SUBJECT : Real & Complex Analysis (Math)

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MAA OMWATI DEGREE COLLEGE, HASSANPUR (PALWAL)

Chapter:- Ist

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Definition:-

If $u_1, u_2, \ldots u_n$ are functions of n independent variables $x_1, x_2, \ldots, x_n$ then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}.$$

is called the Jacobian of $u_1, u_2, \ldots u_n$ with respect to $x_1, x_2, \ldots x_n$.

This Jacobian is denoted by

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots x_n)} \text{ or } J\left(\frac{u_1, u_2, \ldots u_n}{x_1, x_2, \ldots x_n}\right)$$

Thm:- If the functions $u_1, u_2, \ldots, u_n$ are of the forms $u_1 = F_1(x_1)$, $u_2 = f_2(x_1, x_2) \ldots u_n = F_n(x_1, x_2, \ldots, x_n)$ then

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdots \frac{\partial u_n}{\partial x_n}$$

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Proof:- Here $u_1 = f_1(x_1)$

$u_2 = f_2(x_1, x_2)$

$\vdots$

$u_n = f_n(x_1, x_2, \dots x_n)$

$$\therefore \frac{\partial(u_1, u_2, \dots u_n)}{\partial(x_1, x_2, \dots x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \dots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \dots & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdots \frac{\partial u_n}{\partial x_n}$$

Hence the result.

Thm:- If $u = f(x,y)$ and $v = g(x,y)$ are functions of x and y and if the inverse function:

$x = \phi(u,v)$ $y = \psi(u,v)$ exist then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Proof:-

Since u, v are function of x, y then we have

$u = u(x,y)$ and $v(x,y)$

Diff partially w.r.t. u and v we have

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$$1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Now. $$\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} & \frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} & \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Exp. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ then evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

Sol $x = r\cos\theta$

$$\therefore \frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial z} = 0$$

Also $y = r\sin\theta$

$\therefore \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta, \quad \frac{\partial y}{\partial z} = 0$

Again $z = z$

$\frac{\partial z}{\partial r} = 0 \quad \frac{\partial z}{\partial \theta} = 0 \quad \frac{\partial z}{\partial z} = 1$

By definition of Jacobian we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding by IIIrd row

$= r(\cos^2\theta + \sin^2\theta) = r$

Exp. Find the Jacobian of u, v, w with respect to x, y, z when $x+y+z = u$, $y+z = uv$ and $z = uvw$

Sol:- Let $r = u$, $s = uv$, $t = uvw$

$\therefore$ r, s, t are functions of u, v, w

$$\therefore \frac{\partial(r,s,t)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} & \frac{\partial r}{\partial w} \\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} & \frac{\partial s}{\partial w} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} & \frac{\partial t}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = u^2v$$

Now it is given that $x+y+z = u$

$y+z = uv$, $z = uvw$

$\therefore r = x+y+z$, $s = y+z$, $z = t$

Here r, s, t are functions of x, y, z

$$\therefore \frac{\partial(r,s,t)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$ Expanding by Ist C1

$1(1-0) = 1$ Ans

Hence $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \cdot \frac{\partial(r,s,t)}{\partial(x,y,z)}$

$$= \frac{1}{\frac{\partial(r,s,t)}{\partial(u,v,w)}} \cdot \frac{\partial(r,s,t)}{\partial(x,y,z)}$$

$= \frac{1}{u^2v} \cdot 1 = \frac{1}{u^2v}$ Ans

Functional Dependence (or Non-Independence)

⇒ The m functions $u_1, u_2, \ldots u_m$ of n independent variables $x_1, x_2, \ldots x_n$ are said to be functionally dependent if there exist a relation of the form $F(u_1, u_2, \ldots u_m) = 0$. Also we say that the functions $u_1, u_2, u_3, \ldots, u_m$ are not independent if there exist a relation of the form $F(u_1, u_2, \ldots u_m) = 0$

Exp. $u = x$ and $v = x^3$ be two functions of one variable x

But $v = x^3 = u^3$

$\therefore v - u^3 = 0$ or $F(u, v) = 0$

Hence the functions u, v are functionally dependent.

2. Let $u = \sin x$ and $v = \sin^4 x$

Here u and v are function of one variable x

Also $v = \sin^4 x = u^4$

$\therefore v - u^4 = 0$ or $F(u, v) = 0$

Hence the functions u, v are functionally dependent

Exp. Let $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ be two functions. Evaluate $\frac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related (dependent)? If so, find the relation b/w them.

Sol. Here $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

We first find out $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{(1-xy) - (x+y)(-y)}{(1-xy)^2}$$

$$= \frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy) - (x+y)(-x)}{(1-xy)^2} = \frac{1 - \cancel{xy} + \cancel{xy} + x^2}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1+y^2}{(1-xy)^2} \times \frac{1}{1+y^2} - \frac{1+x^2}{(1-xy)^2} \times \frac{1}{1+x^2}$$

$$= 0$$

$\Rightarrow$ u and v are functionall related (depende

$$v = \tan^{-1}x + \tan^{-1}y$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$= \tan^{-1}u$$

Hence $v = \tan^{-1}u$ is the required functional relation b/w the functions u and v.

Exp Show that the functions $u = x^2+y^2+z^2$, $v = xy - yz - xz$, $w = x+y-z$ are dependent. Also find the relation connecting them.

Sol. $u = x^2+y^2+z^2$

$v = xy - xz - yz$

and $w = x+y-z$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial u}{\partial z} = 2z$

$\frac{\partial v}{\partial x} = y-z$, $\frac{\partial v}{\partial y} = x-z$, $\frac{\partial v}{\partial z} = -x-y$

$\frac{\partial w}{\partial x} = 1$, $\frac{\partial w}{\partial y} = 1$, $\frac{\partial w}{\partial z} = -1$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2 \\ y-z & x-z & -x-y \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 + C_3$

Taking common out 2 from R_1

$$2\begin{vmatrix} x-y & y+z & 2 \\ y-x & -y-z & -x-y \\ 0 & 0 & 1 \end{vmatrix}$$

Taking out $(x-y)$ common from C_1 and $(y+z)$ from C_2

$$= 2(x-y)(y+z)\begin{vmatrix} 1 & 1 & 2 \\ -1 & -1 & -x-y \\ 0 & 0 & -1 \end{vmatrix}$$

$= 0$ ($\because C_1$ and C_2 are identical

Thus we have $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

$\therefore$ The functions u, v and w are functionally dependent

Also $w^2 = (x+y-z)^2$

$= x^2+y^2+z^2+2(xy-xz-yz)$

$= u + 2v$

Hence u, v and w are connected by relation $w^2 = u+2v$

Beta And Gamma Function

Chapter - 2

Beta Function:-

The First Eulerian Integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ where $m>0$, $n>0$ is called a Beta function and is denoted by $B(m,n)$

Exp

$\int_0^1 x^5(1-x)^3\,dx$ is Beta function denoted by $B(6,4)$

(ii) $\int_0^1 x^{1/3}(1-x)^{5/4}\,dx$ is a Beta function denoted by $B\left(\frac{4}{3},\frac{9}{4}\right)$

(iii) $\int_0^\infty x^{2/3}(1-x)^{-1/2}\,dx$ is a Beta function and is denoted by $B\left(\frac{5}{3},\frac{1}{2}\right)$

Properties of Beta function:-

To show that $B(m,n) = B(n,m)$

Proof:- For all $m, n > 0$ we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad -(1)$$

Using the property

$$\int_0^a F(x)\,dx = \int_0^a F(a-x)\,dx \text{ in } (1)$$

we get

$$B(m,n) = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$$B(n,m)$$

Thus $B(m,n) = B(n,m)$ for all $m,n>0$

Another form of Beta function:-

To show that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \text{ where } m,n>0$$

Proof:- For all $m,n>0$ we have

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

But $x = \frac{t}{1+t}$ — ②

$$\therefore dx = \frac{1+t-t}{(1+t)^2} = \frac{dt}{(1+t)^2}$$

From $x(1+t) = t$

$\therefore x + xt = t$

$t - xt = x$

$t(1-x) = x$

$$t = \frac{x}{1-x}$$

when $x=0$, $t=0$ and when $x=1$ $t \to \infty$

From ① we have

$$B(m,n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1} \left[1-\frac{t}{1+t}\right]^{n-1} \frac{dt}{(1+t)^2}$$

Or

$$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}} \left(\frac{1}{1+t}\right)^{n-1} \frac{dt}{(1+t)^2}$$

$$\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \text{— ③}$$

Similarly we can prove that

$$B(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

But

$$B(m,n) = B(n,m)$$

$$\therefore B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

1. Express the given integrals in terms of Beta functions:

$$\int_0^2 \sqrt{x}\,(4-x^2)^{-1/4}\,dx$$

put $x^2 = 4z$ — (1)

