 0$

$a = b$

Subspaces:-

A non empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V .

Th^m

A non empty subset W of a vector space $V(F)$ is a subspace of V if and only if $au + bu \in W$ for $a, b \in F$ and $u \in W$.

Proof

Condition is necessary -

Let W be a subspace of $V(F)$. W is also closed under vector addition and scalar multiplication

$a \in F, u \in W \Rightarrow au \in W$
 $b \in F, u \in W \Rightarrow bu \in W$
 $\therefore$ for any $a \in F$ and $u \in W$,
 $au + bu \in W$.

Condition is sufficient.

Let W be a non empty subset of $V(F)$ satisfying condition for all $a, b \in F$ and $u \in W$.

$1 \in F$ so $a = 1, b = 1$

$1 \cdot u + 1 \cdot u \in W$.

$u + u \in W$

W is closed, under vector addition

$a \in F, u \in W$

$au = au + 0u \in W$

$au = au + 0u \in W$

$a \in F, u \in W \Rightarrow au \in W$.

W is closed under scalar multiplication

W is subspace of $V(F)$

W is subspace of $V(F)$

Thm

The intersection of two subspaces W_1 and W_2 of a vector space

Proof

$V(F)$ is also subspace of $V(F)$

Let W_1 and W_2 be two subspaces of a vector space $V(F)$

$0 \in W_1$ and $0 \in W_2 \therefore 0 \in W_1 \cap W_2$
 Let $u \in W_1, v \in W_2$ and $a \in F$
 $u \in W_1, v \in W_2 \Rightarrow au \in W_1$ and $av \in W_2$
 $u \in W_1, v \in W_2 \Rightarrow u + v \in W_1$ and $u + v \in W_2$

W_1 and W_2 are subspaces of V .
 $a \in F$ and $u \in W_1, v \in W_2 \Rightarrow au \in W_1$ and $av \in W_2 \Rightarrow au + av \in W_1 \cap W_2$

$au + av \in W_1 \cap W_2$

$a \in F$ and $u \in W_1, v \in W_2$
 $au + av \in W_1 \cap W_2$
 $W_1 \cap W_2$ is a subspace of $V(F)$

Thm

The union of two subspaces of a vector space $V(F)$ may not be a subspace of $V(F)$

Proof

Let $W_1 = \{x \in V : x = (x, 0)\}$ and $W_2 = \{y \in V : y = (0, y)\}$ are subspaces of the vector space $V = R^2$.

$u \in W_1, v \in W_2$
 $u + v = (x, 0) + (0, y) = (x, y)$

$$w, v, w = \{w : w = (x, 0, 0) \text{ or } w = (0, y, 0) \mid x, y \in \mathbb{R}\}$$

$$w \in w, v, w$$

$$w \in (x, 0, 0) + (0, y, 0) = (x, y, 0)$$

$$(x, y, 0) \notin w, \text{ and } w, \text{ both,}$$

little $\neq w, v, w \Rightarrow w, v, w$ is not closed under vector addition.

w, v, w is not a subspace of $V(\mathbb{R}^3)$.

Ques 1. let $V \subseteq \mathbb{R}^3$ be the Euclidean space. let

$$w = (a, y, z) : a + b + c = 0.$$

$x, y, z \in \mathbb{R}$ a, b, c being fixed real numbers

Show that w is subspace of V .

Proof. let $w = (x, y, z)$ $w \in (x, y, z)$ be any fixed element of w .

$$x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$$

$$ax_1 + by_1 + cz_1 = 0 \quad ax_2 + by_2 + cz_2 = 0$$

$$\begin{aligned} a \in \mathbb{R} \quad a(w_1 + w_2) &= a(x_1, y_1, z_1) + a(x_2, y_2, z_2) \\ &= (ax_1, ay_1, az_1) + (ax_2, ay_2, az_2) \\ &= (ax_1 + ax_2, ay_1 + ay_2, az_1 + az_2) \end{aligned}$$

$$ax_1 + ax_2, ay_1 + ay_2, az_1 + az_2 \in \mathbb{R}$$

$$\begin{aligned} a(ax_1 + ax_2) + b(ay_1 + ay_2) + c(az_1 + az_2) \\ = a(ax_1 + ax_2) + b(ay_1 + ay_2) + c(az_1 + az_2) \\ = a(ax_1 + ax_2) + b(ay_1 + ay_2) + c(az_1 + az_2) \end{aligned}$$

from ② and ③ $a(w_1 + w_2)$ is subspace of V .

Linear Span of Subspace.

If w_1 and w_2 are two subspaces of a vector space V , then we define the linear span of w_1 and w_2 as $w_1 + w_2 = \{w_1 + w_2 : w_1 \in w_1, w_2 \in w_2\}$.

Thm

The linear span of two subspaces w_1 and w_2 of a vector space V is a subspace of V generated by union of w_1 and w_2 .

Proof. obviously $w_1 \subseteq w_1 + w_2$ and $w_2 \subseteq w_1 + w_2$. $w_1 + w_2$ is the smallest of V containing $(w_1 + w_2)$.

$\langle w_1, w_2 \rangle \subseteq \langle w_1 + w_2 \rangle$ — (1)

Let $u = w_1 + w_2$ be any element.

$w_1, w_2 \in \langle w_1, w_2 \rangle$ $w_2 \in \langle w_1, w_2 \rangle$

$w_1 + w_2 = \langle w_1 + w_2 \rangle$

$u = w_1 + w_2$ is a linear combination of the elements $w_1, w_2 \in \langle w_1, w_2 \rangle$

$u \in \langle w_1, w_2 \rangle$

$w_1 + w_2 \subseteq \langle w_1, w_2 \rangle$ — (2)

$u \in \langle w_1, w_2 \rangle$ and $u \in \langle w_1, w_2 \rangle$

$w_1 + w_2 = \langle w_1, w_2 \rangle$

Direct Sum of Subspace — A Vector space V is said to be direct sum of its

Subspace w_1 and w_2 if every $v \in V$ can be uniquely

$v = w_1 + w_2$ $w_1 \in w_1, w_2 \in w_2$

Disjoint Subspace — Let V be a vector space. Two Subspace w_1 and w_2 of V are said to be disjoint if

their intersection is the zero Subspace

$w_1 \cap w_2 = \{0\}$

Thm — The necessary and sufficient condition for a Vector space

V to be a direct sum of its Subspace w_1 and w_2 are that

(1) $V = w_1 + w_2$ (2) $w_1 \cap w_2 = \{0\}$

Proof — The condition are necessary.

$V = w_1 + w_2$ — (1)

If possible $0 \neq v \in w_1 \cap w_2$ — (2)

$v \in w_1$ and $v \in w_2$

$v = 0 + v$ $0 \in w_1$ and $v \in w_2$

$v = v + 0$ $v \in w_1$ and $0 \in w_2$

only vector which is common to both.

$w_1 \cap w_2 = \{0\}$

The condition is necessary.

$V = w_1 + w_2$

$w_1 \cap w_2 = \{0\}$

The representation is unique

$v = w_1 + w_2$ $w_1 \in w_1, w_2 \in w_2$

$v = w_1' + w_2'$ $w_1' \in w_1, w_2' \in w_2$

$w_1 + w_2 = w_1' + w_2'$

$w_1 - w_1' = w_2' - w_2$

w_1, w_2 a Subspace

$w_1 + w_1' \in w_1 \Rightarrow w_1 - w_1' \in w_1$

$w_2 + w_2' \in w_2 \Rightarrow w_2' - w_2 \in w_2$

$w_1 - w_1' = w_2' - w_2 \in w_1 \cap w_2$

$w_1 \cap w_2 = \{0\} \Rightarrow w_1 = w_1'$

$$u_1' - u_2' = 0 \Rightarrow u_2' = u_1'$$

$$V = u_1 \oplus u_2$$

Q2

Let u_1, u_2, u_3 be Subspace of R^3
 $u_1' = (xy, z) \in R^3, x+y+z=0$
 $u_2' = (xy, z) \in R^3, x=z$
 $u_3' = (0, 0, z) : z \in R$

Prove that $R^3 = u_1' + u_2'$ and
 $R^3 = u_2' + u_3'$. Which of these
 sums is direct?

And

Each of $u_1' + u_3'$ and $u_2' + u_3'$
 are Subspace of R^3 .

For any $(a, b, c) \in R^3$
 $(a, b, c) = (a, b, -a-b) + (0, 0, a+b)$

$$R^3 = u_1' + u_3'$$

$$u_1 \cap u_3 = \{(x, y, z) \in R^3 : x+y+z=0, x=y=z=0\}$$

$$= \{(0, 0, 0)\}$$

$$R^3 = u_1' \oplus u_3'$$

$$(a, b, c) \in R^3$$

$$(a, b, c) = (a, b, a) + (0, 0, -a)$$

$$R^3 = u_2' + u_3'$$

$$u_2' \cap u_3' = \{(x, y, z) \in R^3 : x=y=z=0\}$$

$$= \{(0, 0, 0)\}$$

$$R^3 = u_2' \oplus u_3'$$

u_1, u_2, u_3 and u_1', u_2', u_3' are both

Linear Combination of Vectors

A Vector $v \in V$ is said to be a

linear combination of vectors

$u_1, u_2, \dots, u_n \in V$ if v can be expressed as

$$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$$

Linearly dependent - Let $\{v_i\}$ be Vector

space. A set of vectors $u_1, u_2, \dots, u_n \in V$

is said to be linearly dependent if there

$$a_1, a_2, \dots, a_n \in F, a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0$$

linearly independent :- Let $\{v_i\}$ be a

vector space. A set of vectors $u_1, u_2, \dots, u_n \in V$

is said to be linearly independent.

$$a_1 u_1 + \dots + a_n u_n = 0$$

$$a_1 = a_2 = \dots = a_n = 0$$

Th^m

Every subset of a linearly dependent

set of vectors is linearly dependent.

Let $S' = \{u_1, u_2, \dots, u_r\}$ be a linearly

dependent set of vectors

$S' \subseteq \{u_1, u_2, \dots, u_n\}$ $n > r$ be a linearly

independent set of vectors

S_1 is linearly dependent set of

$$vectors, \exists scalars a_1, a_2, \dots, a_r$$

$$a_1 u_1 + a_2 u_2 + \dots + a_r u_r = 0$$

$$a_1 u_1 + a_2 u_2 + \dots + a_r u_r + 0 u_{r+1} + \dots + 0 u_n = 0$$

Ex 11 - The set of non zero vectors $u_1, u_2, \dots, u_n$ of a vector space $V(F)$ is linearly independent if and only if one of the vectors u_i is a linear combination of preceding vectors.

Proof - let the set of vectors $(u_1, u_2, \dots, u_n)$ be linearly dependent.

$\exists$ scalars $a_1, a_2, \dots, a_n$ not all zero
 $a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0$
 let r be the largest integer for which $a_r \neq 0$
 $a_r u_r = -a_{r+1} u_{r+1} - \dots - a_n u_n$

Coroll - If $n=1$, then $u_1 = 0$

$a_1 u_1 = 0$, $u_1 = 0$
 $u_1, u_2, \dots, u_n$ is set of non zero vectors

Coroll - If $n > 1$.

$a_1 u_1 + \dots + a_r u_r = 0$ (1)
 divide (1) by a_r
 $u_r = (-\frac{a_1}{a_r}) u_1 - \dots - (-\frac{a_{r-1}}{a_r}) u_{r-1}$

u_r is linear combination of preceding vectors

Conversely let u_r is a linear

Combination of the preceding vectors

$u_1, u_2, \dots, u_{r-1}$
 $u_r = b_1 u_1 + b_2 u_2 + \dots + b_{r-1} u_{r-1}$
 $b_1 u_1 + b_2 u_2 + \dots + b_{r-1} u_{r-1} - u_r = 0$
 $a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0$

$a_1 = b_1, a_2 = b_2, \dots, a_{r-1} = b_{r-1}, a_r = -1, a_{r+1} = \dots = a_n = 0$
 $(u_1, u_2, \dots, u_n)$ is linearly dependent

Q2 - Express the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ as a linear combination of matrices

$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$
 let $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} + b \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

$+ a_3 \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0+2a_3 & -3a_1+2a_3 \\ 2a_1+2a_3 & a_2+5a_3 \end{bmatrix}$
 Equality of two matrices

$2 = 2a_3$ (1) $0 = -3a_1 + 2a_3$ (2)
 $0 = 2a_1 + 2a_3$ (3) $-5 = a_2 + 5a_3$ (4)

$a_3 = 1$
 $a_1 = a_3 = 1$
 $0 = 2a_1 + 2a_3 = 4$ $a_2 = 2$
 $a_1 = 1, a_2 = 2, a_3 = 1$

$a_1 = 1, a_2 = 2, a_3 = 1$

Spanning set - Let $S = \{u_1, u_2, \dots, u_n\}$ be a set of vectors in a vector space V . The vector space V is spanned by S or the set S spans V if every vector in V is linear combination of vectors in S .

Linear Span

Let $S = \{u_1, u_2, \dots, u_n\}$ be a non empty subset of vector space V . The set of linear combination of finite set of element of S .

$$LS) \text{ or } S = \{u_1, u_2, \dots, u_n\} \\ = \left\{ \sum_{i=1}^n a_i u_i \mid a_i \in F \right\}$$

Linearly generated vector space

A vector space V is said to be linearly generated vector space if a finite subset S of V is all elements of V are generated by elements of S $V = LS$.

In a vector space V LS is the smallest subspace containing S .

Proof - Let S be non empty subset of V . For any $u_1, u_2 \in S \Rightarrow u_1 = 1u_1 + 0u_2$ or $u_2 = 0u_1 + 1u_2$ or $u_1, u_2 \in LS$.

Let $u, v \in LS \Rightarrow u = a_1u_1 + \dots + a_nu_n$

$$u = a_1u_1 + \dots + a_nu_n$$

$$u + v = a_1u_1 + \dots + a_nu_n + b_1u_1 + \dots + b_nu_n$$

Linear combination of $u_1, u_2, \dots, u_n$ belong to LS .

$u, v \in LS$ and $u + v \in LS$ or $u + v \in LS$.

LS is subspace of V containing S .

LS is smallest subspace of V containing S .

$$LS \subseteq W$$

$$u \in LS, \quad u = a_1u_1 + \dots + a_nu_n$$

$$u \in W, \quad u \in W$$

$$a_1u_1 + \dots + a_nu_n \in W$$

$$u \in W$$

$$W \subseteq LS \subseteq W$$

(1) Show that the set $(1, 2, 3), (0, 1, 2), (0, 0, 1)$ is a subset of $V_3(R)$ spans the entire vector space V_3 .

Ans -

$$S = \{(1, 2, 3), (0, 1, 2), (0, 0, 1)\}$$

If (xyz) be an arbitrary vector $(xyz) = a(1, 2, 3) + b(0, 1, 2) + c(0, 0, 1)$

$(xyz) = (a, b, c)$
Equating elements
 $a = x$

$a + b + c = z$

Value above xy

$a = x, b = y, c = z$

$(xyz) \in V$ can be expressed as combination of vectors $(1, 2, 3)$

$(0, 1, 2)$ and $(0, 0, 1)$

$(xyz) \in V$

$V \subseteq L(S)$

$L(S) \subseteq V$

$V \subseteq L(S), V \subseteq (S)$

Basis of a vector space:-

Let V be a vector space. A subset

$S = \{u_1, u_2, \dots, u_n\}$ of V is said to be a

Basis of V , if

(i) $u_1, u_2, \dots, u_n$ are linearly independent vectors

(ii) $u_1, u_2, \dots, u_n$ span V

Ordered Basis:-

A Basis of a vector space is said to be an ordered basis if its

vectors are written in certain

specific order.