$x = \sqrt{4z}$, $x = 2\sqrt{z}$

$dx = z^{-1/2}\,dz$

From (1), $z = \frac{x^2}{4}$

when $x=0$, $z=0$ and when $x=2$, $z=\frac{4}{4}=1$

$$\therefore \int_0^2 \sqrt{x}\,(4-x^2)^{-1/4}\,dx = \int_0^1 (2^{1/2} z^{1/4})\,[4-4z]^{-1/4} z^{-1/2}\,dz$$

$$= \int_0^1 2^{1/2}\cdot z^{1/4}\cdot 4^{-1/4}\,[1-z]^{-1/4} z^{-1/2}\,dz$$

$$= \int_0^1 2^{\frac{1}{2}-\frac{1}{2}} z^{\frac{1}{4}-\frac{1}{2}} (1-z)^{-1/4}\,dz$$

$$= \int_0^1 z^{-1/4} (1-z)^{-1/4}\,dz$$

$$= B\left[-\tfrac{1}{4}+1, -\tfrac{1}{4}+1\right]$$

$$= B\left(\tfrac{3}{4}, \tfrac{3}{4}\right)$$

Exp. Prove that

$$\int_a^b (x-a)^m (b-x)^n\,dx = (b-a)^{m+n+1} B(m+1, n+1)$$

Sol. Let $I = \int_a^b (x-a)^m (b-x)^n\,dx$

put $x = a + (b-a)z$ so that

$dx = (b-a)\,dz$

when $x=a$, $z=0$ and when $x=b$, $z=1$

$$\therefore I = \int_0^1 [(b-a)z]^m\,[b-a-(b-a)z]^n (b-a)\,dz$$

$$= \int_0^1 (b-a)^m z^m (b-a)^n [1-z]^n (b-a)\,dz$$

$$= (b-a)^{m+n+1} \int_0^1 z^m (1-z)^n\,dz$$

$$= (b-a)^{m+n+1} B(m+1, n+1)$$

Hence $\int_a^b (x-a)^m (b-x)^n\,dx$

$$= (b-a)^{m+n+1} B(m+1, n+1)$$

Exp. Express $\int_0^1 x^m (1-x^n)^p\,dx$ in terms of Beta function and hence evaluate $\int_0^1 x^5 (1-x^3)^3\,dx$

put $x^n = z$

$\therefore x = z^{1/n}$ and $dx = \frac{1}{n} z^{\frac{1}{n}-1} dz$

At $x=0, z=0$ and at $x=1, z=1$

$$\therefore \int_0^1 x^m (1-x^n)^p dx = \int_0^1 z^{m/n} (1-z)^p \cdot \frac{1}{n} z^{\frac{1}{n}-1} dz$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right) \quad \text{--- (1)}$$

comparing $\int_0^1 x^m (1-x^n)^p dx$ with $\int_0^1 x^5 (1-x^3)^3 dx$ we get

$m=5, \; n=3, \; p=3$

putting $m=5, n=3, p=3$ in (1) we get

$$\int_0^1 x^5 (1-x^3)^3 dx$$

$$= \frac{1}{3} B\left(\frac{5+1}{3}, 3+1\right)$$

$$= \frac{1}{3} B(2,4)$$

$$= \frac{1}{3} \cdot \frac{1!\,3!}{(2+4-1)!}$$

$$\left[\because B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}\right]$$

$$= \frac{1}{3} \cdot \frac{3!}{5!}$$

$$\frac{1}{3} \cdot \frac{3!}{5 \cdot 4 \cdot 3!} = \frac{1}{3 \cdot 5 \cdot 4} = \frac{1}{60}$$

Hence $\int_0^1 x^5 (1-x^3)^3 dx = \frac{1}{60}$ Ans

✱ Gamma Function :-

The Second Eulerian integral $\int_0^\infty e^{-x} x^{n-1} dx$ when $n>0$ is called Gamma function and is denoted by Γn

Thus $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ for all $n>0$

Illustrations :-

(i) $\int_0^\infty x^6 e^{-x} dx$ is Gamma function and is denoted by $\Gamma 7$

(ii) $\int_0^\infty x^{-1/5} e^{-x} dx$ is a Gamma function and is denoted by $\Gamma \frac{4}{5}$

✡ Recurrence Formula for Gamma Function

1. If $n>1$ is any real number, then $\Gamma n = (n-1)\Gamma(n-1)$

2. If $n>1$ is any integer, then $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!$

Remarks:-

The value of Γn can be easily evaluated if n is a positive integer >1

Exp. $\Gamma(4) = 3! = 6$

$\Gamma(5) = 4! = 4\times3\times2 = 24$.

2. If n is a +ve fraction then

$\Gamma n = (n-1)\,\Gamma n-1$

$= (n-1)(n-2)\,\Gamma n-2$

$\vdots$

$= (n-1)(n-2)\ldots(n-r)\,\Gamma(n-r)$

This process is continued till $(n-r)>0$ but $(n-r-1)<0$

For example $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}$

$\Gamma\frac{7}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}$

$\Gamma\frac{9}{2} = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}$

✡ To prove that $B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$ where $m>0, n>0$

Proof:- By definition, $\Gamma n = \int_0^\infty x^{n-1}e^{-x}\,dx$

put $x = tz$ so that $dx = t\,dz$

when $x=0$, $z=0$ and when $x\to\infty$, $z\to\infty$

From (1) we have

$\Gamma n = \int_0^\infty (tz)^{n-1}e^{-tz}(t\,dz)$

$= \int_0^\infty t^n z^{n-1}e^{-tz}\,dz$

Multiplying both side by $e^{-t}t^{m-1}$ we have.

$e^{-t}t^{m-1}\Gamma n = e^{-t}t^{m-1}\int_0^\infty t^n z^{n-1}e^{-tz}\,dz$

$= \int_0^\infty t^{m+n-1}z^{n-1}e^{-tz}e^{-t}\,dz$

$= \int_0^\infty t^{m+n-1}z^{n-1}e^{-t(z+1)}\,dz$

Integrating both side w.r.t t b/w the limit 0 to ∞ we have

$$\int_0^\infty e^{-t} t^{m-1} \Gamma(n)\, dt = \int_0^\infty \int_0^\infty t^{m+n-1} z^{n-1} e^{-t(z+1)}\, dt\, dz$$

$$= \int_0^\infty z^{n-1} \left[\int_0^\infty t^{m+n-1} e^{-t(z+1)}\, dt \right] dz \quad \text{— (2)}$$

put $t(z+1) = y$ so that $t = \frac{y}{z+1}$

$$dt = \frac{dy}{z+1}$$

when $t=0$, $y=0$ when $t\to\infty$, $y\to\infty$

$\therefore$ From (2) we have

$$\Gamma n \int_0^\infty e^{-t} t^{m-1}\, dt = \int_0^\infty z^{n-1} \left[\int_0^\infty \left(\frac{y}{z+1}\right)^{m+n-1} e^{-y} \frac{dy}{z+1} \right] dz$$

$$= \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}} \left[\int_0^\infty y^{m+n-1} e^{-y}\, dy \right] dz$$

$$= \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+1}} \Gamma(m+n)\, dz$$

$$\Gamma n\, \Gamma m = \Gamma(m+n) \cdot \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}}\, dz$$

$$\Gamma m\, \Gamma n = \Gamma(m+n)\, B(m,n)$$

Hence we have $B(m,n) = \dfrac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$

Exp. Show that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

put $x^2 = t$ so that $2x\,dx = dt$

$$dx = \frac{dt}{2x} \Rightarrow \frac{dt}{2\sqrt{t}}$$

when $x=0$, $t=0$ and when $x\to\infty$, $t\to\infty$

$$I = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2}\, dt$$

$$= \frac{1}{2}\Gamma\left(-\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Exp. Prove that : $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx$

$$= \frac{\Gamma m\, \Gamma n}{a^n (1+a)^m \Gamma(m+n)}$$

Sol. Let $\displaystyle I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx$

$$= \int_0^1 \left(\frac{x}{a+x}\right)^{m-1} \left(\frac{1-x}{a+x}\right)^{n-1} \frac{dx}{(a+x)^2}$$

put $\dfrac{x}{a+x} = \dfrac{z}{a+1}$

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when $x=0,\ z=0$ and when $x=1, z=1$

Diff w.r.t x (2) w.r.t x we have

$$\frac{a+x-x}{(a+x)^2}\,dx = \frac{dz}{a+1}$$

$$\frac{dx}{(a+x)^2} = \frac{dz}{a(a+1)}$$

From (2) $\quad x(a+1) = (a+x)z$

$$\therefore x(a+1-z) = az$$

$$x = \frac{az}{a+1-z}$$

Now $\quad \dfrac{1-x}{a+x} = \dfrac{1-\dfrac{az}{a+1-z}}{a+\dfrac{az}{a+1-z}}$

$$\frac{a+1-z-az}{a^2+a-\cancel{az}+\cancel{az}} = \frac{a(1-z)+(1-z)}{a(a+1)}$$

$$\therefore \frac{\cancel{(a+1)}(1-z)}{a\cancel{(a+1)}} = \frac{1-z}{a}$$

$\therefore$ From (1), we have $I = \displaystyle\int_0^1 \left(\frac{z}{a+1}\right)^{m-1}\left(\frac{1-z}{a}\right)^{n-1}\frac{dz}{a(a+1)}$

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$$= \frac{1}{a^n(a+1)^m}\int_0^1 z^{m-1}(1-z)^{n-1}\,dz$$

$$= \frac{1}{a^n(1+a)^m}\,B(m,n)$$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{1}{a^n(1+a)^m}\,\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

Duplication Formula:-

To prove that $\quad \Gamma\left(n+\frac{1}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma n+1}$

Proof:- We have $\quad B\left(n+\frac{1}{2}, n+\frac{1}{2}\right)$

$$= \int_0^1 x^{n-\frac{1}{2}}(1-x)^{n-\frac{1}{2}}\,dx$$

put $x = \sin^2\theta$ so that

$dx = 2\sin\theta\cos\theta\,d\theta$

when $x=0,\ \theta=0$ and when $x=1$ $\theta=\frac{\pi}{2}$

$$\therefore B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^{\pi/2} (\sin^2\theta)^{n-\frac{1}{2}}(1-\sin^2\theta)^{n-\frac{1}{2}}\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} (\sin\theta)^{2n-1}(\cos^2\theta)^{n-\frac{1}{2}}\,2\sin\theta\cos\theta\,d\theta$$

$$2\int_0^{\pi/2} (\sin\theta)^{2n-1}(\cos^2\theta)^{n-\frac{1}{2}}\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2n}\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2n}\, d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{\sin 2\theta}{2}\right]^{2n} d\theta$$

$$= \frac{2}{2^{2n}} \int_0^{\pi/2} (\sin 2\theta)^{2n}\, d\theta$$

put $2\theta = t$

$2\,d\theta = dt$

At $\theta = 0$, $t = 0$, at $\theta = \frac{\pi}{2}$, $t = \pi$

$$\frac{2}{2^{2n}} \int_0^{\pi} (\sin t)^{2n} \frac{dt}{2} = \frac{1}{2^{2n}} \int_0^{\pi} (\sin t)^{2n}\, dt$$

$$= \frac{2}{2^{2n}} \int_0^{\pi/2} (\sin t)^{2n}\, dt$$

$$\left[\because \int_0^{\pi} f(t)\,dt = 2\int_0^{\pi/2} f(t)\,dt \quad \text{if } f(\pi - t) = f(t)\right]$$

$$= \frac{2}{2^{2n}} \int_0^{\pi/2} (\sin t)^{2n} (\cos t)^0\, dt$$

$$= \frac{2}{2^{2n}} \cdot \frac{1}{2} \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\frac{1}{2}}{\Gamma\left(\frac{2n+2}{2}\right)}$$

$$\left[\because \int_0^{\pi/2} \sin^m t \cos^n t\, dt = \frac{1}{2} \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\left(\frac{m+1}{2} + \frac{n+1}{2}\right)}\right]$$

$$= \frac{1}{2^{2n}} \frac{\Gamma\left(\frac{2n+1}{2}\right)\sqrt{\pi}}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2^{2n}} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$$

putting

$$B\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}\right) = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+n+\frac{1}{2}\right)}$$

$$\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} = \frac{\sqrt{\pi}}{2^{2n}} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}} \frac{\Gamma(2n+1)}{\Gamma(n+1)} \quad \text{Ans}$$

Exp.

Show that:

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}.$$

Let $I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$= \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta)^{1/2} (\cos\theta)^{-1/2}\, d\theta$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+1}{2} + \frac{-\frac{1}{2}+1}{2}\right)}$$

$$\left[\because \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}\right]$$

$$\frac{\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$I = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \quad -(1)$$

By Duplication Formula

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

putting $m = \frac{1}{4}$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}}{2^{1/2-1}}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}.\sqrt{\pi}}{2^{-1/2}} = \sqrt{2}\pi$$

$\therefore$ From (1)

$$I = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{2}(\sqrt{2}.\pi)$$

$$= \frac{\pi}{\sqrt{2}} \text{ Ans}$$

$$\therefore \int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \frac{\pi}{\sqrt{2}} \text{ Ans}$$

Ex. Prove that

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad \text{when}$$

$$\int_0^{\infty} \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$$

Sol. $\Gamma n\,\Gamma n-1 = \frac{\Gamma n\,\Gamma(1-n)}{\Gamma 1} = B(n, 1-n)$

$$= \int_0^1 x^{n-1}(1-x)^{-n}dx \quad -(1)$$

put $x = \frac{y}{1+y}$ —(2) $\therefore\ 1-x = 1-\frac{y}{1+y}$

$$= \frac{1+y-y}{1+y} = \frac{1}{1+y}$$

Also $x = \frac{y}{1+y} = x+xy = y$

$$x = y-xy$$

$$x = y(1-x) \Rightarrow y = \frac{x}{1-x}$$

At $x=0$, $y=0$ and $x=1$, $y\to\infty$

Diff (2) w.r.t x we get

$$dx = \frac{(1+y)-y}{(1+y)^2}dy$$

$$\frac{1}{(1+y)^2}dy$$

From (1)

$$\Gamma n\,\Gamma(1-n) = \int_0^{\infty}\left(\frac{y}{1+y}\right)^{n-1}\left(\frac{1}{1+y}\right)^{-n}\frac{dy}{(1+y)^2}$$

$$= \int_0^{\infty}\frac{y^{n-1}}{(1+y)^{n-1-n+2}}dy$$

$$= \int_0^{\infty}\frac{y^{n-1}}{1+y}dy = \frac{\pi}{\sin n\pi}$$

$$\left[\because \int_0^{\infty}\frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}\right. \text{(Given)}$$

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Chapter - 3rd

Double And Triple Integral

① ① When y_1, y_2 are functions of x and x_1, x_2 are constant then $f(x,y)$ is first integrated w.r.t y keeping x fixed within the limits y_1, y_2 and then the resulting expression is integrated w.r.t x within the limits x_1, x_2

$$I_1 = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right] dx$$

2. When x_1, x_2 are functions of y and y_1, y_2 are constant, then $f(x,y)$ is first integrated w.r.t x keeping y fixed, within the limit x_1, x_2 and then the resulting expression is integrated w.r.t y within the limits y_1, y_2

$$I_2 = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right] dy$$

3. When both pairs of limits are constants, then it does matter whether we first integrate w.r.t x and then w.r.t y or vice versa.

Exp. Verify that $\int_1^2\int_3^4 (xy+e^y)\,dy\,dx = \int_3^4\int_1^2 (xy+e^y)\,dx\,dy$

Sol. $\int_1^2\int_3^4 (xy+e^y)\,dy\,dx = \int_1^2 \left[x\cdot\frac{y^2}{2}+e^y\right]_3^4 dx$

$= \int_1^2\left[(8x+e^4)-\left(\frac{9}{2}x+e^3\right)\right]dx$

$= \int_1^2\left[\frac{7}{2}x+(e^4-e^3)\right]dx$

$= \left[\frac{7x^2}{4}+(e^4-e^3)x\right]_1^2$

$= \left[7+(e^4-e^3)2\right]-\left[\frac{7}{4}+e^4-e^3\right]$

$= \frac{21}{4}+(e^4-e^3)$ — ①

Now $\int_3^4\int_1^2 (xy+e^y)\,dx\,dy = \int_3^4\left[\left(\frac{x^2}{2}y+e^y x\right)\right]_1^2 dy$

$= \int_3^4\left[(2y+2e^y)-\left(\frac{y}{2}+e^y\right)\right]dy$

$= \int_3^4\left(\frac{3}{2}y+e^y\right)dy = \left[\frac{3y^2}{4}+e^y\right]_3^4$

$= (12+e^4)-\left(\frac{27}{4}+e^3\right)$

$= \frac{21}{4}+(e^4-e^3)$ — ②

From ① and ② we have

$$\int_1^2\int_3^4 (xy+e^y)\,dy\,dx = \int_3^4\int_1^2 (xy+e^y)\,dx\,dy$$

Exp. Evaluate $\iint \sqrt{a^2-x^2-y^2}\,dx\,dy$ over the semi-circle $x^2+y^2=ax$ in the positive quadrant.

Sol. Let R be the region of integration

$$R=\{(x,y): x^2+y^2\le ax,\ x\ge 0, y\ge 0\}$$

Put $x=r\cos\theta$ and $y=r\sin\theta$

$\therefore$ $x^2+y^2\le ax$ becomes $r^2\le a\,r\cos\theta$

$$r\le a\cos\theta$$

The region R of integration transforms into region R' where

$$R'=\{(r,\theta): 0\le r\le a\cos\theta,\ 0\le\theta\le\tfrac{\pi}{2}\}$$

$\therefore$ Given integral $=\iint_R \sqrt{a^2-(x^2+y^2)}\,dx\,dy$

$$=\int_0^{\pi/2}\int_0^{a\cos\theta}\sqrt{a^2-r^2}\,|J|\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{a\cos\theta}\sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{\pi/2}\int_0^{a\cos\theta}(a^2-r^2)^{1/2}(-2r\,dr)\,d\theta$$

$$=-\frac{1}{2}\int_0^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_{r=0}^{a\cos\theta}d\theta$$

$$=-\frac{1}{2}\cdot\frac{2}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$=-\frac{1}{3}\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$=-\frac{1}{3}\int_0^{\pi/2}\left[a^3\sin^3\theta-a^3\right]d\theta$$

$$=-\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta-1)\,d\theta$$

$$=-\frac{a^3}{3}\left[\int_0^{\pi/2}\sin^3\theta\,d\theta-\int_0^{\pi/2}d\theta\right]$$

$$=-\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right]\Rightarrow\frac{a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right].$$

Exp. Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$

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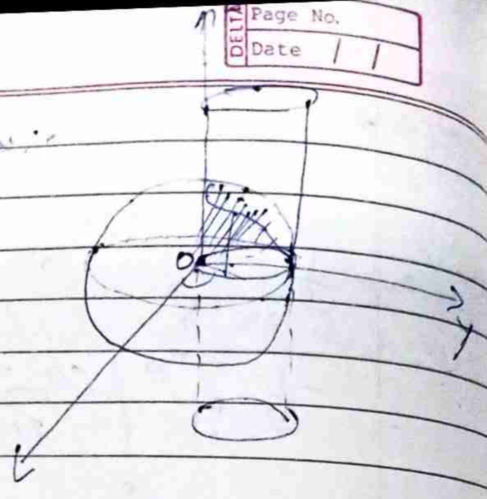
Q. Find the volume :



Sol. The required volume is the part of the sphere $x^2+y^2+z^2=a^2$ lying within the cylinder $x^2+y^2=ay$. Due to symmetry of the sphere, half of the volume lies above the xy plane and half below it.

$$x^2+y^2+z^2=a^2$$
$$\therefore z^2 = a^2-x^2-y^2$$
or $z = \sqrt{a^2-x^2-y^2}$

$\therefore$ Required volume $= 2\iint_R z\,dy\,dx$

where R is the region of integration bounded by the circle $x^2+y^2=ay$ in xy plane.

Due to symmetry, the region of integration in first and fourth quadrant are equal.

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$\therefore$ Required volume $= 2\times 2\iint_{R_1} z\,dy\,dx$

where R_1 is half of the circle $x^2+y^2=ay$ which lies in the first quadrant.