Coordinates of a vector:- The numbers $a_1, a_2, \dots, a_n$ in the linear combination of

invariant of the number of element of a Basis.

Thm If V is a finitely generated vector space, then any two bases of V have same number of element

Proof

Let $S = \{u_1, u_2, \dots, u_n\}$

$S' = \{w_1, w_2, \dots, w_m\}$ be two bases of V if $n \neq m$

$V \subseteq V$ is a basis of V

$u_1 = a_{11}u_1 + \dots + a_{nm}u_m$

$u_2 = a_{21}u_1 + \dots + a_{2m}u_m$

$u_n = a_{n1}u_1 + \dots + a_{nm}u_m$

a system of m equations in n unknowns

$a_{11}x_1 + \dots + a_{1m}x_m = 0$

$a_{21}x_1 + \dots + a_{2m}x_m = 0$

$a_{n1}x_1 + \dots + a_{nm}x_m = 0$

the above system of equations has non zero sol.

$a_{11}x_1 + \dots + a_{1m}x_m = 0$

$a_{21}x_1 + \dots + a_{2m}x_m = 0$

$a_{n1}x_1 + \dots + a_{nm}x_m = 0$

$$a_1 u_1 + \dots + a_{n-1} u_{n-1} = 0$$

$$a_1 u_1 + \dots + a_{n-1} u_{n-1} + a_n u_n = 0$$

$$\text{Ex: } a_1 u_1 + \dots + a_m u_m = 0$$

$$a_1 (a_1 u_1 + a_2 u_2 + \dots + a_m u_m) + \dots + a_n (a_1 u_1 + a_2 u_2 + \dots + a_m u_m) = 0$$

$$a_1 u_1 + \dots + a_n u_n = 0$$

$$a_1 = a_2 = \dots = a_n = 0$$

maximal linearly independent set.

Let S be a subset of a vector space V . Then S is said to be maximal.

linearly independent subset of V .

if no. Superset of S other than S itself is linearly independent.

minimal generating set:-

A subset S of a vector space V is said to be a minimal generating set.

(1) S is a generating set V

(2) V cannot be generated by any proper subset of S

Dimension of vector space:-

$\dim(V)$ = The number of element in any basis of a finite dimensional vector space V .

Extension Theorem:-

Every linearly independent subset of a finitely generated vector space V can be extended to form a basis of V .

If $\{u_1, u_2, \dots, u_n\}$ spans V , then it itself forms a basis of V and $\{u_1, u_2, \dots, u_n\}$ is a maximal linearly independent subset of V .

Let $u \in V$ be arbitrary.

Then $S' = \{u_1, u_2, \dots, u_n, u\}$ is linearly independent.

$$a_1 u_1 + \dots + a_n u_n + a_{n+1} u = 0$$

$$u_1, \dots, u_n \text{ is linearly independent.}$$

$$a_1 = a_2 = \dots = a_n = 0$$

$$\text{Then } a = 0 \text{ implies } a_i = 0. \forall i \text{ using the fact}$$

$$u \in V$$

$$u \in L(S)$$

$$V \subseteq L(S)$$

$$L(S) \subseteq V$$

$$V = L(S)$$

Th^m.

If W is a subspace of a finite dimensional vector space V , then W is finite dimensional and $\dim W \leq \dim V$.

Proof

Let $\dim V = n$ is maximum number of linearly independent vectors in V .

Let W being a subset of V has at most n number of vectors.

Let $w_1, w_2, \dots, w_m$ be maximum number of linearly independent vectors in W .

$$a_1 w_1 + \dots + a_m w_m = 0$$

(1)

$$a_1 w_1 + \dots + a_m w_m = 0 \implies a_1 = 0, \dots, a_m = 0$$

But $w_1, w_2, \dots, w_m$ is linearly independent $a_1 = a_2 = \dots = a_m = 0$.

(2) by a. divided.

$$W = \left\{ \left(\frac{-a_1}{a} \right) w_1 + \dots + \left(\frac{-a_m}{a} \right) w_m \right\}$$

Let's linear combination of vectors $w_1, w_2, \dots, w_m$ is finite dimensional.

$$(w_1, w_2, \dots, w_m) \text{ is basis of } W$$

$$\dim W = m \quad m \leq n = \dim V$$

$$\dim V = \dim W = n$$

Let $A = (w_1, \dots, w_n)$ be basis of W $W = L(A)$ and A is linearly independent.

Subset of W $\dim W = n$ and A is basis of W .

$$v = a_1 w_1 + \dots + a_n w_n$$

$$v \in W$$

$$W \subseteq W$$

$$W = W$$

Conversely $W = L$ $\dim W = \dim V$.

Quest Prove that $S = (1000) (110) (111) (010)$

spans vector space $R^3(R)$ But is not a basis set.

Proof S spans $R^3(R)$

Let $u = (a, b, c)$ be any element of R^3

$$(a, b, c) = x(1000) + y(110) + z(111) + t(010)$$

$$= (x+0+y+z+0, 0+y+z+t+0, 0+0+z+0) = (x+y+z, y+z+t, z)$$

$$a = x+y+z, \quad b = y+z+t, \quad c = z$$

$$b = y+z+t, \quad c = z$$

Solve the above equation.

$$x = c, \quad y = b - z - t = b - c - t, \quad z = c$$

$$x = a - y - z = a - (b - c - t) - c = a - b + t$$

$t=0$ then (2) gives.

Eqn (1) can be expressed as,

$$(a, b, c) = (a-b)(100) + (b-c)(110) + c(111) + 0(010)$$

(a,b) can be expressed as a linear combination.

$$x(100) + y(110) + z(111) + t(1010)_{10}$$

$$x+y+z, y+z+t, z=0$$

$$y+z+t=0$$

$$z=0,$$

$$x=0, y=-t, x=t,$$

$$t=1, x=0, y=-1, x=1.$$

$$(1100) + (-1)(110) + 0(111) + 1(1010)_{10}$$

$$1101 + 0101 + 0101 = 0.$$

is not a linearly independent set
is span vector space $\mathbb{R}^3$ But
is not a basis set.

Ques

Show that the set $(214)(118)(312)$ forms a basis of $\mathbb{R}$

Ans

a finite dimensional vector space V of dimension n . any set of linearly independent vector. in V forms

$$\text{let } S = (214)(118)(312)$$

$$(100)(1010)(101) \text{ is a basis of } \mathbb{R}$$

$$\text{and } \dim(\mathbb{R}^3) = 3, \text{ it is sufficient to}$$

Show it is linearly independent

$$\text{let } a,b,c \in \mathbb{R} \text{ such that}$$

$$a(214) + b(118) + c(312) = (000)$$

$$2a+b+c, a+b+c, 4a+b-2c = (000)$$

$$a-b+c=0 \quad \text{--- (1)}$$

$$4a+b-2c=0 \quad \text{--- (2)}$$

Add (1) and (2)

$$5a+3b-c=0 \quad a=-4c/3$$

$$b=-c/3$$

$$a=-4c/3, b=-c/3 \text{ don't satisfy (1)}$$

$$\text{for any value of } c, a=0, b=0, c=0$$

Ques.

find the coordinates of the vector (1234) relative to the ordered basis $B = (0110)$

$$(0011)(1001)(0100) \text{ for } v_1, v_2, v_3$$

B is a basis for V_4 by proving that B spans V_4 and B is linearly independent set

The coordinates of $(1,2,3,4)$ relative to B

$$(1,2,3,4) = a_1(0110) + a_2(0011) + a_3(1001) + a_4(0100)$$

$$= (a_3, a_1+a_4, a_1+a_2, a_2+a_4)$$

$$a_3=1, a_1+a_4=2, a_1+a_2=3, a_2+a_4=4$$

$$(1,2,3,4) \text{ relative to basis } B$$

$$\text{are } (-2, 5, -1, 4)$$

Ques Extend the set of vectors $(0, 1, 2)$ $(2, -1, 1)$

to form a basis of $\mathbb{R}^3$.

Ans Let $A = (0, 1, 2)$ $(2, -1, 1)$
Standard Basis for $\mathbb{R}^3$ is $(1, 0, 0)$
 $(0, 1, 0)$ $(0, 0, 1)$

$S = (0, 1, 2)$ $(2, -1, 1)$ $(1, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$
A's linearly independent set.

The first three vectors and they form a linearly independent set.

$$a(0, 1, 2) + b(2, -1, 1) + c(1, 0, 0) = (0, 0, 0)$$

$$(8b + 9a - b, 2a + 4b) = (0, 0, 0)$$

$$8b + 12a + b(2, -1, 1) + c(1, 0, 0) = (0, 0, 0)$$

$$8b + 12a + b(2, -1, 1) + c(1, 0, 0) = (0, 0, 0)$$

Ques. Find the dimension of the vector space $(0, 0, 2)$ word

$$a(0, 0, 2) = (0, 0, 2a)$$

$$a \text{ set } S = \{(1, 0, 2)\}$$

$$2a + 1(0, 0, 2) \text{ is arbitrary } 2a + b(0, 0, 2) = 1(0, 0, 2)$$

$$S \text{ is a Basis } \checkmark \text{ dim } (0, 0, 2) = 2$$

Identical spaces - Two vector spaces V and W of the same dimension are said to be identical if each is a subspace of the other.

Ques

Show that the subspace spanned by vectors u_1, u_2, u_3, u_4 and the subspace spanned by vectors v_1, v_2 are identical

Ans

$$u_1 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_2 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_3 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_4 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_5 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_6 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_7 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_8 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_9 = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_{10} = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_{11} = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_{12} = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_{13} = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_{14} = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$$u_{15} = (1, 2, 1) = a(0, 0, 1) + b(1, 2, 5)$$

$a_2 - 1$ bag
 $u_2 = u_1 + 2u_2$ which is
 a linear combination

Thm

If u_1, u_2 are two subspaces of
 a finite dimension vector space V then
 $\dim(u_1 + u_2) = \dim u_1 + \dim u_2$
 $\iff \dim(u_1 \cap u_2) = 0$

Proof

Let $\dim(u_1 \cap u_2) = 1$ So u_1, u_2 are
 is basis of $u_1 \cap u_2$
 S is a subset of u_1 and S is a
 linearly independent set.
 It can be extended form a
 basis of u_1
 $A = \{u_1, \dots, u_k, u_{k+1}, \dots, u_m\}$
 be basis of set u_1
 $\dim u_1 = k+m$

S can be extended to form a
 basis of set u_2
 $\dim u_2 = k+m$

$\dim(u_1 + u_2) = \dim u_1 + \dim u_2$
 $1 + m + k + m = k + m + k + m$
 $\dim(u_1 + u_2) = 2k + 2m$
 $S_1 = \{u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_m\}$
 $u_1, \dots, u_m$

is linearly independent

Let $a_1 u_1 + \dots + a_k u_k + b_1 u_{k+1} + \dots + b_m u_m = 0$
 $a_1 u_1 + \dots + a_k u_k = -b_1 u_{k+1} - \dots - b_m u_m$
 $(a_1 u_1 + \dots + a_k u_k) \in u_1 \cap u_2$
 $b_1 u_{k+1} + \dots + b_m u_m \in u_1 \cap u_2$

Again we can separate
 $a_1 u_1 + \dots + a_k u_k = 0$
 $b_1 u_{k+1} + \dots + b_m u_m = 0$

$a_1 u_1 + \dots + a_k u_k = 0$ implies $a_1 = \dots = a_k = 0$
 $b_1 u_{k+1} + \dots + b_m u_m = 0$ implies $b_1 = \dots = b_m = 0$
 $\therefore u_1, \dots, u_m$ can be separated as
 linear combination of basis of u_1

$a_1 u_1 + \dots + a_k u_k = 0$ implies $a_1 = \dots = a_k = 0$
 a linear combination of element of B_2
 is linearly independent
 $b_1 u_{k+1} + \dots + b_m u_m = 0$ implies $b_1 = \dots = b_m = 0$

$a_1 u_1 + \dots + a_k u_k = 0$ implies $a_1 = \dots = a_k = 0$
 a linear combination of element of B_2
 is linearly independent
 $b_1 u_{k+1} + \dots + b_m u_m = 0$ implies $b_1 = \dots = b_m = 0$

S_1 is linearly independent set
 $u_1, \dots, u_m$ be arbitrary elements
 $u_1 \in u_1, u_2 \in u_2$ is a linear combination
 of elements of S_1
 $u_1 = a_1 u_1 + \dots + a_k u_k + b_1 u_{k+1} + \dots + b_m u_m$
 $\implies u_1 = 0$

$$= (a_1, u_1, \dots, a_{p+q} + b_1, u_1, \dots, b_{p+q}) \\ + (0, u_1, \dots, 0, u_1 + u_2, \dots, u_{n-1}) \\ = 2xy$$

$$u \in W_1 + W_2$$

$$L(S_1) \subseteq W_1 + W_2$$

$$v \in W_1 + W_2$$

$$v \in W_1 + W_2 \quad v \in W_1$$

$$v \in L(S_1)$$

$$W_1 + W_2 \subseteq L(S_1)$$

$$L(S_1) = W_1 + W_2$$

S_1 is a linearly independent set of vectors. S_1 is a basis of W_1 .

But S_1 has 1 element.

$$\dim(W_1 + W_2) = 1 + \dim W_2$$

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2$$

$$= \dim W_1 + \dim W_2$$

$$W_1 \cap W_2 = \{0\}$$

$$\dim(W_1 \cap W_2) = 0$$

$$\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$$

Thm.

Complementary Subspace - $W_1 \subseteq V(F)$, $W_2 \subseteq V(F)$. $W_1 \perp W_2$ is complementary vector space $V(F)$ then a complementary subspace W_1 and $\dim W_1 = \dim V - \dim W_2$.

Proof

Let $B = \{w_1, w_2, \dots, w_n\}$ is a basis of W_2 .

W_1 is linearly independent. Set.

Let $S = \{u_1, u_2, \dots, u_n\}$ is a basis of W_1 .

be basis of V .

$$B' = \{w_1, w_2, \dots, w_n\}$$

$$W_1 \cap W_2 = \{0\}$$

$$V = W_1 + W_2$$

$$W_1 \cap W_2 = \{0\}$$

Ques.