Changing to polar coordinate we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$\therefore x^2+y^2=ay$ becomes
$$r^2 = ar\sin\theta$$
$$r = a\sin\theta$$

Thus θ varies from 0 to $\frac{\pi}{2}$ and r varies from 0 to $a\sin\theta$

$\therefore$ Required volume $= 4\iint_{R_1} z\,dy\,dx$

$$= 4\iint \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 4\int_0^{\pi/2}\int_0^{a\sin\theta}\sqrt{a^2-r^2}\,|J|\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{a\sin\theta}(a^2-r^2)^{1/2}\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{a\sin\theta}-\frac{1}{2}(a^2-r^2)^{1/2}(-2r)\,dr\,d\theta$$

$$= -2\int_0^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\sin\theta}d\theta$$

$$= -\frac{2v2}{3}\int_0^{\pi/2}\left[(a^2-a^2\sin^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}\left[(a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}\left[a^3\cos^3\theta - a^3\right]d\theta$$

$$= -\frac{4}{3}a^3\int_0^{\pi/2}(\cos^3\theta - 1)\,d\theta$$

$$= -\frac{4}{3}a^3\left[\int_0^{\pi/2}\cos^3\theta\, d\theta - \int_0^{\pi/2}d\theta\right]$$

$$= -\frac{4}{3}a^3\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{2}{9}a^3(3\pi-4)$$

Exp. Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the plane $x=0$, $y=0$, $z=0$, and $x+y+z=1$

Sol.

Let R be the region bounded by the give tetrahedron

$R\ (x,y,z) : x+y+z\le 1,\ x\ge 0,$
$y\ge 0,\ z\ge 0$

Here $x\le 1 \Rightarrow 0\le x\le 1$

$x+y\le 1 \Rightarrow y\le 1-x \Rightarrow 0\le y\le 1-x$

$x+y+z\le 1 \Rightarrow z\le 1-x-y \Rightarrow$
$0\le z\le 1-x-y$

Let $I = \iiint (x+y+z)\,dx\,dy\,dz$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{(x+y+z)^2}{2}\right]_{z=0}^{1-x-y}dy\,dx$$

$$= \int_0^1\int_0^{1-x}\frac{1}{2}\left[(x+y+1-x-y)^2-(x+y)^2\right]dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x}\left[1-(x+y)^2\right]dy\,dx$$

$$= \frac{1}{2}\int_0^1\left[y-\frac{(x+y)^3}{3}\right]_{y=0}^{1-x}dx$$

$$= \frac{1}{2}\int_0^1\left[1-x-\frac{(x+1-x)^3}{3}-0+\frac{x^3}{3}\right]dx$$

$$= \frac{1}{2}\int_0^1\left(1-x-\frac{1}{3}+\frac{x^3}{3}\right)dx$$

$$= \frac{1}{2}\int_0^1\left(\frac{2}{3}-x+\frac{x^3}{3}\right)dx$$

$$\frac{1}{2}\left| \frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12} \right|_{x=0}^{1}$$

$$\frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12} \right] = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$\therefore \iiint_{x+y+z \le 1} (x+y+z)\,dx\,dy\,dz = \frac{1}{8} \text{ Ans}$$

*** Substitution method for triple integral:**

Exp. Evaluate $\iiint_R z(x^2+y^2)\,dx\,dy\,dz$ where

$R = (x,y,z) : -1 \le z \le 1,\ x^2+y^2 \le 4$

Sol. Changing to cylindrical co-ordinate we have

$x = r\cos\theta,\ y = r\sin\theta$ and $z = z$

$x^2+y^2 = r^2$

$x^2+y^2 \le 4$

$r^2 \le 4 \Rightarrow |r| \le 2$

Here the space R is changed to space R' when

$R' = \{(r,\theta,z)\ \ 0 \le r \le 2,$

$0 \le \theta \le 2\pi$

$-1 \le z \le 1$

$$\therefore \iiint_R z(x^2+y^2)\,dx\,dy\,dz$$

$$= \iiint_{R'} z(r^2)\,|J|\,dr\,d\theta\,dz$$

$$= \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2} z r^2 r\,dr\,d\theta\,dz.$$

$$= \int_{-1}^{1}\int_{0}^{2\pi} z\left|\frac{r^4}{4}\right|_0^2 d\theta\,dz$$

$$= \int_{-1}^{1}\int_{0}^{2\pi} 4z\,d\theta\,dz = \int_{-1}^{1} 4z\,|\theta|_0^{2\pi}\,dz$$

$$= \int_{-1}^{1} 8\pi z\,dz = 8\pi\left|\frac{z^2}{2}\right|_{-1}^{1}$$

$$= 0 \text{ Ans}$$

Exp. Evaluate $\iiint_{x^2+y^2+z^2 \le 1} (x^2+y^2+z^2)\,dx\,dy\,dz$

Sol. Let $V = (x,y,z) : x^2+y^2+z^2 \le 1$

Changing to cylindrical co-ordinates we have

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$\therefore x^2+y^2+z^2 = r^2$

Since $x^2+y^2+z^2 \le 1$

$\therefore r^2 \le 1 \Rightarrow |r| \le 1$

Also $|J| = r^2 \sin\theta$

the V transforms to V' when

$$V' = (r,\theta,\phi) : (0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi)$$

$$\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz = \iiint_{V'} r^2 |J|\, d\theta\, d\phi\, dr$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} r^2 \cdot r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \int_0^1 \int_0^{2\pi} -[r^4 \cos\theta]_0^{\pi}\, d\phi\, dr$$

$$= \int_0^1 \int_0^{2\pi} 2r^4\, d\phi\, dr \Rightarrow \int_0^1 2r^4 [\phi]_0^{2\pi}\, dr$$

$$= \int_0^1 2r^4 \cdot 2\pi\, dr = 4\pi \left[\frac{r^5}{5}\right]_0^1$$

$$= \frac{4\pi}{5} \text{ Ans}$$

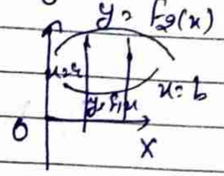
Application of Double and Triple Integrals for finding are and volume of Surfaces.

Consider the rectangle $0 \le x \le b$, $c \le y \le d$ its area is $(b-a)(d-c)$

Consider $\int_c^d \int_a^b 1\,dx\,dy = \int_c^d (b-a)\,dy$

$(d-c)(b-a)$

Area of the rectangle



Again we know that the area b/w the curves $y = f_1(x)$, $y = f_2(x)$ and the co-ordinates $x = a$, and $x = b$ is

$$= \int_a^b f_2(x) - f_1(x)\, dx$$

Consider the integral

$$\int_a^b \int_{f_1(x)}^{f_2(x)} 1\,dy\,dx = \int_a^b [f_2(x) - f_1(x)]\,dx$$

this leads us to the idea that the area of the region T is given by $\iint_T 1\,dx\,dy$.

Similarly volume of the space V is given $\iiint_V 1\,dx\,dy\,dz$.

Exp. Find the volume of sphere of radius a

Or

Find the volume of sphere $x^2+y^2+z^2 < a^2$, $a>0$

Sol. The eqn of sphere is $x^2+y^2+z^2 = a^2$

$$\text{Volume of Sphere} = \iiint_{x^2+y^2+z^2 < a^2} 1\,dx\,dy\,dz$$

Volume of sphere $= \iiint\limits_{x^2+y^2+z^2 \le a^2} 1 \, dx\,dy\,dz$

Changing to spherical polar co-ordinate

put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$
and $z = r\cos\theta$

$\therefore x^2+y^2+z^2 = r^2$

But $x^2+y^2+z^2 \le a^2$

$\therefore r^2 \le a^2 \Rightarrow |r| \le a \Rightarrow 0 \le r \le a$

Also

$|J| = r^2 \sin\theta$

the region of integration V' (in terms of r, θ, ϕ) is given by

$V' = \{ (r,\theta,\phi) : 0 \le r \le a,\ 0 \le \theta \le \pi,$
$0 \le \phi \le 2\pi \}$

$\therefore$ Volume of sphere $= \iiint\limits_{x^2+y^2+z^2 \le a^2} 1 \, dx\,dy\,dz$

$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} |J| \, dr\,d\theta\,d\phi$

$= \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, dr\,d\theta\,d\phi$

$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \left| \frac{r^3}{3} \right|_0^a d\theta\,d\phi$

$\int_0^{2\pi}\int_0^{\pi} \frac{a^3}{3} \sin\theta \, d\theta\,d\phi$

$= \int_0^{2\pi} \frac{a^3}{3} (-\cos\theta)_0^{\pi} \, d\phi$

$= \int_0^{2\pi} \frac{a^3}{3} (-\cos\pi + \cos 0) \, d\phi$

$= \int_0^{2\pi} \frac{2a^3}{3} \, d\phi$

$= \frac{2a^3}{3} [\phi]_0^{2\pi} = \frac{2a^3}{3} \cdot 2\pi = \frac{4\pi a^3}{3}$

$\therefore$ Volume of sphere $= \frac{4\pi a^3}{3}$ Ans

* Dirichlet Integral

the triple integral $\iiint\limits_V x^{p-1} \cdot y^{m-1} \cdot z^{n-1} \, dx\,dy\,dz$

$= \frac{\Gamma p \, \Gamma m \, \Gamma n}{\Gamma(1+p+m+n)}$ where V is the region given by $x \ge 0$, $y \ge 0$, $z \ge 0$, $x+y+z \le 1$ is called Dirichlet's integral.

* To establish the Dirichlet integral

$\iiint\limits_V x^{p-1} \cdot y^{m-1} \cdot z^{n-1} \, dx\,dy\,dz = \frac{\Gamma p \, \Gamma m \, \Gamma n}{\Gamma(1+p+m+n)}$

where V is the region $x \ge 0$, $y \ge 0$, $z \ge 0$ and $x+y+z \le 1$

Proof:- $V = \{(x,y,z) : x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$

$V = (x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y$

Let $I = \iiint_V x^{p-1} \cdot y^{m-1} \cdot z^{n-1}\, dx\, dy\, dz$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{p-1} y^{m-1} z^{n-1}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x^{p-1} \cdot y^{m-1} z^{n-1}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x^{p-1} y^{m-1} \left|\frac{z^n}{n}\right|_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x^{p-1} y^{m-1} \frac{(1-x-y)^n}{n}\, dy\, dx$$

put $y = (1-x)t$ so that $dy = (1-x)\,dt$

when $y=0$, $t=0$ when $y = 1-x$, $t=1$

putting these value in (1) we have

$$I = \int_0^1 \int_0^1 x^{p-1} (1-x)^{m-1} t^{m-1} \frac{[1-x-(1-x)t]^n}{n} (1-x)\,dt\,dx$$

$$= \int_0^1 \int_0^1 x^{p-1} (1-x)^{m-1} t^{m-1} \frac{[(1-x)(1-t)]^n}{n} (1-x)\,dt\,dx$$

$$= \frac{1}{n} \int_0^1 \int_0^1 x^{p-1} (1-x)^{m+n} t^{m-1} (1-t)^n\, dt\, dx$$

$$= \frac{1}{n} \int_0^1 x^{p-1} (1-x)^{m+n} B(m, n+1)\, dx$$

$$= \frac{1}{n} B(m, n+1) \int_0^1 x^{p-1} (1-x)^{m+n}\, dx$$

$$= \frac{1}{n} B(m, n+1)\ B(p, m+n+1)$$

$$= \frac{1}{n} \frac{\Gamma m\, \Gamma n+1}{\Gamma(m+n+1)} \cdot \frac{\Gamma p\, \Gamma(m+n+1)}{\Gamma(p+m+n+1)}$$

$$= \frac{1}{n} \frac{\Gamma m\, \Gamma n+1 \cdot \Gamma p}{\Gamma(1+p+m+n)} = \frac{1}{n} \frac{\Gamma m \cdot n \Gamma n\, \Gamma p}{\Gamma(1+p+m+n)}$$

$$= \frac{\Gamma p\, \Gamma m\, \Gamma n}{\Gamma(1+p+m+n)}$$

Thus we have $\iiint_V x^{p-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$

$$= \frac{\Gamma p\, \Gamma m\, \Gamma n}{\Gamma(1+p+m+n)}.$$

\# Liouville's extension of Dirichlet's Integral.

If x, y, z are all positive such that $h_1 \leq x+y+z \leq h_2$ then

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$$\iiint F(x+y+z)\,x^{p-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$$

$$= \frac{\Gamma p\,\Gamma m\,\Gamma n}{\Gamma(p+m+n)} \int_{h_1}^{h_2} F(h)\cdot h^{p+m+n-1}\,dh.$$

Exp. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, and C. Apply Dirichlet's integrals to find the volume of the tetrahedron OABC.

Sol. The given eqn of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

put $x = au$, $y = bv$, $z = cw$ so that

$dx = a\,du$, $dy = b\,dv$ and $dz = c\,dw$

$$dx\,dy\,dz = abc\,du\,dv\,dw$$

The condition are $u \geq 0$, $v \geq 0$, $w \geq 0$ and $u+v+w \leq 1$

Volume of tetrahedron OABC

$$= \iiint_{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\leq 1} 1\,dx\,dy\,dz$$

$$= \iiint (abc)\,du\,dv\,dw$$

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$$abc \iiint_{u+v+w\leq 1} u^{1-1}v^{1-1}w^{1-1}\,du\,dv\,dw$$

Applying Dirichlet's integral, we have

Volume of tetrahedron

$$= abc\,\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)} = \frac{abc}{\Gamma 4}$$

$$\frac{abc}{3!} = \frac{abc}{3\times 2} = \frac{abc}{6}$$

Exp. Using Dirichlet's theorem, find the volume bounded by the surface.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}=1$$

Let $V = \left\{(x,y,z) : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}\leq 1\right\}$

put $\frac{x^2}{a^2}=X$, $\frac{y^2}{b^2}=Y$ and $\frac{z^4}{c^4}=Z$

$x = aX^{1/2}$, $y = bY^{1/2}$ and $z = cZ^{1/4}$

$dx = \frac{a}{2}X^{-1/2}dX$, $dy = \frac{b}{2}Y^{-1/2}dY$

and $dz = \frac{c}{4}Z^{-3/4}dZ$

$$dx\,dy\,dz = \frac{abc}{16}X^{-1/2}Y^{-1/2}Z^{-3/4}\,dX\,dY\,dZ$$

Required volume $= 8 \times$ Volume bounded by the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}=1$ in the positive octant

$$8\iiint dx\,dy\,dz$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}\le 1$$

$$= 8\iiint_{x+y+z\le 1} \frac{abc}{16} X^{-1/2} Y^{-1/2} Z^{-3/4}\, dX\,dY\,dZ$$

$$= \frac{abc}{2} \cdot \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{4}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+1\right)}$$ using Dirichlet's

$$\frac{abc}{2} \cdot \frac{\sqrt{\pi}\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{9}{4}\right)} = \frac{\pi abc}{2}\,\frac{\Gamma\frac{1}{4}}{\Gamma\frac{9}{4}}$$

$$= \frac{\pi abc}{2} \cdot \frac{\Gamma\frac{1}{4}}{\frac{5}{4}\cdot\frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right)} \quad (\because \Gamma n = (n-1)\Gamma(n-1)$$
$$= (n-1)(n-2)\Gamma(n-2)]$$

$$= \frac{8\pi abc}{5}$$

Ex. Show that the mass of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density at any point being $\rho = kxyz$ is $\frac{k a^2 b^2 c^2}{48}$

Sol. Let $V = \{x, y, z : x\ge 0, y\ge 0, z\ge 0,$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\}$$

put $\frac{x^2}{a^2} = X$, $\frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z$

$x = aX^{1/2}$, $y = bY^{1/2}$ and $z = cZ^{1/2}$

$dx = \frac{a}{2}X^{-1/2}dX$, $dy = \frac{b}{2}Y^{-1/2}dY$ and $dz = \frac{c}{2}Z^{-1/2}dZ$

$$dx\,dy\,dz = \frac{abc}{8} X^{-1/2} Y^{-1/2} Z^{-1/2}\, dX\,dY\,dZ$$

Required mass $= \iiint \rho\, dx\,dy\,dz$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

$$= \iiint (kxyz)\, dx\,dy\,dz$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

$$= \iiint_{x+y+z\le 1} k\,(aX^{1/2})(bY^{1/2})(cZ^{1/2})\,\frac{abc}{8}\,X^{-1/2}Y^{-1/2}Z^{-1/2}\, dX\,dY\,dZ$$

$$= \frac{k a^2 b^2 c^2}{8}\iiint_{x+y+z\le 1} X^{1-1} Y^{1-1} Z^{1-1}\, dX\,dY\,dZ$$

$$= \frac{k a^2 b^2 c^2}{8} \cdot \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)}$$

$$= \frac{k a^2 b^2 c^2}{8\,\Gamma 4} = \frac{k a^2 b^2 c^2}{48} \text{ Pr.} \quad (\Gamma 4 = 3!$$
$$3! = 3\times 2 = 6$$

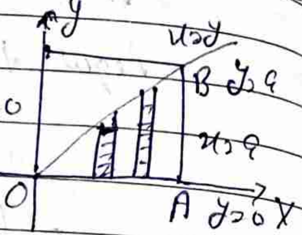
Change of order of integration:

Exp. Change the order of integration in $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ and hence evaluate the same.

Sol. Let $I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$

Limits of integration are $0 \le y \le a$, $y \le x \le a$

∴ Region of integration is bounded by the lines $y=0$ (x-axis), $y=a$ and $x=y$, $x=a$



Region OAB is the region of integration. In the given order, we have to integrate first w.r.t. x and then w.r.t. y.

After change of order of integration we have to integrate first w.r.t. y and then w.r.t. x:-

Divide the region OAB into vertical strips.

Each vertical strip starts from line $y=0$ and ends on line $y=x$

$0 \le y \le x$

Also these strips varies from $x=0$ to $x=a$

$0 \le x \le a$

After change of order of integration we have

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a x \left(\int_0^x \frac{dy}{x^2+y^2} \right) dx$$

$$= \int_0^a x \times \frac{1}{x} \left| \tan^{-1} \frac{y}{x} \right|_{y=0}^{x} dx$$

$$= \int_0^a 1 \left[\tan^{-1} 1 - \tan^{-1}(0) \right] dx$$

$$= \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4} [x]_0^a = \frac{\pi a}{4} \text{ Ans}$$

Exp. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$. Change the order of integration and verify the result.

Sol. Let $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_{x^2}^{2-x} dx$$

$$= \int_0^1 \frac{x}{2} \left[(2-x)^2 - x^4 \right] dx$$

$$= \int_0^1 \frac{x}{2} (4 - 4x + x^2 - x^4)\,dx$$

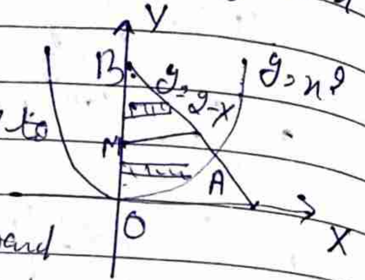
$$= \frac{1}{2} \int_0^1 (4x - 4x^2 + x^3 - x^5)\,dx$$

$$= \frac{1}{2} \left[2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4} - \frac{x^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[2 - \frac{4}{3} + \frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{9}{24}$$

the region of integration is bounded by $x=0$, $x=1$, $y=x^2$ and $y=2-x$



(i) $x=0$ is y axis

(ii) $x=1$ is a line // to y-axis

(iii) $y=x^2$ is an upward parabola whose vertex is at (0,0)

(iv) $y=2-x$ is a straight line passing through point (0,2), (1,1) and (2,0)

The parabola $y=x^2$ and the line $y=2-x$ intersect at point (1,1)

∴ Region OAB is the region of integration.