If W_1 and W_2 are subspaces of vector space $V(F)$ spanned by $S_1 = \{u_1, u_2, \dots, u_n\}$ and $S_2 = \{v_1, v_2, \dots, v_m\}$ find $\dim(W_1 + W_2)$.

$$S_1 = \{u_1, u_2, \dots, u_n\} \quad S_2 = \{v_1, v_2, \dots, v_m\}$$

$$S_1 = \{u_1, u_2, \dots, u_n\} \quad S_2 = \{v_1, v_2, \dots, v_m\}$$

Proof for $W_1 + W_2$ and $W_1 \cap W_2$

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$$S_1 \cup S_2 = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m\}$$

$$S_1 \cup S_2 = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m\}$$

$$W_1 + W_2 = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m\}$$

$$W_1 + W_2 = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m\}$$

$W_1 + W_2$ is a basis of V

max

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & -1 \\ 3 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

By performing various suitable elementary rows.

$$A_2 \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1(-1) \rightarrow R_1(-2) \rightarrow R_1(-1) \rightarrow R_1(-1)$
 $R_2(-1) \rightarrow R_2(-2) \rightarrow R_2(-1) \rightarrow R_2(-1)$
 $R_3(-1) \rightarrow R_3(-2) \rightarrow R_3(-1) \rightarrow R_3(-1)$
 $R_4(-1) \rightarrow R_4(-2) \rightarrow R_4(-1) \rightarrow R_4(-1)$
 $R_5(-1) \rightarrow R_5(-2) \rightarrow R_5(-1) \rightarrow R_5(-1)$
 $\dim W_1 + \dim W_2 = 3$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & -1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{R_1(-1), R_2(-1), R_3(-1)} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3(2)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of $W_1 = \{(1, 0, -1), (0, 1, 3, 1)\}$

$$\dim W_1 = 2$$

$$\dim W_1 + \dim W_2 = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$3 = 2 + 2 - \dim(W_1 \cap W_2)$$

$$\dim(W_1 \cap W_2) = 1$$

Ques Find a complement of subspace W generated by $(1, 0, 1)$ and $(1, 2, 3)$ in $V = \mathbb{R}^3$.

$W = \langle (1, 0, 1), (1, 2, 3) \rangle$
 $(1, 0, 1), (1, 2, 3)$ is a linearly independent set of vectors in $\mathbb{R}^3$.
 the set $(1, 0, 1), (1, 2, 3)$ is a basis of W .
 Standard basis of $\mathbb{R}^3$ is $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.
 the set $(1, 0, 1), (1, 2, 3), (0, 0, 1)$ is a basis of $\mathbb{R}^3$.
 $a(1, 0, 1) + b(1, 2, 3) + c(0, 0, 1) = (0, 0, 0)$
 $(a+b, 2b, a+3b+c) = (0, 0, 0)$
 $a+b=0, 2b=0, a+3b+c=0$
 $a=b=0, c=0$
 from (1) and (2) $a=b=c=0$
 $(1, 0, 1), (1, 2, 3), (0, 0, 1)$ is linearly independent set and forms a basis of $\mathbb{R}^3$.
 Extension of Basis is $(0, 0, 1)$
 $W = W_1 \cup W_2 = (0, 0, 1)$

the set $(1, 0, 1), (1, 2, 3), (0, 0, 1)$ is a basis of $\mathbb{R}^3$.
 $a(1, 0, 1) + b(1, 2, 3) + c(0, 0, 1) = (0, 0, 0)$
 $(a+b, 2b, a+3b+c) = (0, 0, 0)$
 $a+b=0, 2b=0, a+3b+c=0$
 $a=b=0, c=0$
 from (1) and (2) $a=b=c=0$
 $(1, 0, 1), (1, 2, 3), (0, 0, 1)$ is linearly independent set and forms a basis of $\mathbb{R}^3$.
 Extension of Basis is $(0, 0, 1)$
 $W = W_1 \cup W_2 = (0, 0, 1)$

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Subhash

Thm If W is a subspace of a vector space V and $u, v \in V$ then $W + W = W$

$$\text{catd} = \text{catd} + 1$$

$$2\omega + \mu \neq \omega' - \omega''$$

$$= w + u' - \mu' + u'$$

2nd full

$$W + H = W' + H'$$

with a with

with 5 with

$\log_2 \leq \log_4 + \log_8 = \log_{16}$

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Dimension of a Quotient Space.

The set is the subspace of $\mathbb{R}^n$

dem $M_{12} =$ die V -dim.

S is linearly independent subset of V , it can be extended to form a basis of V .

Barb. eg. R.

dim V = Card $\mathcal{B}$.

• a set T_2 (write, listen)

It is a good book of VLM.

$$w + \langle a_1 u_1, \dots, a_n u_n \rangle \in \langle w \rangle$$

and on 6-10

all _____ ends is linear combination
of _____ cm.

$$a_1, a_2, \dots, a_n = b_1, b_2, \dots, b_n$$

$\frac{d}{dt} \left(\frac{1}{\rho} \right) = - \frac{1}{\rho^2} \frac{d\rho}{dt}$

$$b_1 \xrightarrow{b_2} b_{m+1}$$

It is linearly independent.

$$u_2 = a_1u_1 + b_1u_4 \text{ --- basis}$$

$$b_1u_2 = b_1(a_1u_1 + b_1u_4) \text{ --- basis}$$

$$= a_1b_1u_1 + b_1^2u_4 \text{ --- basis}$$

(not basis) --- (not basis)
by Cramer's --- by Cramer's

$$\dim V_{u_2} = \dim V - \dim u_1$$

Ques.

If u_2 is a subspace of u_2 & generated by $(1,2)$ find V_{u_2} and its basis.

Ans

$(1,2)$ being a singleton set is linearly independent and so it forms a basis of u_2 .

We extend $(1,2)$ to get a Basis of u_2 consider a set $(1,2)(1,0)$

$$(1,2) \neq a(1,0)$$

Neither of $(1,2)$ and $(1,0)$ is a multiple of other

$(1,2)$ and $(1,0)$ is linearly independent and forms a basis for V .

$$V_{u_2} = \{u_2 + u_1 : u_2 \in u_2\}$$

$$= \{(1,2) + a(1,0) : a \in \mathbb{R}\}$$

$$= \{(1,2) + (a,0) : a \in \mathbb{R}\}$$

Basis of V_{u_2} is $(1,2) + (1,0)$

Ques

If u_2 is a subspace of u_2 generated by $(1,0)$ find V_{u_2} and its basis

Ans

The set $(1,0)$ $(1,1)$ is linearly independent because neither of $(1,0)$ and $(1,1)$ is a multiple of other.

$(1,0)$ $(1,1)$ form a basis of u_2 . Basis can be extended to form a basis of V by considering an additional vector.

The set $(1,0)$ $(1,1)$ $(0,1)$ is linearly independent.

That are used for extension of Basis of u_2 to Basis of V .

$$V_{u_2} = \{(1,0) + u : u \in u_2\}$$

$$= \{(1,0) + a(1,0) : a \in \mathbb{R}\}$$

The Basis of V_{u_2} is $(1,0) + (1,0)$

linear transformation

defn. let U and V be two vector spaces over the same field F . Then a map

$$T: U \rightarrow V$$

$$T(u+u') = T(u) + T(u')$$

$$T(au) = aT(u)$$

identity transformation $\rightarrow$ The identity map $I: V \rightarrow V$ s.t. $I(u) = u$ $\forall u \in V$ is a linear transformation

zero transformation $\rightarrow$ The zero map $0: V \rightarrow V$ s.t. $0(u) = 0 \forall u \in V$ is a linear transformation

Thm

let U and V be two vector spaces over same field F . Then map $T: U \rightarrow V$ is a linear transformation iff

$$T(au + bu') = aT(u) + bT(u') \forall a, b \in F, u, u' \in U$$

Proof let $T: U \rightarrow V$ be a linear function

$$T(u+u') = T(u) + T(u')$$

$$T(au) = aT(u)$$

$$T(au + bu') = T(au) + T(bu') = aT(u) + bT(u')$$

$$= aT(u) + bT(u')$$

Conversely let $T(au + bu') = aT(u) + bT(u')$

$$\text{Taking } a=1, b=0 \text{ in } \textcircled{1}$$

$$T(u+0) = T(u) + T(0)$$

$$T(u) = aT(u)$$

T is a linear transformation

Thm

Properties of linear transformation.
let $T: U \rightarrow V$ be a linear transformation

$$T(0_U) = 0_V$$

$$T(-u) = -T(u)$$

$$T(u-0) = T(u) - T(0)$$

$$T(u_1, u_2, \dots, u_n) = (T(u_1), T(u_2), \dots, T(u_n))$$

$$T(u_1 + u_2, \dots, u_n) = T(u_1) + T(u_2), \dots$$

$$T(0_U) = T(0_U) \text{ for both sides}$$

$$0_V = T(0_U)$$

$$T(0_U) = 0_V$$

③

$$u \in U$$

$$T(-u) = T((-1)u)$$

$$= (-1)T(u)$$

$$= -T(u)$$

④

$$T(u-0) = T(u) - T(0)$$

$$= T(u) + T(0)$$

$$= T(u) - T(0)$$

Ques Show that the function $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

given by $T(x, y) = (x+y, y, y)$ is a linear transformation

Pro

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$

$T(u+v) = T(x_1+x_2, y_1+y_2)$

$= (x_1+y_1+x_2+y_2, y_1+y_2, y_1+y_2)$

$= (x_1+y_1, y_1+y_2, y_1+y_2) + (x_2+y_2, y_2, y_2)$

$= T(x_1, y_1) + T(x_2, y_2)$

$T(a u_1 + b u_2) = T(ax_1 + by_1, ay_1 + by_2)$

$= (ax_1 + by_1 + ay_1 + by_2, ay_1 + by_2, ay_1 + by_2)$

$= a T(x_1, y_1) + b T(x_2, y_2)$

Ques. Show that the map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

defined by $T(x, y, z) = (x+y, y-z)$ is not a linear transformation.

Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$

$T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (x_1+y_1+x_2+y_2, y_1+y_2-z_1-z_2)$

$= (x_1+y_1, y_1-z_1) + (x_2+y_2, y_2-z_2)$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$T(u) + T(v) = T(u+v)$ is not a linear transformation.

Vector Space Isomorphism

One one linear transformation $T: U \rightarrow V$ is said to be one to one if $u_1 \neq u_2 \implies T(u_1) \neq T(u_2)$

onto linear transformation

A linear transformation T is said to be onto if for each $v \in V$, there exists $u \in U$ such that $T(u) = v$

vector space isomorphism

Let U and V be two vector spaces over same field F , then a linear transformation $T: U \rightarrow V$ is said to be vector space isomorphism if T is one one and U is isomorphic to V .

Equality of two linear transformations

If $T_1: U \rightarrow V$ and $T_2: U \rightarrow V$ are two linear transformations then T_1 and T_2 are said to be equal if $T_1(u) = T_2(u)$ for all $u \in U$.

T_m Two finite dimensional vector space over same field are isomorphic if and only if they have same dimension. Let U and V be finite dimensional vector space over same field F .

Q. Let $U \cong V$ we have to show $\dim U = \dim V$.
 $U \cong V$ a one to one onto linear transformation T from $U \rightarrow V$.
 Let $A = \{u_1, u_2, \dots, u_m\}$ is basis of U .
 $\dim U = m$.

$B = \{v_1, v_2, \dots, v_n\}$ is a basis for V .

$T(u_1) = a_1 v_1 + a_2 v_2 + \dots + a_m v_m$

$T(a_1 u_1) = a_1 m = 0$

$a_1 u_1 = a_1 m = 0$

B is linearly independent.

B spans V .
 $u \in V$ be arbitrary. This u can be written as

$T(u) = 0$

A is a basis of U $u \in U$.

$u = b_1 u_1 + b_2 u_2 + \dots + b_m u_m$

$u \in U$ can be written as

$\dim V = n$
 $\dim U = \dim V$

Conversely let $\dim U = \dim V$.
 If $\{u_1, \dots, u_m\}$ and $\{v_1, \dots, v_n\}$ are bases of U and V . $\exists$ a linear map T from $U \rightarrow V$ $T(u_i) = v_i$

$u = a_1 u_1 + \dots + a_m u_m$ $a_i \in F$

$T(u) = T(a_1 u_1 + \dots + a_m u_m)$

$= a_1 T(u_1) + \dots + a_m T(u_m)$

$T(u) \in V$

$T(u) = T(u_1) + \dots + T(u_m)$

$(a_1 - a_1) T(u_1) + \dots + (a_m - a_m) T(u_m) = 0$

$a_i = a_i$

This one to one

$u \in V$ $u = b_1 u_1 + \dots + b_m u_m$

$u = b_1 u_1 + \dots + b_m u_m$

$u \in U$

T_m From that every n dimensional vector space $U \subseteq \mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ let $\{u_1, u_2, \dots, u_n\}$ be basis of U

$$u_1, u_2, \dots, u_n \xrightarrow{T|_U} \mathbb{R}^n$$

$$T(u_1) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \mathbb{R}^n$$

$$a_1 u_1 + b_1 u_2 = a_1 (a_1 u_1 + b_1 u_2) + b_1 (b_1 u_1 + a_1 u_2) = (a_1^2 + b_1^2) u_1 + (a_1 b_1 + b_1^2) u_2$$

$\therefore T$ is linear transformation

$$u_1, u_2, \dots, u_n \xrightarrow{T|_U} \mathbb{R}^n$$

$$T(u_1) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \mathbb{R}^n$$

$$(a_1 - a_2) = (b_1 - b_2)$$

$$a_1 u_1 - a_2 u_2 = b_1 u_1 - b_2 u_2$$

$$T(u_1) = T(u_2)$$

T is onto

$$T(a_1) = a_1 u_1 + b_1 u_2 \in \mathbb{R}^n$$

$$T(a_1 u_1) = a_1 u_1 + b_1 u_2 \in \mathbb{R}^n$$

$a_1 u_1, \dots, a_n u_n$ are basis image of $u_1, \dots, u_n$
 T is an isomorphism $U \cong \mathbb{R}^n$

Ques Show that transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined $T(x_1, x_2, x_3) = (x_1, x_2)$ is a linear transformation and is onto but not one to one

$$u_1 = (x_1, x_2, x_3) \xrightarrow{T} u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$$

$$T(a_1 u_1 + b_1 u_2) = T(a_1 (x_1, x_2, x_3) + b_1 (y_1, y_2, y_3)) = (a_1 x_1 + b_1 y_1, a_1 x_2 + b_1 y_2) = a_1 (x_1, x_2) + b_1 (y_1, y_2) = a_1 T(u_1) + b_1 T(u_2)$$

T is a linear transformation

$$T(1, 0, 0) = (1, 0) \in \mathbb{R}^2$$

$$T(1, 0, 0) = (1, 0) \in \mathbb{R}^2$$

T is not one to one

$$T(1, 0, 0) = (1, 0) \in \mathbb{R}^2$$

$$T(1, 0, 0) = (1, 0) \in \mathbb{R}^2$$

The linear transformation -
The images of the element of Basis set under a transformation are

Ques. Let $u_1 = (1, 1, -1)$, $u_2 = (1, 1, 1)$, $u_3 = (1, 1, 1)$ be a Basis of R^3 let $T: R^3 \rightarrow R^3$ be the linear transformation such that

$T(u_1) = (1, 0, 1)$, $T(u_2) = (1, 1, 1)$, $T(u_3) = (1, 1, 1)$ find

A. Let $u = (x_1, x_2, x_3)$ GR be arbitrary.

Given that (u_1, u_2, u_3) is a basis

u can be expressed as linear

combination of vectors of Basis

$\exists$ scalars a_1, a_2, a_3

$u = a_1 u_1 + a_2 u_2 + a_3 u_3$

$(x_1, x_2, x_3) = a_1 (1, 1, -1) + a_2 (1, 1, 1) + a_3 (1, 1, 1)$

$a_1 + a_2 + a_3 = x_1$

$a_1 + a_2 - a_3 = x_2$

$-a_1 + a_2 + a_3 = x_3$

To find the values of a_1, a_2, a_3

The above system of equations can

be written in matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = I$

A^{-1} exist and is given by

$A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

Putting the value of A^{-1} in (2)

$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$

Equating element of two matrices

$a_1 = x_1 + x_2 + x_3$

$a_2 = x_1 + x_2 - x_3$

$a_3 = x_1 + x_2 + x_3$

$T(u) = T(a_1 u_1 + a_2 u_2 + a_3 u_3)$

$= a_1 T(u_1) + a_2 T(u_2) + a_3 T(u_3)$

$= (x_1 + x_2 + x_3) T(u_1) + (x_1 + x_2 - x_3) T(u_2) + (x_1 + x_2 + x_3) T(u_3)$

$T(u_1) = (1, 0, 1)$, $T(u_2) = (1, 1, 1)$, $T(u_3) = (1, 1, 1)$

$(x_1 + x_2 + x_3)(1, 0, 1) + (x_1 + x_2 - x_3)(1, 1, 1) + (x_1 + x_2 + x_3)(1, 1, 1)$

$(x_1 + x_2 + x_3 - x_1 - x_2 - x_3, 0 + x_1 + x_2 - x_1 - x_2 - x_3, x_1 + x_2 + x_3 + x_1 + x_2 - x_1 - x_2 - x_3)$

$(0, 0, 0)$

Ques find a linear transformation which maps $(1,1,1)$ to $(1,2,0)$ in $\mathbb{R}^3$ to $(2,1,1)$ in $\mathbb{R}^3$.