In the region OAM each horizontal strip starts from line $x=0$ and ends on curve $x=\sqrt{y}$. Also these strips vary from $y=0$ to $y=1$. In the region OAM, each horizontal strips starts from line $x=0$ and ends on line $x=2-y$. Also these strips vary from $y=1$ to $y=2$

Thus after change of order of integration, we have

$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left|\frac{x^2}{2}\right|_0^{\sqrt{y}} dy + \int_1^2 y\left|\frac{x^2}{2}\right|_0^{2-y} dy$$

$$= \int_0^1 \frac{y}{2}(y)\,dy + \int_1^2 \frac{y}{2}(2-y)^2\,dy$$

$$= \left|\frac{y^3}{6}\right|_0^1 + \frac{1}{2}\int_1^2 y(4-4y+y^2)\,dy$$

$$= \frac{1}{6} + \frac{1}{2}\int_1^2 (4y-4y^2+y^3)\,dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8-\frac{32}{3}+4\right) - \left(2-\frac{4}{3}+\frac{1}{4}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3} - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\times\frac{5}{12} = \frac{9}{24}$$ which verifies the result.

Chapter - 4th

Fourier series

Some Definitions:-

(1) Even function:- A function f is said to be an even function if $f(-x) = f(x)$ for all x and the graph of the function is symmetrical about y-axis.

$\cos x$, x^2, $x^3 \sin x$, $x^2 \cos x$ etc are even function.

(2) Odd function:- A function f is said to be an odd function if $f(-x) = -f(x)$ for all x and the graph of this function is symmetrical about origin. $\sin x$, $x^3 \cos x$, $x^2 \sin x$ etc

(3) Periodic function:- A function $f: R \to R$ is said to be periodic if

$f(x+\lambda) = f(x)$ for all $x \in R$ (λ ...

Some Important results on Integrals:-

If f is integrable on $[c, c+2\pi]$ and has period 2π then

(i) $\int_c^{c+2\pi} f(x) = \int_c^{c+2\pi} f(x+a)\,dx$ for any $a \in R$

(ii) $\int_c^{c+2\pi} \sin nx\,dx = 0 = \int_c^{c+2\pi} \cos nx\,dx$ for all n

(iii) $\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0$ for $m \neq n$ and $m, n = 0, 1, 2, 3 \ldots$

when $m = n$ then $\int_c^{c+2\pi} \cos mx \cos nx\,dx$

$= \int_c^{c+2\pi} \cos^2 nx\,dx = \pi$ for $n = 0, 1, 2, \ldots$

(iv) $\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0$ for all $m, n = 0, 1, 2, 3$

(v) $\int_c^{c+2\pi} \sin mx \sin nx\,dx = 0$ for $m \neq n$

$m, n = 0, 1, 2, 3 \ldots$

Some Important result to be Remembered:-

(1) $\sin(n\pi + \alpha) = (-1)^n \sin\alpha$ $n \in N$

(2) $\cos(n\pi \pm \alpha) = (-1)^n \cos\alpha$ $n \in N$

(3) $\sin\pi = \sin 2\pi = \sin 3\pi = \sin n\pi = 0$

(4) $\sin\frac{\pi}{2} = 1$, $\sin\frac{3\pi}{2} = -1$, $\sin\frac{5\pi}{2} = 1$

$\sin\frac{7\pi}{2} = -1$

(5) $\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \cos\frac{7\pi}{2}$

$= \cos(2n+1)\frac{\pi}{2} = 0$

(vi) $\cos 0 = \cos 2\pi = \cos 4\pi = \cos 6\pi = \cos 2n\pi = 1$

(vii) $\cos \pi = \cos 3\pi = \cos 5\pi = \cos (2n+1)\pi = -1$, $n \in \mathbb{N}$

Fourier series:-

Let f be a real valued bounded and integrable function (Riemann integrable) defined on $(c, c+2\pi)$ such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \quad n=(1,2,3 \ldots)$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \quad (n=1,2,3\ldots)$$

Then the expression ① is known as Fourier series and the constant a_n, b_n are called Fourier coff of f, the values of a_0, b_n, a_n are known as Euler's formulae.

Fourier series for even and odd functions:-

Case I: If $f(x)$ is an even function defined in the interval $[-\pi, \pi]$ then $f(x)\cos nx$ is an even function and $f(x)\sin nx$ is odd function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \text{ for } n=0,1,2 \ldots \quad ①$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \text{ for } n=1,2,3 \ldots$$

Thus the Fourier series corresponding to an even function in $[-\pi, \pi]$ consist of terms with cosine only so that we have

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad ②$$

$a_n (n \geq 0)$ are given by ①

The series given by ② is known as cosine series.

Case II If $f(x)$ is an odd function defined in the interval $[-\pi, \pi]$ the function $f(x)\cos nx$ and $f(x)\sin nx$ are respectively the odd and even function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad \text{for } n \geq 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ for $n=1,2,3$

This the fourier series corresponding to an odd function in $[-\pi,\pi]$ consists of terms with sines only, so that we have

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (4)}$$

when $b_n\ (n\geq 1)$ are given by 3

The series given by 4 is known as sine series.

Exp. Find the Fourier series expansion of $f(x) = \frac{1}{4}(\pi-x)^2 \quad 0 \le x \le 2\pi$

Let $f(x) = \frac{1}{4}(\pi-x)^2$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi-x)^2\,dx$$

$$\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi}\left[-\pi^3 - \pi^3\right]$$

$$= -\frac{1}{12\pi}\cdot 2\pi^3 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n} + 2(\pi-x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi\cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi\cos 0}{n^2} + 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] \quad (\because \cos 2n\pi = 1)$$

$$= \frac{1}{4\pi}\cdot\frac{4\pi}{n^2} = \frac{1}{n^2}$$

Also

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - \frac{2}{n^2}(\pi-x)\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(-\frac{\pi^2}{n} - 0 + \frac{2}{n^3}\cos 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\times 0 = 0$$

Hence the Fourier series for $\frac{1}{4}(\pi-x)^2$ is

$$\frac{1}{4}(\pi-x)^2 \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$\frac{1}{4}(\pi-x)^2 \sim \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} \cdots$$

Eg. Find the Fourier series (or expansion) for the function $f(x) = |x|, -\pi \le x \le \pi$. Also deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Sol. Since $f(x) = |x|$ is an even function, therefore $b_n = 0$ for $n = 1, 2, 3, \ldots$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} -x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \pi$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx$

$[\because 0 \le x \le \pi,\ f(x) = |x| = x]$

$$= \frac{2}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[nx\sin nx + \cos nx\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[n\pi\cos n\pi + \cos n\pi - 0 - 1\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right] \text{ for } n = 1, 2, 3, \ldots$$

$$a_n = \begin{cases} -\frac{4}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus we obtain $f(x) = |x|$

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

when $x = 0$

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \text{ Pr}$$

★ Fourier expansion of Function Having point of Discontinuity

At a point of finite discontinuity x=a there is a finite jump in the graph of the function. Both the limit on the left $f(a-0)$ and the limit on the right $f(a+0)$ exist and are diff. At such a point Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits at x=a

$$f(x) = \frac{1}{2}\left[f(a-0) + f(a+0)\right]$$

Exp. Obtain the Fourier series expansion of the function f in $(0, 2\pi)$ defined as:-

$$f(x) = \begin{cases} x & \text{for } 0 < x < \pi \\ 0 & \text{for } \pi < x < 2\pi \end{cases}$$

Sol. $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx + \int_0^{2\pi} 0\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{n^2}\left[\cos nx\right]_0^{\pi} = \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} f\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left(\frac{-x\cos nx}{n}\right)_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{(-1)^{n+1}}{n}$$

$\therefore b_1 = 1,\ b_2 = -\frac{1}{2},\ b_3 = \frac{1}{3},\ b_4 = -\frac{1}{4}.$

Hence we obtain the Fourier series expansion as:-

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

$$+ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots$$

Ex. If $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$ then

prove that $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$

Hence show that

(i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots = \frac{1}{2}$

(ii) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots = \frac{\pi - 2}{4}$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} \sin x\,dx\right]$

$= \frac{1}{\pi}\left[0 - [\cos x]_0^{\pi}\right] = \frac{2}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} \sin x \cos nx\,dx\right]$

$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos nx\,dx$

$= \frac{1}{2\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx$

$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$

$\frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

therefor $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{2}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$

If $n=1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\,dx$

$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\,dx$

$= \frac{1}{2\pi}\left[\frac{\cos 2x}{2}\right]_0^{\pi} = 0$

$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} \sin x \sin nx\,dx\right]$

$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos nx\,dx$

$= \frac{1}{2\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]\,dx$

$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0$

Therefore $b_n = 0$ $n \neq 1$

If $n=1$ $b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x\,dx$

$= \frac{1}{\pi}\int_0^{\pi} \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) dx$

$\frac{1}{\pi}\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}\right] = \frac{1}{2}$

Also $b_n = 0$ for $n = 2, 3$ ——

Hence we obtain the fourier series expansion as:-

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \quad \text{(i)}$$

put $x = 0$ in (i) we have.

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$$

$$= \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$$

ii. putting $x = \frac{\pi}{2}$ in (i) we have

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\frac{\pi-2}{4} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)}$$

$$= \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \ldots$$

Hence proved that

Change of Interval:-

Exp. Obtain Fourier series for the function $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

Sol. Here $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ — (1) Cx

$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$\pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \pi\frac{1}{2} + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x \cdot \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+ \left[\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left[-\frac{\cos n\pi x}{n^2\pi^2}\right]\right]_1^2$$

$$\left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right]$$

$$\frac{2}{n^2\pi}\left[\cos n\pi - 1\right) = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\begin{cases} 0 & \text{when } n \text{ is even} \\ -\frac{4}{n^2\pi} & \text{when } n \text{ is odd} \end{cases}$$

$$b_n = \int_0^2 f(x) \sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

From (1)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

Hence proved.

Half Range Series:-

Sine Series:- When it is required to express $f(x)$ as a sine series in $0 < x < c$, we extend the function $f(x)$, reflecting it in the origin, so that
$$f(-x) = -f(x)$$
Then the extended function is odd function in $(-c, c)$ and its expansion will give the half range sine series as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where $b_n = \frac{2}{c}\int_0^c f(x) \sin\frac{n\pi x}{c} dx$

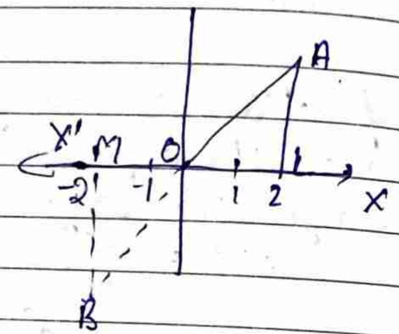
Cosine series:- When it is required to express $f(x)$ as cosine series in $0 < x < c$, we extend the function $f(x)$ reflecting it in the y-axis, so that $f(-x) = f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$

where $a_0 = \frac{2}{c}\int_0^c f(x)\,dx$ and $\frac{2}{c}\int_0^c f(x) \cos\frac{n\pi x}{c} dx$

Exp. Express $f(x) = x$ as a half range sine series in $0 < x < 2$

Let OA be the line which represents the graph $f(x) = x$ in $0 < x < 2$



Let us extend the function $f(x)$ in the interval $-2 < x < 0$ as shown by line OB so that the new function is symmetrical about the origin

$$f(x) = \begin{cases} x & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$$

In this case $f(x)$ is an odd function of x

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

where $b_n = \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx$

$$= \int_0^2 x \sin \frac{n\pi x}{2} dx$$

$$= \left[\frac{-2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right]_0^2$$

$$= -\frac{4(-1)^n}{n\pi}$$

thus $b_1 = \frac{4}{\pi}$, $b_2 = -\frac{4}{2\pi}$, $b_3 = \frac{4}{3\pi}$

$b_4 = -\frac{4}{4\pi}$, ... —

Hence the half range sine series is

$$f(x) = \frac{4}{\pi} \left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + \dots \right)$$

Ans

★ Parseval's Identity for Fourier series:-

$$\int_c^{c+2l} [f(x)]^2 dx = l \left[\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

provided the fourier series for $f(x)$ converges uniformly in $(c, c+2l)$.

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Exp. Let $F(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$ using Parseval's identity, compute the sum $\sum_{k=1}^{\infty} (2k-1)^{-2}$

Sol. By Parseval's identity, we

$$\int_{-\pi}^{\pi} [F(x)]^2 dx = \pi \left[\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] \quad -①$$

$$\int_{-\pi}^{\pi} [F(x)]^2 dx = \int_{-\pi}^{0} [F(x)]^2 dx + \int_{0}^{\pi} [F(x)]^2 dx$$

$$= \int_{-\pi}^{0} (-1)^2 dx + \int_{0}^{\pi} (1)^2 dx$$

$$= [x]_{-\pi}^{0} + [x]_{0}^{\pi}$$

$$= [0+\pi] + [\pi - 0] = 2\pi \quad -②$$

Since $F(x)$ is an odd function therefore $a_n = 0$ for $n = 1, 2 \ldots$

Now $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin nx \, dx$$

$$= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_{0}^{\pi} = \frac{2}{\pi} \left[\frac{-\cos n\pi}{n} + \frac{1}{n} \right]$$

$$= \frac{2}{n\pi} (1 - \cos n\pi)$$

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$$= \begin{cases} 0 & \text{for } n = 2k \text{ (even)} \\ \frac{4}{\pi(2k-1)} & \text{for } n = 2k-1 \text{ (odd)} \end{cases}$$

putting the values of a_n, b_n and using ② in ① we have

$$2\pi = \pi \left[\sum_{k=1}^{\infty} \frac{16}{\pi^2} \frac{1}{(2k-1)^2} \right]$$

$$2\pi = \frac{16}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \quad \therefore$$

$$\sum_{k=1}^{\infty} (2k-1)^{-2} = \frac{\pi^2}{8}$$ which is the required sum.

Chapter - 5th
Calculus of Complex Function.

Method 1. Let $\mathbb{C}$ be the complex plane. In this method we set up a correspondence b/w the points of the complex plane $\mathbb{C}$ and those of a sphere of radius $\frac{1}{2}$ with centre at $(0,0,\frac{1}{2})$ tangent to this plane through the origin construct a line perpendicular to $\mathbb{C}$. Consider the sphere S of radius $\frac{1}{2}$ and centre at $(0,0,\frac{1}{2})$

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 : X^2 + Y^2 + \left(Z - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}$$

Image of point $(x,y,0)$ on the complex plane $\mathbb{C}$ is the point (X,Y,Z) on the sphere S where

$$X = \frac{x}{1+x^2+y^2}, \quad Y = \frac{y}{1+x^2+y^2}$$

$$\text{and } Z = \frac{x^2+y^2}{1+x^2+y^2}$$

Image of point $(x,y,0)$ on the complex plane $\mathbb{C}$ is the point (X,Y,Z) on the sphere S when $X = \frac{2x}{1+x^2+y^2}$

$$Y = \frac{2y}{1+x^2+z^2} \text{ and } Z = \frac{x^2+y^2-1}{1+x^2+y^2}$$

Exp. Determine the image or stereographic projection of the following points on the sphere of radius $\frac{1}{2}$ and centre $(0,0,\frac{1}{2})$

$1+i$

Sol. $x+iy = 1+i$

Equating real and imaginary parts
$x=1$ and $y=1$

We know that image of point $x+iy$ of complex plane is the point (X,Y,Z) on the sphere,

when $X = \frac{x}{1+x^2+y^2}$, $Y = \frac{y}{1+x^2+y^2}$

$$Z = \frac{x^2+y^2}{1+x^2+y^2}$$

put $x=1$ and $y=1$ we have

$$X = \frac{1}{3}, \quad Y = \frac{1}{3} \text{ and } Z = \frac{2}{3}$$

Image of point $1+i$ of the complex plane is the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$ on the sphere

Exp. determine the image or stereographic projections of the following points on the sphere of radius 1 an central $(0,0,0$

(i) $x+iy = 1+i$

Equating real and imaginary part we have $x=1$ and $y=1$

We know that image of points $x+iy$ of the complex plane is the point (X, Y, Z) on the sphere where

$X = \frac{2x}{1+x^2+y^2}$ $Y = \frac{2y}{1+x^2+y^2}$ $Z = \frac{x^2+y^2}{1+x^2+y^2}$

putting $x=1$ and $y=1$ we get

$X = \frac{2}{3}$; $Y = \frac{2}{3}$ $Z = \frac{1}{3}$

Hence image of the point $(1+i)$ of the complex plane is the point $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ on the sphere.

☆ Limit of A Complex function:-

Let $w = f(z)$ be any complex function of z defined in a bounded and closed domain D. we say that $F(z)$ tends to the limit l as z approaches z_0 along any path in D, if to each positive arbitrary number ε there exists a positive number δ such that

$|F(z)-l| < \varepsilon$ whenever $0<|z-z_0|<\delta$

$l-\varepsilon < F(z) = l+\varepsilon$ whenever $0<|z-z_0|<\delta$

$l-\varepsilon < F(z) < l+\varepsilon$ whenever $z_0-\delta<z<z_0+\delta$ ($z \neq z_0$)

$\lim_{z \to z_0} F(z) = l$

☆ Continuity of a complex function:-

A complex function $w = F(z)$ defined in the bounded closed domain D is said to be continuous at a point $z = z_0$ of D, if given any positive number ε, we can find a positive number δ such that $|F(z) - F(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta$

Symbolically,

$\lim_{z \to z_0} F(z) = F(z_0)$

Uniform Continuity:-

A complex function $F(z)$ is said to be uniformly continuous in the domain D if for a given $\varepsilon>0$ it is possible to find a number $\delta>0$ such that the inequality $|F(z_1) - F(z_2)| < \varepsilon$ holds for every pair of points z_1, z_2 of the Domain D for which $|z_1 - z_2| < \delta$

Exp. Show that $f(z) = z^2$ is uniformly continuous in the region $|z|<1$

Sol. $F(z) = z^2$

Suppose $F(z) = z^2$ is uniformly cont. in the region $|z|<1$

Then for a given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|F(z) - F(z_0)| = |z^2 - z_0^2| < \varepsilon$$
whenever $|z - z_0| < \delta$

where δ depends only on ε not on the particular choice of the point z_0, if z and z_0 are any points in $|z|<1$, then

$$|z^2 - z_0^2| = |z+z_0||z-z_0|$$
$$\leq (|z| + |z_0|)\,|z-z_0|$$
$$< 2|z-z_0|$$

Thus if we choose $\delta = \frac{\varepsilon}{2}$ it follows that $|z^2 - z_0^2| < \varepsilon$

This ensures that $F(z) = z^2$ is uniformly cont. in $|z|<1$

☆ **Differentiablity of a complex function:–**

Let $f(z)$ be a single valued complex function defined in a domain D of complex plane

we say that $f(z)$ is diff. at a point $z_0 \in D$ if the increment ratio $\frac{f(z)-f(z_0)}{z-z_0}$ tends to a finite limit as z tends to z_0 in any manner, provided that z always remains a point of D. This finite limit, if exists is termed as the diff. coeff. or the derivative of F at $z = z_0$ and is denoted by $f'(z_0)$

thus $\lim_{z\to z_0} \frac{F(z) - f(z_0)}{z - z_0} = f'(z_0)$

Exp. Show that the function $F(z) = |z|^2$ is cont. everywhere but nowhere diff. except at the origin

Sol. $F(z) = |z|^2 = x^2+y^2$

$u(x,y) = x^2+y^2$ and $v(x,y) = 0$

The function $F(z)$ is obviously cont. everywhere because of the continuity of $u(x,y)$ and $v(x,y)$

$$F'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad z \neq (0,0)$$

$$\lim_{z\to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0}$$

$$= \frac{z\bar{z} - z_0\bar{z}_0}{z - z_0} \quad [\because |z|^2 = z\bar{z}$$

$$\lim_{z\to z_0} \frac{z\bar{z} - \bar{z}z_0 + \bar{z}z_0 - z_0\bar{z}_0}{z-z_0}$$

$$\lim_{z\to z_0} \frac{\bar{z}(z-z_0) + z_0(\bar{z}-\bar{z}_0)}{z-z_0}$$

$$= \lim_{z\to z_0}\left[\bar{z} + z_0\frac{(\bar{z}-\bar{z}_0)}{z-z_0}\right]$$

Take $z-z_0 = r(\cos\theta + i\sin\theta)$

$\overline{z-z_0} = r(\cos\theta - i\sin\theta)$

so that

$$\frac{\overline{z-z_0}}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$$

$$= \cos 2\theta - i\sin 2\theta$$

Thus $= f'(z_0) = \lim_{z\to z_0}\left[\bar{z} + z_0(\cos 2\theta - i\sin\theta)\right]$

Hence the function $F(z)$ is non diff for any non zero value of z. But if $z_0 = (0,0)$ then $F(z_0)$ tends to $\bar{z}$, which of course tends as $z \to z_0$

Exp. Using rules of diff.