1. $\det T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$T(1,1,1) = (2,1,1)$

$T(1,1,0) = (2,1,1)$

$T(1,0,0) = (2,1,1)$

$(1,1,1), (1,1,0), (1,0,0)$ is linearly independent

let $xy \in \mathbb{R}^3$.

$xy = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$ax + by = a + b + c$ $ax = a$

$ay = b + c$ $ay = b + c$

$T(xy) = aT(1,1,1) + bT(1,1,0) + cT(1,0,0)$

$= a(2,1,1) + b(2,1,1) + c(2,1,1)$

$= (2a + 2b + 2c, a + b + c, a + b + c)$

$= (2x + 2y - 2z, x + y - z, x + y - z)$

which is required linear transformation

Ques find linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

such that $T(x) = Ax$ matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

2. $\det A$ Suppose that for $(x_1, x_2, x_3) \in \mathbb{R}^3$.

$(y_1, y_2, y_3) = T(x_1, x_2, x_3)$

$(y_1, y_2, y_3) = (x_1 + 2x_2 - 3x_3, x_2 + x_3, x_1 - x_3)$

$y = Ax$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ x_2 + x_3 \\ x_1 - x_3 \end{bmatrix}$$

$y_1 = x_1 + 2x_2 - 3x_3$

$y_2 = x_2 + x_3$

$y_3 = x_1 - x_3$

the required transformation is $T(x_1, x_2, x_3) = (x_1 + 2x_2 - 3x_3, x_2 + x_3, x_1 - x_3)$

Rank and Nullity

Null space — Let $T: U \rightarrow V$ be a linear transformation. The null space of T is subset of U . Consist of all vectors u whose image under T is zero and is denoted by $\text{Ker } T$.

$$\text{Ker } T = \{u \in U : T(u) = 0\}$$

Thm Let $T: U \rightarrow V$ be a Homomorphism, then $\text{Ker } T$ is a Subspace of U .

Proof Let $T: U \rightarrow V$ be a linear transformation.

$$\text{Let } u \in \text{Ker } T \Rightarrow T(u) = 0$$

$$\text{Let } u_1, u_2 \in \text{Ker } T$$

$$T(au_1 + bu_2) = aT(u_1) + bT(u_2) = a \cdot 0 + b \cdot 0 = 0$$

$$\Rightarrow 0 \in \text{Ker } T$$

$$\text{Let } u_1, u_2 \in \text{Ker } T$$

$$T(u_1 + u_2) = T(u_1) + T(u_2) = 0 + 0 = 0$$

Range of a linear transformation

Let $T: U \rightarrow V$ be linear transformation

The Range of T is set of all vectors u that are image

$$\text{Range } T = \{T(u) : u \in U\}$$

Thm Let $T: U \rightarrow V$ be a linear Transformation

Then Range of T is a Subspace of V .

Proof

$$\text{Let } u_1, u_2 \in \text{Range } T \Rightarrow u_1 = T(u), u_2 = T(v)$$

$$T(au_1 + bu_2) = aT(u_1) + bT(u_2) = aT(u) + bT(v) = T(au + bv) \in \text{Range } T$$

$$au_1 + bu_2 \in \text{Range } T \Rightarrow 0 \in \text{Range } T$$

Thm

Let $T: U \rightarrow V$ be a linear Transformation

Proof

$$\text{Let } u_1, u_2 \in \text{Range } T \Rightarrow u_1 = T(u), u_2 = T(v)$$

$$T(au_1 + bu_2) = aT(u_1) + bT(u_2) = aT(u) + bT(v) = T(au + bv) \in \text{Range } T$$

$$\text{Let } u_1, u_2 \in \text{Range } T \Rightarrow u_1 = T(u), u_2 = T(v)$$

$$T(au_1 + bu_2) = aT(u_1) + bT(u_2) = aT(u) + bT(v) = T(au + bv) \in \text{Range } T$$

$$\text{Let } u_1, u_2 \in \text{Range } T \Rightarrow u_1 = T(u), u_2 = T(v)$$

$$T(au_1 + bu_2) = aT(u_1) + bT(u_2) = aT(u) + bT(v) = T(au + bv) \in \text{Range } T$$

$$\text{Let } u_1, u_2 \in \text{Range } T \Rightarrow u_1 = T(u), u_2 = T(v)$$

is linear Transformation
let $u, v \in V$

$$f(u+v) = f(u) + f(v) = f(u+v)$$

$$= f(u+v)$$

$$= f(u+v)$$

$$= f(u+v)$$

$$f(u+v) = f(u) + f(v) = f(u+v)$$

$$= f(u+v)$$

is one line
let $u, v \in V$

$$f(u+v) = f(u) + f(v)$$

$$= f(u) + f(v)$$

$$= f(u) + f(v)$$

$$= f(u) + f(v)$$

$$= f(u) + f(v)$$

$$= f(u) + f(v)$$

$$= f(u) + f(v)$$

for $u, v \in V$, $f(u+v) = f(u) + f(v)$

$$f(u+v) = f(u) + f(v)$$

Rank and Nullity of linear Transformation
let $T: V \rightarrow V$ be a linear transformation then the dimension of $\text{Im } T$ and $\text{Ker } T$ are related by the Rank-Nullity Theorem. The dimension of $\text{Im } T$ is called Rank.

Use find a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ whose range space is spanned by the vectors

$\{e_1, e_2, e_3\}$ is standard basis of $\mathbb{R}^3$

$\mathbb{R}^3$ is a three dimensional vector space and $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$

$$T(e_1) = (1, 0, 0)$$

$$T(e_2) = (0, 1, 0)$$

$$T(e_3) = (0, 0, 1)$$

$\text{Im } T$ is spanned by $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

$$T(e_1) + T(e_2) + T(e_3)$$

$$= e_1 + e_2 + e_3$$

$$= e_1 + e_2 + e_3$$

$$= e_1 + e_2 + e_3$$

$$= e_1 + e_2 + e_3$$

$$T(e_1, e_2, e_3) = e_1 + e_2 + e_3$$

$$B_1 + 4B_2, 8B_1 + 5B_2, 3B_1 + 6B_2$$

Ques det V the vector space of 2×2 matrix over $\mathbb{R}$ and let $N_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

Ans The basis set for vector space of 2×2 matrix is

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(A) = N(A) \cdot A \in V.$$

$$T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_3(U) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{is zero column} \\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

Algebra of linear transformation.

Sum of linear Transformation:-

Let $T_1: U \rightarrow V$ and $T_2: U \rightarrow V$

be two transformation of U and V are two vector space over field F .

Sum of T_1 and T_2 by $T_1 + T_2: U \rightarrow V$

$$(T_1 + T_2)u = T_1(u) + T_2(u)$$

Thm Let U and V be two vector space

over field F if $T_1: U \rightarrow V$. and

$T_2: U \rightarrow V$ be two linear transformation.

Proof. The sum of T_1 and T_2

$$T_1 + T_2: U \rightarrow V,$$

$$(T_1 + T_2)u = T_1(u) + T_2(u)$$

$$\exists \text{ a vector } u \in U$$

$$u \in (T_1 + T_2)u$$

$$= T_1(u) + T_2(u)$$

$$u \in T_1(u) + T_2(u)$$

$$(T_1 + T_2)u \subseteq T_1(u) + T_2(u)$$

$$\dim(T_1 + T_2)u \leq \dim(T_1(u) + T_2(u))$$

$$\dim(T_1 + T_2)u \leq \dim T_1(u) + \dim T_2(u)$$

$$\rho(T_1 + T_2) \leq \rho(T_1) + \rho(T_2)$$

Composition of two linear Transformation

Let U, V, W three vector

space over field F .

Let $T_1: U \rightarrow V$ and $T_2: V \rightarrow W$

$$(T_2 \circ T_1)u = T_2(T_1(u))$$

Thm Let $T_1: U \rightarrow V$ and $T_2: V \rightarrow W$,

be two linear transformation Then

① If T_1 is one one then $T_2 \circ T_1$

is one one, if T_2 is onto, then $T_2 \circ T_1$ is onto

$$T_1(u) = T_1(u)$$

$$T_2(T_1(u)) = T_2(T_1(u))$$

$$T_2(T_1(u)) = T_2(T_1(u))$$

T_1 is one one

② Let $u \in U$ be an arbitrary element,

Since $T_2: V \rightarrow W$ is onto,

$$T_2(T_1(u)) \in W$$

$$T_2(T_1(u)) = W$$

T_2 is surjective of W .

T_2 is onto

T_2 is onto

Ques Illustrate with help of an ex. that 3 linear transformations $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $T_2 T_1 \neq 0$ $T_1 T_2 \neq 0$

Solⁿ let us $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
 $T_1(xy) = (0, xy)$

$$T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T_2(xy) = (0, xy)$$

$T_1 T_2$ and $T_2 T_1$ both are defined

$$T_2 T_1(xy) = T_2(0, xy)$$

$$= (0, 0)$$

$$T_2 T_1 = 0$$

$$T_1 T_2(xy) = T_1(0, xy) = T_1(0)$$

$$= (0, xy) \neq 0(xy)$$

$$T_1 T_2 \neq 0$$

Ques Let $T_1: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ QSt $T_1(xy, z) = (x, y)$

$$T_2: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T_2(xy, z) = (2x + z, xy)$$

$$T_3: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T_3(xy, z) = (x, y)$$

$T_1 T_2 T_3$ are linear transformations

$$2T_1 - 3T_2 + 4T_3$$

$$(2T_1 - 3T_2 + 4T_3)(xy, z) =$$

$$2T_1(xy, z) - 3T_2(xy, z) + 4T_3(xy, z)$$

$$= 2(x + z, xy) - 3(2x + z, xy) + 4(x, y)$$

$$= (-4x + 10y + z, 3x - y)$$

image of $(-1, 0, 3)$ under the map is given by

$$(2T_1 - 3T_2 + 4T_3)(-1, 0, 3) =$$

$$(-4(-1) + 10(0) + 3, 3(-1) - 0) = (1, -3)$$

$T_1 T_2 T_3$ are linearly independent

$$(aT_1 + bT_2 + cT_3)(1, 0, 0) = b(1, 0, 0)$$

$$(aT_1 + bT_2 + cT_3)(1, 0, 0) = (0, 0)$$

$$a(1, 0) + b(2, 1) + c(1, 0) = (0, 0)$$

$$a + 2b + c = 0$$

$$b = 0$$

Again operating this transformation

$$(aT_1 + bT_2 + cT_3)(0, 1, 0) = b(0, 1, 0)$$

$$(aT_1 + bT_2 + cT_3)(0, 1, 0) = (0, 0)$$

$$(a + 2c, b) = (0, 0)$$

$$a + b + c = 0$$

$$a + b = 0$$

$$a + b + c = 0$$

$$T_1, T_2, T_3 \text{ are } \mathbb{R}$$

Singular and non singular

A linear transformation $T: U \rightarrow V$ is said to be non singular if

only $0 \in U$ maps into $0 \in V$
or $0 \in U$ is only vector where
image under T is 0 .

Singular - A linear Transformation
 $T: U \rightarrow V$ is said to be non singular

if only $0 \in U$ maps into $0 \in V$
 $T: U \rightarrow V$ is said to be non singular
if for $u \in U$ $T(u) = 0 \Rightarrow u = 0$

Singular $\rightarrow$ A linear transformation
 $T: U \rightarrow V$ is said to be singular if
there exists $u \in U$ s.t. $T(u) = 0$

Thm A linear transformation $T: U \rightarrow V$

is non singular iff T is one to one

Proof The non singular

0 is only vector $T(u) = 0$

$$T(u) = T(v)$$

$$T(u) - T(v) = 0$$

$$T(u - v) = 0$$

$$u - v = 0$$

$$u = v$$

T is one to one

Conversely - let T is not one to one
then image of atleast one element
of U

$$\text{let } u \in N(T)$$

$$T(u) = 0$$

$$u = 0$$

$$N(T) = \{0\}$$

T is non singular

Thm A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$
given by $Y = AX$ is non singular
if and only if the matrix A
of transformation is non singular

Proof matrix A of Transformation is non singular

$$|A| \neq 0 \text{ if } A^{-1} \text{ exists}$$

$$\text{let } x_1, x_2 \in \mathbb{R}^n$$

$$Ax_1 = Ax_2$$

$$A(x_1 - x_2) = 0$$

$$A(x_1 - x_2) = 0$$

$$A^{-1} A(x_1 - x_2) = A^{-1} 0$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

T is singular. Conversely let the transformation be non singular

let matrix A be singular

$$Ax = 0$$

$$Ax = 0$$

$$Ax = 0$$

image of X and 0 is same

$$T(x) = T(0)$$

$$x = 0$$

which is a contradiction as $x \neq 0$

A is non singular.

Ques Show that linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$T(x, y, z) = (x, y, z+t, 0)$$

singular transformation

for any arbitrary element

$$T(x, y, z) = (x, y, z+t, 0)$$

$$(x, y, z+t, 0) = (x, y, z, 0)$$

$$x = 0, y = 0, z = t$$

$$x = 0, y = 0, z = t$$

$$T(0, 0, t) = (0, 0, t, 0)$$

$$T(0, 0, t) = (0, 0, t, 0)$$

Ques

$$\text{between linear transformation } Y = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} X$$

(a)

it is singular

(b)

The image of 1st vector

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, Y_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A (c)

The given transformation

$$Y_2 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} X$$

$$Y_1 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

The transformation is singular

(d)

$$Y = AX$$

$$Y_1 = AX_1 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$Y_3 = AX_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -2 \end{bmatrix} Y_2 = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$Y_3 = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

y_1, y_2, y_3 are linearly independent.
Let $a_1(2, 1, 2) + a_2(3, 2, 9) + a_3(8, 11, 11) = 0$

$$2a_1 + 3a_2 + 8a_3 = 0$$

$$a_1 + 2a_2 + 11a_3 = 0$$

$$6a_1 + 9a_2 + 19a_3 = 0$$

The system is $AX = 0$ where $AX = 0$

$$B = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 11 \\ 6 & 9 & 19 \end{bmatrix}$$

$$B \sim \begin{bmatrix} 2 & 3 & 8 \\ 0 & 0 & 2 \\ 0 & 0 & 10 \end{bmatrix}$$

$$B \sim \begin{bmatrix} 2 & 3 & 8 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank } B = 2 < 3.$$

(y_1, y_2, y_3) is set of image which is linearly dependent.

Invertible linear Transformation

A linear transformation $T: U \rightarrow V$ is said to be invertible if it is

one-one onto and inverse of T is

$T^{-1}: V \rightarrow U$ s.t. $T^{-1}(T(u)) = u$ if and only if $T(u) = v$.

Ques If $T_1: U \rightarrow V$ and $T_2: V \rightarrow W$ are linear transformations then $T_2 T_1$ is invertible and $(T_2 T_1)^{-1} = T_1^{-1} T_2^{-1}$.

Proof T_2 and T_1 are invertible, T_1^{-1} exist

$$T_2^{-1} T_1^{-1} (T_2 T_1) = T_2^{-1} T_1^{-1} T_2 T_1 = I$$

$$T_1^{-1} T_2^{-1} (T_2 T_1) = T_1^{-1} T_2^{-1} T_2 T_1 = I$$

$$T_2^{-1} T_1^{-1} = (T_2 T_1)^{-1}$$

$$T_2^{-1} T_1^{-1} \text{ is } I$$

$$T_2^{-1} T_1^{-1} : U \rightarrow W$$

If $u \in U$, T_2 is onto $T_2^{-1} T_2(u) = u$

$$T_1^{-1}(u) = u$$

$$T_2^{-1} T_1^{-1} (T_2 T_1(u)) = T_2^{-1} T_1^{-1} T_2 T_1(u) = u$$

$$T_2^{-1} T_1^{-1} (u) = u$$

$$T_2^{-1} T_1^{-1} \text{ is onto}$$

$$(T_1 T_2) (x_1^T x_2^T) = T_2 (T_1^T T_2^T x_1^T x_2^T)$$

$$= T_2 (T_1^T T_2^T) T_1^T x_1^T$$

$$= T_2 (T_1^T T_2^T) = T_2 T_1^T = I$$

$$(T_1^T T_2^T) (T_1 T_2) = T_1^T (T_2^T T_1 T_2) T_1$$

$$= T_1^T (I T_1) = T_1^T T_1 = I$$

$$(T_2 T_1)^T = T_1^T T_2^T$$

Ques. If $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear operator defined by $T(xyz) = (xyz, xyz)$ show that T is invertible or find T^{-1}

Ans.

T is one one

Let $u, v \in \mathbb{R}^3$.

$u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$

$$T(u) = T(v)$$

$$T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$$

$$(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$x_1 + x_2 = x_1 + x_2, y_1 + y_2 = y_1 + y_2, z_1 + z_2 = z_1 + z_2$$

$$x_1 = x_2, y_1 = y_2, z_1 = z_2$$

$$x_1 = x_2, y_1 = y_2, z_1 = z_2$$

$$(x_1, y_1, z_1) = (x_2, y_2, z_2)$$

$$u = v$$

T is 1-1

T is onto

for $(xyz) \in \mathbb{R}^3$ T is onto

$$T(xyz) = (xyz, xyz)$$

$$x + t = x, y + t = y, z + t = z$$

$$x = x, y = y, z = z$$

$$xyz \in \mathbb{R}^3$$

$$xyz \in \mathbb{R}^3$$

$$xyz \in \mathbb{R}^3$$

$$T(xyz) = T(xyz)$$

T is an auto

$$T(xyz) = (xyz)$$

$$T^T(xyz) = (xyz) = (x_1, y_1, z_1)$$

Matrix of a linear transformation -

Co-ordinate vector $\rightarrow$ If $B = \{u_1, \dots, u_n\}$ is a Basis for vector space V (V.E.), then a vector $v \in V$ can be represented as $a_1 u_1 + \dots + a_n u_n$

Matrix of a linear transformation Relative to ordered Basis $B = \{u_1, \dots, u_n\}$
 Let $T: V \rightarrow V$ be linear transformation

$$T(u_1) = a_{11}u_1 + \dots + a_{n1}u_n$$

$$T(u_n) = a_{1n}u_1 + \dots + a_{nn}u_n$$

The coefficient matrix

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

$$(T(B), B) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Matrix of linear transformation with standard Basis

$$T(u_1 + u_2) = T(u_1) + T(u_2)$$

$$T(u_1) = A T(u_1)$$

The standard Basis

$$e_1 = (1 \ 0 \ \dots \ 0)$$

$$e_1 = (0 \ 0 \ \dots \ 1 \ 0)$$

$$e_n = (0 \ 0 \ \dots \ 0 \ 1)$$

Let $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$

$$u = a_1 e_1 + \dots + a_n e_n$$

$$T(u) = T(a_1 e_1 + \dots + a_n e_n) = a_1 T(e_1) + \dots + a_n T(e_n)$$

$$T(e_1) = (a_{11}, a_{21}, \dots, a_{n1})$$

$$T(u) = a_1 (a_{11}, a_{21}, \dots, a_{n1}) + \dots + a_n (a_{1n}, a_{2n}, \dots, a_{nn})$$

$$y_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$y_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$

$$Y_2 = AX$$

$$Y_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$y_1 = b_{11}x_1 + b_{12}x_2 + b_{13}x_3$$

$$y_2 = b_{21}x_1 + b_{22}x_2 + b_{23}x_3$$

Ques Find matrix representing transformation

T: R³ → R⁴ defined by

T(x₁, x₂, x₃) = (x₁ + x₂, 2x₁ + x₂ + x₃, 2x₂ + x₃) relative to standard Basis of R³

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(1, 0, 0) = (1, 2, 0)$$

$$T(0, 1, 0) = (1, 1, 2)$$

$$T(0, 0, 1) = (0, 0, 1)$$

$$e_1 = (1, 0, 0) \quad e_2 = (0, 1, 0) \quad e_3 = (0, 0, 1)$$

$$e_4 = (0, 0, 0, 1)$$

$$T(1, 0, 0) = (1, 2, 0, 0) = 1e_1 + 2e_2 + 0e_3 + 0e_4$$

$$T(0, 1, 0) = (1, 1, 2, 0) = 1e_1 + 1e_2 + 2e_3 + 0e_4$$

$$T(0, 0, 1) = (0, 0, 1, 1) = 0e_1 + 0e_2 + 1e_3 + 1e_4$$

$$[T(B, B')] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Ques Find matrix representing transformation

T: R³ → R³ given by

$$T(x_1, x_2, x_3) = (3x_1 - 2x_2, 2x_1 + 4x_2, 2x_3)$$

relative to standard Basis of R³ and R³

A

The ordered standard Basis of R³ is B = {e₁, e₂, e₃} and R³ is B' = {e₁, e₂, e₃}

$$T(e_1, e_2, e_3) = (3e_1 - 2e_2, 2e_1 + 4e_2, 2e_3)$$

$$T(1, 0, 0) = (3, 2, 0) = 3e_1 + 2e_2 + 0e_3$$

$$T(0, 1, 0) = (-2, 4, 0) = -2e_1 + 4e_2 + 0e_3$$

$$T(0, 0, 1) = (0, 0, 2) = 0e_1 + 0e_2 + 2e_3$$

$$T(1, 0, 0) = (3, 2, 0) = 3(1, 0, 0) + 2(0, 1, 0) + 0(0, 0, 1)$$

$$T(0, 1, 0) = (-2, 4, 0) = -2(1, 0, 0) + 4(0, 1, 0) + 0(0, 0, 1)$$

matrix = $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

Ques If $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is $\mathbb{R}^3 \rightarrow \mathbb{R}^4$ are two linear transformation show that the function $T+S: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ by

$f(x) = T(x) + S(x)$ is a linear transformation. If $A = (a_{ij})$ $B = (b_{ij})$ be two matrix associated to T and S respectively. find the matrix $T+S$.
let $u, v \in \mathbb{R}^3$ be arbitrary

$$\begin{aligned} (T+S)(u+v) &= T(u+v) + S(u+v) \\ &= T(u) + T(v) + S(u) + S(v) \\ &= T(u) + S(u) + T(v) + S(v) \\ &= (T+S)u + (T+S)v \end{aligned}$$

$$\begin{aligned} (T+S)(au) &= T(au) + S(au) \\ &= aT(u) + aS(u) \\ &= a(T(u) + S(u)) \\ &= a(T+S)u \end{aligned}$$

$T+S$ is linear transformation. If $(a_{ij})_{3 \times 4}$ is a matrix

$$T(x) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$S(x) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$T(x) + S(x) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$(T+S)x = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

To prove let U and V be finite dimensional vector spaces and let $B = (u_1, u_2, \dots, u_n)$ and $B' = (v_1, \dots, v_m)$ be ordered bases for U and V . If T is a linear transformation from U into V for all $u \in U$

$$(T(u), B') = (T(u), B') (u, B)$$

Proof $(T(u), B') = (T(u), B') (u, B)$

$$(T(u), B') = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$(u, B) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{aligned} T(u) &= T(b_1 u_1 + \dots + b_n u_n) \\ &= b_1 T(u_1) + \dots + b_n T(u_n) \\ &= \underbrace{(b_1 u_1 + \dots + b_n u_n)}_{(u)} \underbrace{(T(u_1) \dots T(u_n))}_{(u_1 \dots u_n)} \end{aligned}$$

$$T(u, B') = \begin{bmatrix} a_{11} b_1 & \dots & a_{1n} b_n \\ \vdots & & \vdots \\ a_{m1} b_1 & \dots & a_{mn} b_n \end{bmatrix}$$

$$T(u, B') = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$T(u, B') = [T; B, B'] (u, B)$$

Que. Let $T: R^3 \rightarrow R^2$ be linear

Transformation

$T(u_1, u_2, u_3) = (2x+4y-z, 3x-2y+2z)$

Find matrix of T w.r.t. to

Basis $B_1 = \{(1, 1, 1)^T, (1, 1, 0)^T, (1, 0, 0)^T\}$

$B_2 = \{(1, 3)^T, (1, 4)^T\}$ of R^2

Ans

$$[T; B_1, B_2] [u, B_1] = [T; B_1, B_2]$$

The given linear transformation is

$$T(x, y, z) = (2x+4y-z, 3x-2y+2z)$$

$$T(u_1) = (2, 3)$$

$$T(u_2) = (2, 1)$$

$$T(u_3) = (2, 3)$$

$$\begin{aligned} (2, 3) &= a(1, 3) + b(1, 4) = (a+b, 3a+4b) \\ (2, 1) &= c(1, 3) + d(1, 4) = (c+d, 3c+4d) \\ (2, 3) &= e(1, 3) + f(1, 4) = (e+f, 3e+4f) \end{aligned}$$

$$\begin{aligned} a+b &= 2, \quad 3a+4b = 3 \\ 3c+d &= 1, \quad e+f = 2, \quad 3e+4f = 3 \end{aligned}$$

$$a=3, \quad b=-1, \quad c=1, \quad d=2, \quad e=2, \quad f=-3$$

$$\begin{aligned} T(u_1) &= (2, 3) = 3(1, 3) + (-1)(1, 4) \\ T(u_2) &= (2, 1) = 1(1, 3) + 2(1, 4) \\ T(u_3) &= (2, 3) = 5(1, 3) - 3(1, 4) \end{aligned}$$

$$[T; B_1, B_2] = \begin{bmatrix} 3 & 1 & 5 \\ -1 & -8 & -3 \end{bmatrix}$$

$$A(1, 3) = a_1, \quad C(1, 1) = a_2, \quad D(1, 1) = a_3$$

$$B = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$$

$$a_1 = c \quad b_1 = b - c \quad c_1 = a - b$$

$$u_2 = (1 \ 1 \ 1) + (b - c) (1 \ 1 \ 0) + (a - b) (1 \ 0 \ 0) \quad (2)$$

$$(u_1, u_2) = \begin{pmatrix} c \\ b - c \\ a - b \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$(1 \ 1 \ 1) \cdot (b_2)$$

$$T(u_1) = c T(1 \ 1 \ 1) + (b - c) T(1 \ 1 \ 0) +$$

$$= b(1 \ 3 \ 1) + (c - 1)(1 \ 4) + (b - 1)$$

$$(1 \ 1 \ 1) - 8(1 \ 4) + (a + b)(1 \ 1 \ 3) - 3$$

$$= (5a + 6b - 8c)(1 \ 1 \ 3) + (3a - 5b + 7c)(1 \ 1 \ 3)$$

$$(1 \ 1 \ 3) \cdot (u_1, u_2) = \begin{pmatrix} 3 & 1 & 5 \\ -1 & -8 & -3 \end{pmatrix} \cdot \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$

$$= \begin{pmatrix} 3a + 6b - 8c \\ 3a - 5b + 7c \end{pmatrix}$$

$$= (1 \ 1 \ 1, u_2)$$

Change of Basis
Transition matrix
let $B_1 = (u_1, \dots, u_m)$ be Basis

$$u_1 = a_{11} u_1 + \dots + a_{1m} u_m$$

$$u_2 = a_{21} u_1 + \dots + a_{2m} u_m$$

$$u_m = a_{m1} u_1 + \dots + a_{mm} u_m$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

$$P = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

T_B

If $B_1 = B_X$ is a linear transformation from R_n into R_n and $B_2 = B_Y$ is same linear transformation with respect to another Basis B_2 then the matrix A and B are similar

But $Y_1 = BY_1$ is a linear combination of B_1 vectors B_1

a non-singular matrix

$$Y_2 = P \times B_1$$

$$Y_3 = P \times B_2$$

$$P^{-1} \times B_1 =$$

$$P^{-1} Y_2 = B_1$$

$$P^{-1} (B_2 \times B_1)$$

$$Y_3 = P \times B_1$$

$$= P (B \times B_1)$$

$$= (P \times B) \times B_1$$

$$= (P \times B) (P^T \times B_2)$$

$$= (P \times P^T) (B \times B_2)$$

$$Y_3 = B \times B_2 \quad B = P \times P^T$$

$$B = P \times P^T$$

Given Y_1 and Y_2 the co-ordinate of a vector X_1 with B_1

$$B_1: Y_1 = (110) \quad Y_2 = (101) \quad Y_3 = (111)$$

$$B_2: Y_2 = (112) \quad Y_3 = (201) \quad Y_4 = (122)$$

find matrix P $X_{B_2} = P \times B_1$

$$B_1$$

$$A = (Y_1 Y_2 Y_3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B = (Z_1 Z_2 Z_3) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}$$

Applying suitable elementary operations on $B = AB$

$$B^T = \frac{1}{3} \begin{bmatrix} 2 & -3 & 2 \\ 2 & 0 & -1 \\ -3 & 3 & 0 \end{bmatrix}$$

$$P = B^T A = \frac{1}{3} \begin{bmatrix} 2 & -3 & 2 \\ 2 & 0 & -1 \\ -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$T \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 0 & -3 & 0 \end{pmatrix}$$

Dual space

Defⁿ → let V be a vector space over a field F . Then the set V^* of all linear functional on V is also a vector space over field F and this vector space V^* is dual space.

$$V^* = \{f: V \rightarrow F\}$$

Th^m let U and V be two vector space over field F . let $L(U, V)$ denote the set of all linear transformation from U to V . Then $L(U, V)$ forms vector space over F .

$$(T_1 + T_2)u = T_1(u) + T_2(u)$$

$$aT_1(u) = aT_1(u)$$

Proof $L(U, V)$ is an abelian group as

$$(T_1 + T_2)u = T_1(u) + T_2(u)$$

$$= T_2(u) + T_1(u)$$

$$= (T_2 + T_1)u$$

$$(T_1 + T_2) = (T_2 + T_1)$$

$$(3) \text{ let } T_1, T_2, T_3 \in L(U, V)$$

$$T_1 + (T_2 + T_3)u = T_1u + (T_2 + T_3)u$$

$$\begin{aligned} T(u) + T_2(u) + T_3(u) \\ = T_1(u) + T_2(u) + T_3(u) \\ = (T_1 + T_2 + T_3)(u) \\ = (I_1 + I_2 + I_3)(u) \end{aligned}$$

③ $0: U \rightarrow V$ $0(u) = 0$
 $(T+0)(u) = T(u) + 0(u) = T(u)$
 $= (T+0)(u)$

④ $(-T): U \rightarrow V$

$$\begin{aligned} (-T+T)(u) &= -T(u) + T(u) \\ &= 0 \end{aligned}$$

$$-T+T=0 = T+(-T)$$

Thm $\{u_1, u_2, \dots, u_n\}$ be basis of V
 $\{f_1, f_2, \dots, f_n\}$ be dual basis

① Any vector $u \in V$ $u = \sum_{i=1}^n f_i(u) u_i$

② Any linear function $f \in V^*$

$$f = \sum_{i=1}^n f_i(u) f_i$$

Proof $\{u_1, \dots, u_n\}$ is a basis of V and $\{f_1, f_2, \dots, f_n\}$ is dual basis of u_i

$$f(u) = \sum_{i=1}^n f_i(u) f_i(u)$$

① $u = \sum_{i=1}^n f_i(u) u_i$

$$u = \sum_{i=1}^n q_i u_i \text{ --- given}$$

operating linear functional f_i on both side

$$f(u) = f_i \left(\sum_{j=1}^n q_j u_j \right)$$

$$\sum_{j=1}^n q_j f_i(u_j) = \sum_{j=1}^n \delta_{ij} q_j = q_i$$

$$f(u) = \sum_{i=1}^n q_i f_i(u)$$

$$u = \sum_{i=1}^n f_i(u) u_i$$

② $\{f_1, f_2, \dots, f_n\}$ is a basis for V^*

$$f_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = f_2 \begin{pmatrix} 1+1 \\ 1 \end{pmatrix} = -b_1 + b_2 + b_3 = 0$$

$$f_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = f_2 \begin{pmatrix} 1+1 \\ 1+1 \end{pmatrix} = b_1 - b_2 + b_3 = 1$$

$$f_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = f_3 \begin{pmatrix} 1+1 \\ 1+1 \end{pmatrix} = b_1 + b_2 - b_3 = 0$$

$$\frac{b_1}{-1+1} = \frac{b_2}{-1+1} = \frac{b_3}{-1+1}$$

$$\frac{b_1}{1} = \frac{b_2}{0} = \frac{b_3}{1}$$

$$b_1 = k \quad b_2 = 0 \quad b_3 = k$$

$$R = 0 + k = k \quad Y = k$$

$$b_1 = 1/2 \quad b_2 = 0 \quad b_3 = 1/2$$

$$f_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2} + 0 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$f_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + 1 + 1 = 3$$

$$f_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = f_3 \begin{pmatrix} 1+1 \\ 1+1 \end{pmatrix} = 4 + 4 + 4 = 12$$

$$f_3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = f_3 \begin{pmatrix} 1+1+1 \\ 1+1 \end{pmatrix} = 6 + 4 + 4 = 14$$

$$C_1 = Y_1 \quad C_2 = Y_2 \quad C_3 = 0$$

$$f_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 1 + 4 + 4 = 9$$

The Dual Basis is (f_1, f_2, f_3)

$$= \frac{1}{2} (y+2) + \frac{1}{2} (x+2) - \frac{1}{2} (x+y)$$

Ques

A

Let f be a linear functional on $\mathbb{R}^2$ defined by $f(1,3) = -4$ and $f(2,1) = 7$
 Let $f(x,y) = a_1x + a_2y$ where $(x,y) \in \mathbb{R}^2$
 $f(1,3) = -4 \Rightarrow a_1 + 3a_2 = -4$
 $f(2,1) = 7 \Rightarrow 2a_1 + a_2 = 7$
 Solving
 $a_2 = -3, a_1 = 5$

$$f(x,y) = 5x - 3y$$

Thm

Let V be a finite dimensional vector space over the field F if $u \in V$ is arbitrary, the functional f on V is

$Lu(f) = f(u)$ where $f \in V^*$ is a linear functional on V and $u \in V$

Also prove that the function $u \mapsto Lu$ is an isomorphism of V onto V^* if $u \in V$ then $f(u)$ is a unique element of F

$$Lu(f) = f(u) \Rightarrow (a_1 + 3a_2)u$$

Amphilateral of an Amphilateral-

Let W be a subset of a vector space V . Then AW is a subspace of V .

$$A(AW) = \{w \in V \mid w = Af \text{ for } f \in AW\}$$

Then AW and W are subsets of vector space V . If W is a subspace then $AW \subseteq W$.

Self

Let $f \in AW$ be arbitrary.

$$f \in W, \quad w \in W, \quad f \in W \rightarrow w \in W$$

$$f \in AW$$

$$AW \subseteq AW$$

$$AW \subseteq AW$$

Then let f, g be three linear functions on $\mathbb{R}^4$

$$f(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 3x_3 + 4x_4$$

$$g(x_1, x_2, x_3, x_4) = -2x_1 + 4x_3 + 3x_4$$

$$f(x) = 0 \quad 12123$$

$$f(x_1, x_2, x_3, x_4) = 0 \quad 12123$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$-2x_1 + 4x_3 + 3x_4 = 0$$

System of E_3^n can be expressed

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Applying elementary rows

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 + 2x_3 = 0$$

$$(x_1, x_2, x_3, x_4) = (-2x_3, 0, x_3, 0)$$

Eigen value and Eigen vector.

Eigen value of $T \rightarrow$ Let T be a linear operator on a vector space $V(F)$ i.e.

$T: V \rightarrow V$ is a linear Transformation
 $\exists$ a non zero vector $v \in V$ such that
 $T(v) = \lambda v$ for some $\lambda \in F$.

Eigen space - Let $T: V \rightarrow V$ be a linear transformation on V over F . For each real number λ , the set of all vectors $v \in V$ st. $T(v) = \lambda v$ is used to be eigen space of T associated with eigen value λ of T .

$$V_\lambda = \{v \in V : T(v) = \lambda v\}$$

Characteristic polynomial of linear Transformation $T \rightarrow$

Let $T: V \rightarrow V$ be a linear transformation on a finite dimensional vector space over F . The characteristic polynomial of T , to be characteristic polynomial of any matrix of T .

Thm Let T be a linear operator on $V(F)$. If λ is an eigen value of T , then V_λ is a subspace of $V(F)$.

Proof Let $v, w \in V_\lambda$ be any element
 $T(v) = \lambda v$
 $T(w) = \lambda w$

Let $a, b \in F$

$$\begin{aligned} T(av + bw) &= aT(v) + bT(w) \\ &= a\lambda v + b\lambda w \\ &= \lambda(av + bw) \end{aligned}$$

$av + bw \in V_\lambda$

Thm Let A be a non matrix over a field F . Then $\lambda \in F$ is eigen value of A iff $(A - \lambda I)$ is singular i.e. $|A - \lambda I| = 0$
 Let A be an eigen value of square matrix A .

$$\begin{aligned} A - \lambda I &= 0 \\ A - \lambda I &= 0 \end{aligned}$$

$$|A - \lambda I| = 0$$

Conversely $|A - \lambda I| = 0$

$$(A - \lambda I)u = 0$$

$$(A - \lambda I)u = 0$$

$$(A - \lambda I)u = 0$$

$$Au = \lambda u$$

Ques.

Proof

If λ is an eigen value of A , then, if only one eigen value λ for which λ is eigen vector.
Let u be an eigen vector of A .
Then $\exists$ scalar λ called the eigen value as
 $Av = \lambda u$

λ is unique.

Eigen value λ .

$$Av = \lambda u \quad A^T v = \lambda u$$

$$\lambda u = \lambda u$$

$$(A - \lambda I) u = 0$$

$$u = 0$$

$$A - \lambda I = 0$$

$$A = \lambda I$$

Ques

Ques

Let A be a real matrix of type $n \times n$ and λ be its real eigen value. Show that $\exists$ an eigen vector x of A corresponding to eigen value λ .
Let x be also real
Let x be. Eigen vector of A
 $Ax = \lambda x$

$$Ax = \lambda x$$

If x is real

If λ is not real x is not real.

λ and x are real column vectors

$$Ax = \lambda x$$

$$A(x + \lambda x) = A(x + \lambda x)$$

$$A(x + \lambda x) = A(x + \lambda x)$$

$$Ax = \lambda x$$

$$Ax = \lambda x$$

λ is real eigen vector of A .

λ is real eigen vector of A .

Similar matrices $\rightarrow$ A matrix B is said to be similar to a matrix A if $\exists$ a non singular matrix P st $B = P^{-1}AP$.

Prove that similar matrices have same characteristic polynomial

Proof

$$B = P^{-1}AP$$

$$|B - \lambda I| = |P^{-1}AP - \lambda I|$$

$$= |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}| |A - \lambda I| |P|$$

$$= |P^{-1}| |P| |A - \lambda I|$$

$$= |P^{-1}P| |A - \lambda I|$$

$$= |I| |A - \lambda I| = |A - \lambda I|$$

A and B have same characteristic

T_m

If A and B are similar matrices, and if λ is an eigen vector of A corresponding to eigen

Value λ then PX is an eigen vector of B corresponding to eigenvalue λ and B are similar matrices. $\exists$ a non singular matrix P s.t. $B = P \lambda P^{-1}$.

$$\begin{aligned} B - \lambda I &= P \lambda P^{-1} - \lambda I \\ &= P \lambda P^{-1} - \lambda I \parallel P \\ &= P \lambda P^{-1} \parallel A - \lambda I \\ &= P \lambda P^{-1} \parallel A - \lambda I \\ &= I \parallel A - \lambda I = A - \lambda I \end{aligned}$$

Thm Let T be a linear operator on finite dimensional vector space V . Let $\lambda_1, \dots, \lambda_r$ be distinct eigen values of T and $v_1, \dots, v_r$ be corresponding eigen vectors of T . Then $(v_1, \dots, v_r)$ is linearly ind.

Proof The independence of $(v_1, \dots, v_r)$ by showing that dependence would imply that two of eigen values were equal $v_1, v_2, \dots, v_r$ is dependent. Let n be largest integer for $v_1, \dots, v_n$ is independent. $v_{n+1} = q_1 v_1 + \dots + q_n v_n$ on $T v_{n+1} = q_1 T v_1 + \dots + q_n T v_n$ on $T v_{n+1} = q_1 \lambda_1 v_1 + \dots + q_n \lambda_n v_n$

$$\begin{aligned} &= \lambda_{n+1} (q_1 v_1 + \dots + q_n v_n) \\ &= q_1 \lambda_{n+1} v_1 + \dots + q_n \lambda_{n+1} v_n \end{aligned}$$

$$\begin{aligned} 0 &= q_1 (\lambda_{n+1} - \lambda_1) v_1 + \dots + q_n (\lambda_{n+1} - \lambda_n) v_n \\ v_1, \dots, v_n &\text{ are l.i.} \\ q_1 (\lambda_{n+1} - \lambda_1) &= 0 \text{ and } (\lambda_{n+1} - \lambda_n) = 0. \\ \lambda_1 &= \lambda_{n+1} \\ \text{all eigen values are distinct.} \\ (v_1, \dots, v_r) &\text{ cannot be dependent and is set.} \end{aligned}$$

Diagonalisation - A linear operator T on a finite dimensional vector space is said to be diagonalisable if $\exists$ an ordered basis B of V s.t. matrix of T wrt to B is a diagonal matrix.

Diagonalizable matrix - Let A be an matrix A is diagonalizable if $\exists$ a norm singular matrix P s.t. $A = P D P^{-1}$ D is diagonal matrix.

Ex: Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given a field $\mathbb{R}$, the linear operator s.t. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ wrt matrix of T wrt to the $\{e_1, e_2, e_3\}$

Standard ordered Basis Debate
Eigen values and Eigen vectors

Solⁿ

$$A = \begin{bmatrix} -9 & 4 & 9 \\ -8 & 3 & 4 \\ -16 & 8 & 4 \end{bmatrix} \text{ is a matrix}$$

used Standard Basis (100)
(0 10) (00 1)

The corresponding Eigen is

$$(A - \lambda I) = \begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix} = 0$$

$$C_1 + C_2 + C_3$$

$$-(1+\lambda) \begin{bmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{bmatrix} = 0$$

$$-(1+\lambda) \begin{bmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{bmatrix} = 0$$

Expanding by first column
 $-(1+\lambda) (1-\lambda) (3-\lambda) - 0 = 0$

$$\lambda = -1, -1, 3$$

If λ is the Eigen value do
Eigen value λ

$$AX = \lambda X$$

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

when $\lambda = -1$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1(-1) \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3(-2) \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y + z = 0$$

$$y = 1, z = 1$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3$$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 (-1) \begin{pmatrix} -4 & 4 & 0 \\ -8 & 0 & 4 \\ -8 & 8 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_3 (-2) \begin{pmatrix} -4 & 4 & 0 \\ -8 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x + 4y = 0, -8x + 4z = 0$$

$$x = y, z = 2x$$

$$x = 1, y = 1, z = 2$$

$$x = 2, y = 1, z = 2 \text{ are Eigen vectors}$$

Ques Show that matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable over $\mathbb{C}$ field

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

The characteristic eqn is

$$(1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

$$(1-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 1$$

Distinct Eigen value

If $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is Eigen vector

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0x + 2y = 0$$

$$0x + 0y = 0$$

$$x = 1, y = 0 \Rightarrow X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Minimal Polynomial

Let T be a linear operator on a finite dimensional vector space V

$T: V \rightarrow V$ is a linear transformation

m_T is a monic polynomial

m_T annihilates T $m_T(T) = 0$

all polynomials which annihilate T

m_T is of lowest degree

T^m

The characteristic polynomial of an operator T have same irreducible

factor 1

Let $m(t)$ be minimal polynomial for T
 let α be a scalar. α is an eigenvalue of T
 $m(\alpha) = 0$.

$$m(t) = (t - \alpha)^k \quad q(t)$$

$q(t)$ is a polynomial $q(t) \in \mathbb{R}[t]$
 $\deg q(t) < \deg m(t)$
 $q(T) \neq 0$.

$$u \in V \quad q(T)u \neq 0 \quad u \neq 0$$

$$t = 2 \quad q(T)u \neq 0$$

$$m(T)u = 0$$

$$(T - \alpha I)^k q(T)u = 0$$

$$(T - \alpha I)^k \neq 0 \quad t \neq 0$$

α is an eigen value of T

α is a root of characteristic equation of T

Conversely - let α be an eigen value of T
 if α is non zero root of $p(t)$

$$T u = \alpha u$$

$$m(T)u = m(\alpha)u$$

$$m(T)u = 0 \quad u \neq 0$$

$$m(\alpha) = 0$$

α is a root of minimal polynomial $m(t)$

Ques -

let V be the vector space of functions which have $(\sin \theta, \cos \theta)$ as a basis.
 let D be differential operator.
 Then find characteristic polynomial for D .

A -

The matrix of differential operator D

$$B = (\sin \theta, \cos \theta)$$

$$D(\sin \theta) = \cos \theta = 0 \cdot \sin \theta + 1 \cdot \cos \theta$$

$$D(\cos \theta) = -\sin \theta = -1 \cdot \sin \theta + 0 \cdot \cos \theta$$

$$D = A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the characteristic polynomial for

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

Inner product space

Let V be a vector space over a field F . Let $u, v, w \in V$ be arbitrary. The vector space V

- (1) $\langle u, v \rangle = \overline{\langle v, u \rangle}$
- (2) $\langle u, u \rangle \geq 0$ iff $u \neq 0$
- (3) $\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$

Let $u = (q_1, \dots, q_n)$ and $v = (b_1, \dots, b_n)$ be arbitrary members of C^n . Define $\langle u, v \rangle = u_1 \overline{v_1} + \dots + u_n \overline{v_n}$. Show that C^n is an inner product space.

Ans. all three axioms

$$(1) \langle u, v \rangle = \overline{\langle v, u \rangle} \\ = \overline{v_1 \overline{q_1} + \dots + v_n \overline{q_n}} \\ = \overline{v_1} q_1 + \dots + \overline{v_n} q_n \\ = \langle v, u \rangle$$

$$(2) \langle u, u \rangle = q_1 \overline{q_1} + \dots + q_n \overline{q_n} \\ = |q_1|^2 + \dots + |q_n|^2 \\ \geq 0$$

$$\langle u, u \rangle = 0 \iff |q_1|^2 + \dots + |q_n|^2 = 0 \\ \iff q_1 = \dots = q_n = 0 \\ \iff u = 0$$

$$(3) \langle au + bv, w \rangle = \langle aq_1 + bq_1, q_1 + \dots + q_n \rangle \\ = a \langle q_1, q_1 + \dots + q_n \rangle + b \langle q_1, q_1 + \dots + q_n \rangle \\ = a \langle u, w \rangle + b \langle v, w \rangle$$

Q13

Let V be a vector space over C of all complex valued continuous function on interval (a, b) . Define inner product $\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$. Prove that V is an inner product space.

$$(1) \langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt \\ = \overline{\int_a^b g(t) \overline{f(t)} dt} = \overline{\langle g, f \rangle}$$

$$\langle f, f \rangle = \int_a^b |f(t)|^2 dt \geq 0$$

$$(2) \langle f, f \rangle = 0 \iff \int_a^b |f(t)|^2 dt = 0 \iff f(t) = 0$$

$$\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle$$

$$(3) \langle af + bg, h \rangle = \int_a^b (af + bg)(t) \overline{h(t)} dt \\ = a \int_a^b f(t) \overline{h(t)} dt + b \int_a^b g(t) \overline{h(t)} dt \\ = a \langle f, h \rangle + b \langle g, h \rangle$$

Norm of a vector $\rightarrow$
Let V be an inner product space. For any $u \in V$ the norm $\|u\|$ is defined by $\|u\| = \sqrt{\langle u, u \rangle}$

Ques find the norm of vector $u = (2, -3, 1)$

Ans and normalize this vector

$$u = (2, -3, 1)$$

$$\|u\|^2 = \langle u, u \rangle = 2^2 + (-3)^2 + 1^2 = 14$$

$$\|u\| = \sqrt{14} = 7$$

$$\text{Normalized vector } \hat{u} = \frac{u}{\|u\|} = \left(\frac{2}{7}, -\frac{3}{7}, \frac{1}{7} \right)$$

Ques

Ans

To prove that $\|au\| = |a| \|u\|$ for all $a \in \mathbb{R}$ and $u \in V$

$$\|au\|^2 = \langle au, au \rangle = a^2 \langle u, u \rangle = a^2 \|u\|^2$$

$$\|au\| = |a| \|u\|$$

Ques

Ans

Prove that u is always unit vector

$$\left\| \frac{u}{\|u\|} \right\|^2 = \left\langle \frac{u}{\|u\|}, \frac{u}{\|u\|} \right\rangle = \frac{1}{\|u\|^2} \langle u, u \rangle = \frac{1}{\|u\|^2} \|u\|^2 = 1$$

$$\left\| \frac{u}{\|u\|} \right\| = 1$$

$\frac{u}{\|u\|}$ is always a unit vector

Cauchy Schwarz Inequality

Thm let U be an inner product space

$$|u \cdot v| \leq \|u\| \|v\|$$

$$\text{Proof if } u=0 \text{ then } |u \cdot v| = 0 \leq \|u\| \|v\|$$

$$\text{if } u \neq 0 \text{ then } \frac{u}{\|u\|} \text{ is a unit vector}$$

let $u \neq 0$. $\frac{u}{\|u\|} \neq 0$

$$\|u\|^2 = 0 \implies \langle u, u \rangle = 0 \implies u=0$$

$$u \neq 0 \implies \langle u, u \rangle > 0$$

$$\langle u, u \rangle = \left(\frac{\langle u, u \rangle}{\|u\|^2} \right) \|u\|^2$$

$$\langle u, u \rangle = \frac{\langle u, u \rangle}{\|u\|^2} \|u\|^2 = \langle u, u \rangle$$

$$\langle u, u \rangle \geq 0$$

$$\|u\|^2 = \langle u, u \rangle \geq \left(\frac{\langle u, u \rangle}{\|u\|^2} \right) \|u\|^2$$

$$= \langle u, u \rangle - \frac{\langle u, u \rangle \langle u, u \rangle}{\|u\|^2}$$

$$= \langle u, u \rangle - \frac{\langle u, u \rangle^2}{\|u\|^2} \geq 0$$

$$= \langle u, u \rangle - 0 = \langle u, u \rangle$$

$$= \langle u, u \rangle - \frac{\langle u, u \rangle^2}{\|u\|^2}$$

$$= \langle u, u \rangle - \frac{\langle u, u \rangle \langle u, u \rangle}{\|u\|^2}$$

$$= \langle u, u \rangle - \frac{\langle u, u \rangle^2}{\|u\|^2}$$

$$\frac{\|u\|^2 \|u\|^2 - \langle u, u \rangle^2}{\|u\|^2} \geq 0$$

$$\|u\|^2 \|u\|^2 - \langle u, u \rangle^2 \geq 0$$

$$\|u\| \|u\| \geq |u \cdot u|$$

Triangle inequality -
Let U be an inner product space

$$\|u+v\| \leq \|u\| + \|v\|$$

Proof

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v \rangle \\ &= \langle u, u+v \rangle + \langle v, u+v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v \rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\|u+v\| \leq \|u\| + \|v\|$$

which is Triangle inequality

Th^m

Let U be an inner product space
If $u, v \in V$ such that $\|u\| = \|v\|$
then show that $u+v$ and $u-v$ are orthogonal

Proof

Let u, v be vectors in an inner product space V

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v \rangle$$

If either of u or v is zero then $\langle u, v \rangle = 0$
If $u \neq 0, v \neq 0$ then $\frac{\langle u, v \rangle}{\|u\|\|v\|} = \cos \theta$

$$\langle u+v, u-v \rangle = \langle u, u \rangle - \langle u, v \rangle + \langle v, u \rangle - \langle v, v \rangle$$

$$\langle u+v, u-v \rangle = \|u\|^2 - \|v\|^2$$

$$\begin{aligned} &= \|u\|^2 - \|v\|^2 - \frac{\|u\|^2 - \|v\|^2}{\|u\|^2} + \frac{\|u\|^2 - \|v\|^2}{\|u\|^2} \\ &= \|u\|^2 - \|v\|^2 - \frac{\|u\|^2 - \|v\|^2}{\|u\|^2} + \frac{\|u\|^2 - \|v\|^2}{\|u\|^2} \end{aligned}$$

$$= \|u\|^2 - \|v\|^2 - \frac{\|u\|^2 - \|v\|^2}{\|u\|^2} + \frac{\|u\|^2 - \|v\|^2}{\|u\|^2}$$

$$= \frac{0}{\|u\|^2} = 0$$

$$\langle u, u \rangle = 0 \quad u=0$$

$$u - \frac{\langle u, u \rangle}{\|u\|^2} u = 0$$

$$u = \frac{\langle u, u \rangle}{\|u\|^2} u, \quad \frac{\langle u, u \rangle}{\|u\|^2} \in \mathbb{R}$$

u is scalar multiple of u

Th^m

Every inner product space is a metric space

Proof

Let V be an inner product space
Let $x, y \in V$ be arbitrary

$$d(x, y) = \|x - y\|$$

d is always non negative

$$d(x, y) \geq 0$$

$$d(x, y) = 0 \quad \text{iff } \|x - y\| = 0$$

$$\text{iff } \langle x - y, x - y \rangle = 0 \quad \text{iff } x = y$$

$$d(x, y) = 0 \quad \text{iff } x = y$$

Q

$$\|x\|^2 = \|x\|^2 - \|y\|^2 + \|y\|^2$$

$$\| \beta - \alpha \| = d(\beta, \alpha)$$

$$d(\alpha, \beta) = d(\beta, \alpha)$$

③

Let $\alpha, \beta, \gamma \in V$

$$d(\alpha, \gamma) = d(\alpha, \beta) + d(\beta, \gamma) \geq \| \alpha - \beta \| + \| \beta - \gamma \|$$

$$\geq \| \alpha - \gamma \| = d(\alpha, \gamma)$$

$$d(\alpha, \gamma) = d(\gamma, \alpha) \geq d(\alpha, \beta)$$

γ is a metric space

Normed linear space

Let V be a vector space

$$\| u \| \geq 0 \quad \| u \| \geq \| u \|$$

①

②

③

$$\| u \| \geq 0 \quad \| u \| \geq 0$$

$$\| u \| = \| u \|$$

$$\| u \| \leq \| u \| + \| u \|$$

Orthogonal vector $\rightarrow$ Let V be an inner product space A vector $u \in V$ is said to be orthogonal to $v \in V$ if $\langle u, v \rangle = 0$

Thm

Let W be a non empty subset of an inner product space V then $W^\perp$ is a subspace of V

Proof

$$0 \in W^\perp \quad W^\perp \neq \emptyset$$

Let $u_1, u_2 \in W^\perp$ and $\alpha, \beta \in \mathbb{R}$

$$\langle \alpha u_1 + \beta u_2, w \rangle = \alpha \langle u_1, w \rangle + \beta \langle u_2, w \rangle = \alpha \cdot 0 + \beta \cdot 0 = 0$$

$$\alpha u_1 + \beta u_2 \in W^\perp$$

$W^\perp \subseteq V$

Let M and N be subspace of a finite dimensional inner product space V . Then show that $(M+N)^\perp = M^\perp \cap N^\perp$

Proof

$$\text{Let } z \in (M+N)^\perp$$

$$\langle z, u \rangle = 0 \quad \forall u \in M+N$$

$$\langle z, u \rangle = 0 \quad \forall u \in M$$

$$\langle z, u \rangle = 0 \quad \forall u \in N$$

$$\langle z, u \rangle = 0 \quad \forall u \in M$$

$$\langle z, u \rangle = 0 \quad \forall u \in N$$

$$z \perp \text{ to both } M \text{ and } N$$

$$z \in M^\perp \text{ and } z \in N^\perp$$

$$z \in M^\perp \cap N^\perp$$

$$(M+N)^\perp \subseteq M^\perp \cap N^\perp$$

$$\text{Conversely, let } z \in M^\perp \cap N^\perp$$

$$z \in M^\perp \text{ and } z \in N^\perp$$

$$z \perp \text{ to both } M \text{ and } N$$

$$\langle z, u \rangle = 0 \quad \forall u \in M+N$$

$$\langle z, u \rangle = 0 \quad \forall u \in M+N$$

$$\langle z, u \rangle = 0 \quad \forall u \in M+N$$

$$z \in (M+N)^\perp$$

$$\begin{aligned} & \mathbb{R}[x] \text{ to } \mathbb{R}[x] \\ & \mathbb{R}[x] \subset \mathbb{R}[x] \\ & (\mathbb{R}[x])^\perp = \mathbb{R}[x] \end{aligned}$$

Gram Schmidt orthogonalization process —

Every finite dimensional inner product space has an orthonormal basis
Let $V \subset \mathbb{R}^n$ be an inner product space of dimension

Let $u_1, \dots, u_n$ be a basis of V
Every orthonormal set is a subset of a basis

u_1 being a basis is a subset of a basis
Let $u_1, u_2, \dots, u_n$ be a basis of V

$$u_1 = \frac{1}{\|u_1\|} u_1$$

(u_1, u_2) is a subset of a basis

which is a contradiction u_1, u_2 being subset of it set is a basis

$$u_2 = \frac{1}{\|u_2\|} u_2$$

$$u_3 = \frac{1}{\|u_3\|} u_3$$

$$\langle u_1, u_2 \rangle = \frac{\langle u_1, u_2 \rangle}{\|u_1\| \|u_2\|}$$

$$\langle u_1, u_2 \rangle = \langle u_1, u_2 \rangle - \langle u_1, u_2 \rangle = 0$$

u_1, u_2 is orthogonal to u_1, u_2 is orthogonal set.

$$u_3 = \frac{1}{\|u_3\|} u_3 - \frac{\langle u_3, u_1 \rangle}{\|u_1\|} u_1 - \frac{\langle u_3, u_2 \rangle}{\|u_2\|} u_2$$

$$u_1, u_2, u_3 \text{ is a basis.}$$

$$\langle u_3, u_1 \rangle = \langle u_3 - \frac{\langle u_3, u_1 \rangle}{\|u_1\|} u_1, u_1 \rangle = \langle u_3, u_1 \rangle - \frac{\langle u_3, u_1 \rangle}{\|u_1\|^2} \langle u_1, u_1 \rangle$$

$$\langle u_3, u_1 \rangle = \langle u_3, u_1 \rangle - \frac{\langle u_3, u_1 \rangle}{\|u_1\|^2} \langle u_1, u_1 \rangle = 0$$

$$\langle u_3, u_1 \rangle = 0$$

$$\langle u_3, u_2 \rangle = 0$$

$$u_1, u_2, u_3 \text{ is a basis}$$

$$u_1, u_2, u_3 \text{ is a basis}$$

Then u_1, u_2, u_3 is an orthonormal set

Let V be vector space over $\mathbb{R}$ of polynomials in $\mathbb{R}[x]$ of degree ≤ 3

inner product on $C([0,1])$ and
 $\{1, x, x^2, \dots\}$ is an orthonormal basis of V and
 Schmidt's orthogonalization process

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$$u_1 = 1, u_2 = x, u_3 = x^2$$

$$\|u_1\|^2 = \int_0^1 1 \cdot 1 dx = 1$$

$$\|u_2\|^2 = \int_0^1 x^2 \cdot x^2 dx = \frac{1}{5}$$

$$\|u_3\|^2 = \int_0^1 x^4 \cdot x^4 dx = \frac{1}{9}$$

$$= \frac{1}{5} \left(\frac{1}{5} \right)^{\frac{1}{2}} = \frac{1}{5}$$

$$u_1 = 1, u_2 = x, u_3 = \frac{1}{\sqrt{2}}(x^2 - 1)$$

$$u_2 = x - \frac{1}{2} \int_0^1 x \cdot 1 dx$$

$$= x - \frac{1}{2} \left(\frac{x^2}{2} \right)_0^1 = x - \frac{1}{4}$$

$$u_3 = x^2 - \frac{\langle x^2, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle x^2, u_2 \rangle}{\|u_2\|^2} u_2$$

$$= x^2 - \frac{1}{2} \int_0^1 x^2 \cdot 1 dx - \frac{x^3}{3} \int_0^1 x^2 dx$$

$$= x^2 - \frac{1}{6} - \frac{1}{10} x^2 = \frac{2}{3} x^2 - \frac{1}{6}$$

$$u_1 = 1, u_2 = x, u_3 = \frac{1}{\sqrt{2}}(x^2 - 1)$$

$$\|u_3\|^2 = \int_0^1 \left(x^2 - \frac{1}{2} \right)^2 dx = \frac{1}{10}$$

$$\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}, \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}, \frac{1}{9} \cdot \frac{1}{9} = \frac{1}{81}$$

$$u_1 = 1, u_2 = x, u_3 = \frac{1}{\sqrt{10}}(x^2 - 1)$$

The orthonormal basis is

$$\left\{ 1, \frac{1}{\sqrt{2}}(x^2 - 1), \frac{1}{\sqrt{10}}(x^3 - \frac{3}{2}x) \right\}$$

Thm: Let V be a finite dimensional inner product space and $W \subseteq V$.

① $W^\perp$ is a subspace of V .
 ② $W \cap W^\perp = \{0\}$
 ③ $W + W^\perp = V$

finite dimensional inner product space

$$\langle u, u \rangle = 1, \|u\| = 1$$

Let W be a subspace of V .
 Let $W^\perp$ be the orthogonal complement of W .

$$W \oplus W^\perp = V$$

$$u = \sum_{i=1}^n a_i u_i \quad \text{where } \sum_{i=1}^n a_i^2 = 1$$

$q_1, \dots, q_m$ are unique

$$V_2 = W_1 \perp W_1^\perp$$

$$W_1 \cap W_1^\perp = \{0\}$$

$$W_1 \cap W_1^\perp = \{0\}$$

$$W_1 \cap W_1^\perp = \{0\}$$

$$W_1 \cap W_1^\perp = \{0\}$$

(2)

W_1 is subspace of V
Repeating W_1 by $W_1^\perp$

$$V = W_1 \oplus W_1^\perp$$

V is finite dimensional

$$V = W_1 \oplus W_1^\perp$$

$$\dim V = \dim W_1 + \dim W_1^\perp$$

$$\dim V = \dim W_1 + \dim W_1^\perp$$

$$\dim W_1 = \dim W_1^\perp$$

$$\dim W_1 = 0$$

$$\langle W_1 \rangle = 0 \quad W_1 \subseteq (W_1^\perp)^\perp$$

$$W_1 \subseteq W_1^\perp$$

$$W_1 \subseteq W_1^\perp$$

$$W_1 \subseteq W_1^\perp$$

$$W_1 \subseteq W_1^\perp$$

Orthogonal Projection

If V is a finite dimensional vector space and W is its

subspace $V = W \oplus W^\perp$ and

every vector $v \in V$ can be uniquely as $V = W \oplus W^\perp$ and $v \in W$ and $v \in W^\perp$.

Let W be a subspace of $\mathbb{R}^n$

generated by the vectors y_1, y_2, y_3 and $y_4 = (2, 4, 5, -1)$. Obtain a

basis for $W^\perp$

$$\text{Let } B = \{y_1, y_2\}$$

y_1 and y_2 are linearly independent

$$W = \langle B \rangle$$

B is a basis of W and $\dim W = 2$

$$V = W \oplus W^\perp$$

$$\dim V = \dim W + \dim W^\perp$$

$$4 = 2 + \dim W^\perp$$

Let $W^\perp = \{z_1, z_2\}$. be arbitrary

$$W_1 = \langle y_1, y_2, y_3, y_4 \rangle$$

$$\langle W_1 \rangle = 0$$

$$\langle W_1 \rangle = 0 \quad 2y_1 + 4y_2 + 5y_3 - 2y_4 = 0$$

$$y_3 - 3y_4 = 0$$

$$y_3 = 3y_4$$

$$y_3 = 3y_4$$

$$y_3 = 3y_4$$

$$y_3 = 3y_4$$

$$a_3 = 2$$

$$a_2 = -\frac{7}{2}$$

$$w_2 = \begin{pmatrix} 0, -\frac{7}{2}, 18, 1 \end{pmatrix}$$

$$w_2 = (2, 1, 0, 0) \begin{pmatrix} 0, -\frac{7}{2}, 18, 1 \end{pmatrix}$$

Linear operators on inner product spaces

Adjoint operator - let V be an inner product space. A linear transformation

T is a unique operator T^* on V . $\langle Tu, v \rangle = \langle u, T^*v \rangle \forall u, v \in V$. Then T^* is called adjoint of T .

Self adjoint operator $\rightarrow$

A linear operator T on an inner product space V is called self adjoint if $T = T^*$.

iff $\langle Tu, v \rangle = \langle u, Tv \rangle \forall u, v \in V$

Thm: Let T be a linear operator on a unitary space V . Prove that

$T = 0$ iff $\langle Tu, v \rangle = 0 \forall u, v \in V$

Proof: Let T be a linear operator on a unitary space V . Let $u, v \in V$ be arbitrary and for

$$\langle Tu, v \rangle = \langle 0(u), v \rangle = 0$$

$$\text{Conversely} \rightarrow \langle Tu, v \rangle = 0 \forall u, v$$

$$\langle Tu, v \rangle = \langle u, Tv \rangle = 0$$

$$T = 0$$

$$\begin{aligned} \langle T\alpha, \alpha + \beta \rangle + \langle T\beta, \alpha + \beta \rangle &= 0 \\ \langle T\alpha, \alpha \rangle + \langle T\alpha, \beta \rangle + \langle T\beta, \alpha \rangle + \langle T\beta, \beta \rangle &= 0 \\ \langle T\alpha, \alpha \rangle + \langle T\beta, \beta \rangle + \langle T\alpha, \beta \rangle + \langle T\beta, \alpha \rangle &= 0 \\ \langle T\alpha, \alpha \rangle + \langle T\beta, \beta \rangle &= 0 \end{aligned}$$

suppose β is $i\alpha$

$$\langle T\alpha, i\alpha \rangle + \langle T(i\alpha), \alpha \rangle = 0$$

$$\begin{aligned} \langle T\alpha, \alpha \rangle + i\langle T\alpha, \alpha \rangle &= 0 \\ \langle T\alpha, \alpha \rangle + i\langle T\alpha, \alpha \rangle &= 0 \end{aligned}$$

$$\begin{aligned} -\langle T\alpha, \alpha \rangle + \langle T\alpha, \alpha \rangle &= 0 \\ \langle T\alpha, \alpha \rangle - \langle T\alpha, \alpha \rangle &= 0 \end{aligned}$$

$$\begin{aligned} \langle T\alpha, \beta \rangle + \langle T\beta, \alpha \rangle &= 0 \\ \langle T\alpha, \beta \rangle &= 0 \end{aligned}$$

$$\begin{aligned} \langle T\alpha, \alpha \rangle &= 0 \\ T\alpha &= 0 \end{aligned}$$

Thm Let T be a linear operator on a finite dimensional inner product space

$$\text{① } (AT)^* = T^*A^* \quad \text{② } (T^*)^* = T$$

Proof ① is linear transformation

$$\begin{aligned} (AT)u &= A(Tu) \text{ for all } u \in V \\ \langle (AT)u, v \rangle &= \langle A(Tu), v \rangle = \langle Tu, A^*v \rangle \\ &= \langle Tu, (A^*v) \rangle = \langle Tu, (A^*v) \rangle \end{aligned}$$

$$\langle (AT)u, v \rangle = \langle u, (AT)^*v \rangle$$

from ① and ②

$$\langle u, (AT)^*v \rangle = \langle u, (A^*T^*)v \rangle$$

$$AT^* = A^*T$$

②

$$\begin{aligned} \langle u, T(u) \rangle &= \langle T^*(u), u \rangle \\ \langle T(u), u \rangle &= \langle u, (T^*)^*(u) \rangle \\ \langle u, T(u) \rangle &= \langle u, T^*(u) \rangle \\ T^*(T^*(u)) &= u \end{aligned}$$

Thm The product of two self adjoint operators on an inner product space is self adjoint iff. the operators commute

Proof S and T be two self adjoint operators on an inner product space V

$$S^* = S, T^* = T \quad \text{①}$$

Let T be self adjoint

$$(TS)^* = S^*T^* = TS$$

$$S \text{ and } T \text{ commute}$$

Conversely let S and T commute

$$ST = TS.$$

$$(ST)^* = (TS)^* \Rightarrow (ST)^* = S^*T^* = TS$$

Thm Let T be a self adjoint operator on an inner product space V

Then for all α , $\langle T\alpha, \alpha \rangle$ is self adjoint w.r.t α is real.

Proof Let T be a self adjoint operator on an inner product space V (c.f.)

$$T^* = T.$$

Let α be self adjoint.

$$(T\alpha)^* = \alpha^* T^* = \alpha^* T = T\alpha$$

$$\overline{\langle T\alpha, \alpha \rangle} = \langle \alpha, T\alpha \rangle$$

$$\overline{\langle T\alpha, \alpha \rangle} = \langle \alpha, T\alpha \rangle$$

α is real.

Conversely let α be real

$$\overline{\langle T\alpha, \alpha \rangle} = \langle \alpha, T\alpha \rangle$$

$$(T\alpha)^* = \alpha^* T^* = \alpha^* T = T\alpha$$

T is self adjoint

Thm

A linear operator T on a unitary space V is Hermitian w.r.t $\langle T\alpha, \alpha \rangle$.

Proof

Let T be a linear operator on a unitary space V (c.f.)

$$\langle T\alpha, \alpha \rangle = \langle \alpha, T^*\alpha \rangle$$

$$= \langle \alpha, T\alpha \rangle = \langle T\alpha, \alpha \rangle$$

$$\langle T\alpha, \alpha \rangle = \langle T\alpha, \alpha \rangle$$

$\langle T\alpha, \alpha \rangle$ is real & $\langle T\alpha, \alpha \rangle = \langle T\alpha, \alpha \rangle$

Conversely let $\langle T\alpha, \alpha \rangle$ be real & $\langle T\alpha, \alpha \rangle = \langle T\alpha, \alpha \rangle$

$$\langle T\alpha, \alpha \rangle = \langle \alpha, T\alpha \rangle$$

$$\langle T\alpha, \alpha \rangle = \langle \alpha, T\alpha \rangle = \langle T\alpha, \alpha \rangle$$

$$\langle T\alpha, \alpha \rangle = \langle T\alpha, \alpha \rangle = 0.$$

$$\langle T\alpha, \alpha \rangle - \langle T\alpha, \alpha \rangle = 0.$$

$$\langle T - T^*\alpha, \alpha \rangle = 0$$

$$T - T^* = 0$$

$$T = T^* \quad T \text{ is Hermitian}$$

Thm

Let T be a linear operator on an inner product space V . Then the following statements are equivalent

- (i) $T^* = T^{-1}$
- (ii) $\langle Tu, Tv \rangle = \langle u, v \rangle$
- (iii) $\|Tu\| = \|u\| \quad \forall u \in V$

Proof Let T be a linear operator on inner product space V .

Let $u, v \in V$,

(i) $\Rightarrow$ (ii) $T^* = T^{-1}$
 $T^* T = I \Rightarrow T^* T u = I u = u$
 $\langle T^* T u, v \rangle = \langle u, v \rangle$
 $\langle T u, T v \rangle = \langle u, v \rangle$

(ii) $\Rightarrow$ (i) $T^* T = I$
 $\langle T^* T u, v \rangle = \langle u, v \rangle$
 $\langle T u, T v \rangle = \langle u, v \rangle$
 $\langle T u, T v \rangle = \langle u, v \rangle$

- inner product

Similarly $\langle T u, T v \rangle = \langle u, v \rangle$
 $\langle T u, T v \rangle = \langle u, v \rangle$
 $\|T u\|^2 = \|u\|^2$

$\|T u\| = \|u\|$

(ii) $\Rightarrow$ (i) $\|T u\| = \|u\|$
 $\|T u\|^2 = \|u\|^2$
 $\langle T u, T u \rangle = \langle u, u \rangle$
 $\langle T^* T u, u \rangle = \langle u, u \rangle$
 $\langle T^* T - I, u, u \rangle = 0$

$\langle T^* T - I, u, u \rangle = 0$

$(T^* T - I) u = 0$
 $T^* T u = u$
 $T^* T = I$

$T^* T = I$

$T^* T = I$

$T^* = T^{-1}$

Thm

Proof

Let T be a linear operator on an inner product space V . Then, T is normal iff its real and imaginary parts commute.

$$A = \frac{1}{2}(T+T^*) \quad B = \frac{1}{2i}(T-T^*)$$

Are self adjoint operators

$$A^* = A, \quad B^* = B$$

$$T^* = (A+B)(A+B)^* = A^* + B^* = A+B$$

$$T^* = (A+iB)(A+iB)^* = A^2 + B^2 + i(BA-AB)$$

$$T^*T = (A-iB)(A+iB) = A^2 + B^2 + i(AB-BA)$$

$$T \text{ is normal iff } T^*T = TT^*$$

$$A^2 + B^2 + i(CA-AC) = A^2 + B^2 + i(AB-BA)$$

$$CA-AB = AB-BA$$

$$CA = AB$$

as $AB = BA$

Q

Let T be a linear operator on an inner product space V . If $T^2 = 0$ and T is self adjoint or skew symmetric, then show that $T = 0$.

Let T be a linear operator on an inner product space V .

Let u, v be arbitrary. Then

$$\langle T^2u, u \rangle = \langle T(Tu), u \rangle = \langle Tu, T^*u \rangle = \langle Tu, Tu \rangle = 0$$

$$\langle T^2u, u \rangle = 0$$

$$\langle T^2u, u \rangle = \langle 0, u \rangle = 0$$

$$\langle Tu, Tu \rangle = 0$$

(2) Let T is self adjoint

$$\langle Tu, Tu \rangle = 0$$

$$\langle Tu, Tu \rangle = 0$$

(3)

Let T be skew symmetric

$$\langle Tu, Tu \rangle = -\langle Tu, Tu \rangle = 0$$

Ques If T and S are normal operators on an inner product space $V(F)$

(1) TS is normal operator

(2) $T+S$ is a normal operator

if either of the operators commute with adjoint operator to other

solⁿ let T and S be normal operators

$$T T^* = T^* T, \quad S S^* = S^* S$$

$$T S^* = S^* T, \quad T^* S = S T^*$$

$$(1) (TS)^* (TS) = S^* T^* TS = S^* (T^* T) S$$

$$= S^* (T T^*) S = S^* (T^* T) S \\ = (S^* T) (T^* S)$$

$$(T+S)^* (TS) = (S^* T) (T^* S)$$

$$(TS)(TS)^* = TS S^* T^* = T(SS^*)T^*$$

$$= T(S^* S)T^* = (TS)^* (ST^*)$$

$$= (S^* T) (T^* S)$$

$$(TS)^* (TS) = (TS)(TS)^*$$