$F(z) = \cos^2(2z+3i)$

$\frac{d}{dz}F(z) = \frac{d}{dz}\left[\cos^2(2z+3i)\right]$

$= 2.\cos(2z+3i)\left[-\sin(2z+3i)\right.\frac{d}{dz}(2z+3i)$

$= 4\cos(2z+3i)\sin(2z+3i)$ Ans

Analytic function:- A complex function $F(z)$ is said to be analytic at a point z_0 if it is diff. in some neighbourhood of z_0. In other words A function $F(z)$ is said to be analytic at a point z_0 if there exist a neighbourhood $|z-z_0|<\delta$ at all points of which $F'(z)$ exists.

Thm If a function $F(z) = u(x,y) + iv(x,y)$ is diff. at the point $z_0 = x_0 + iy_0$ in a domain D, then the four partial derivatives u_x, u_y, v_x and v_y exist at (x_0, y_0) and satisfy the eqs

$u_x = v_y$ and $u_y = -v_x$

Proof:- Let the function $w = F(z)$ be analytic in a region R, then $F'(z)$ exists uniquely at every point of that region.

Let $\Delta u, \Delta v$ and Δz be the increment in u, v, and z corresponding to the increments Δx, Δy and Δz be the increment in u, v and z corresponding to increment Δx. Δy in x and y resp.

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when $z \neq 0$, u and v are rational functions of x and y with non-zero denominators.

It follows that u and v are cont. when $z \neq 0$

Test the continuity of u and v at $z=0$. Change u and v in polar coordinates by taking $x = r\cos\theta$, $y = r\sin\theta$

as $u = r(\cos^3\theta - \sin^3\theta)$

$v = r(\cos^3\theta + \sin^3\theta)$

As $z \to 0$, $r \to 0$ and $\lim_{r \to 0} u = 0 = \lim_{r \to 0} v$

$\lim_{z \to 0} f(z) = 0$

$= f(0)$

$f(z)$ is continuous at $z=0$

Hence $f(z)$ is continuous for all values of z

Now at origin,

$$\left.\frac{\partial u}{\partial x}\right|_{z=0} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{u(x,0)}{x} = \lim_{x \to 0} \frac{x^3}{x^3} = 1$$

Similarly $\left.\frac{\partial u}{\partial y}\right|_{z=0} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$

$$= \lim_{y \to 0} \frac{u(0,y)}{y} = \lim_{y \to 0} \frac{-y^3}{y^3} = -1$$

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$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = 1$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The C-R eqn are therefore satisfied.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z}$$

$$= \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2+y^2)(x+iy)}$$

Let $z \to 0$ along the path x-axis then

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^2 \cdot x} = 1+i$$

Again let $z \to 0$ along the path $y = x$ then

$$f'(0) = \lim_{x \to 0} \frac{2x^3 i}{2x^3(1+i)} = \frac{i}{1+i}$$

Since limits obtained are diff the function $f(z)$ is not diff at the origin.

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CR eqn in Polar form:-

Statement :- If $F(z) = u+iv$ is an analytic function and $z = re^{i\theta}$ when u, v, r, θ are real, then CR eqn are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof :- $F(z) = u+iv$

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) \quad -(2)$$

Since u and v are functions of x and y and y are function of r and θ, therefore u and v can be treated as function of r and θ.

From eqn (1) and (2) we have

$$u+iv = f(re^{i\theta}) \quad -(3)$$

Diff eqn (3) partially w.r.t. r and θ we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = F'(re^{i\theta})e^{i\theta} \quad -(4)$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = F'(re^{i\theta})(ire^{i\theta}) \quad -(5)$$

using eqn (4) and (5) we have

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

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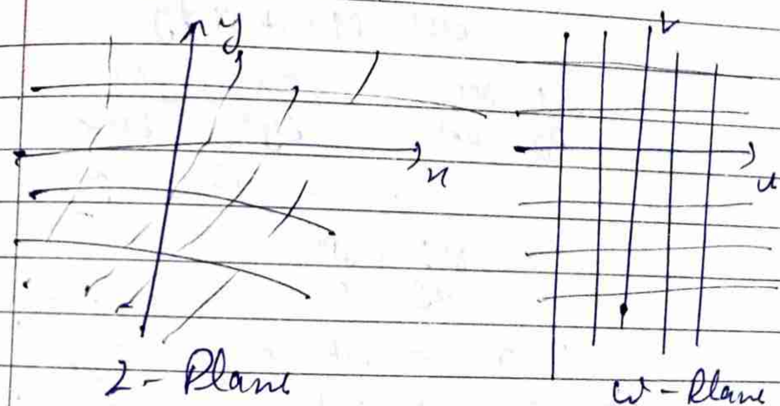
Eq. the real and imaginary part on both side we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \text{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{Pg}$$

Orthogonal System:- H.P

Two families of curves $u(x,y) = C_1$, $v(x,y) = C_2$ are said to form an orthogonal system if they intersect at right angle at each of their points of intersection.



Orthogonal families.

Harmonic Functions:-

A function of x and y which possesses continuous partial derivatives up to second order and satisfies Laplace eqn is called Harmonic function.

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* Exact Diff Method:-

Suppose $f(z) = u + iv$ is analytic and u is known. then we first determine v by using

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\left(\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

[$\because$ u is known and using C-R eqn]

Take $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$ we get

$$\therefore dv = M dx + N dy \quad —(1)$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}$$

$$= -\nabla^2 u = 0$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Consequently (1) is an exact diff. eqn, so eqn (1) can be integrated and v can be determined. Now u and v are known and hence $f(z)$ can be determined from the eqn $f(z) = u + iv$.

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Exp Construct the analytic function of which real part is

$$u(x,y) = e^x(x\cos y - y\sin y)$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = M dx + N dy$$

Here $M = -\frac{\partial u}{\partial y}$ $N = \frac{\partial u}{\partial x}$

$$N = \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y + \cos y)$$

$$M = -\frac{\partial u}{\partial y} = e^x(x\sin y + \sin y + y\cos y)$$

$$\frac{\partial M}{\partial y} = e^x[x\cos y + \cos y + \cos y - y\sin y]$$

$$= e^x[(x+2)\cos y - y\sin y]$$

$$\frac{\partial N}{\partial x} = e^x(x\cos y - y\sin y + \cos y) + e^x\cos y$$

$$e^x[(x+2)\cos y - y\sin y]$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Con. (1) is an exact diff eqn so it can be integrated as:

$$dv = e^x(x\sin y + \sin y + y\cos y)dx + e^x($$

$e^x(x\cos y - y\sin y + \cos y)\,dy$

Integrating we get

$$v = \int e^x(x\sin y + y\cos y + \sin y)\,dx$$

$+ \int$ those terms which do not contain x dy $+c$

$= \sin y \int x e^x dx + (y\cos y + \sin y)\int e^x dx + \int 0\, dy + c$

$= [(x-1)\sin y + y\cos y + \sin y]e^x + c$

$\therefore v = (x\sin y + y\cos y)e^x + c$

$F(z) = u + iv$

$= e^x(x\cos y - y\sin y) + i\,[e^x(x\sin y + y\cos y) + c]$

$= xe^x(\cos y + i\sin y) + iye^x(\cos y + i\sin y) + ic$

$= (x+iy)e^x e^{iy} + c'$

$= (x+iy)\,e^{x+iy} + c'$

$F(z) = z e^z + c'$ where $c' = ic$

Chapter 6th

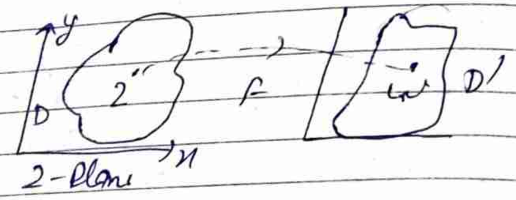
Elementary Functions and Mobius transformations

1. **Multivalued Function:-** A function $w = F(z)$ is said to be multivalued when for given z, we may find more than one value of w. thus a function $F(z)$ is said to be single valued if it is satisfies $F(z) = F(z, (r, \theta))$ $= F(z, r, \theta + 2\pi)$ otherwise, F is classified as multivalued function.

2. **Branch:-** By branch of a multivalued function $F(z)$ defined in a complex plane $\mathbb{C}$ we mean a single-valued function $F(z)$ which is analytic in some domain $D \subset \mathbb{C}$ at each point of which $F(z)$ is one of the values of $F(z)$

Mapping By Elementary Function:-

For $z = x + iy$, the eqn $u = u(x,y)$, $v = v(x,y)$ which are cont. function of x and y and possess continuous partial derivatives define a mapping or transformation by means of which a relation b/w uv-plane and xy plane can be established it is denoted by $w = f(z) = u + iv$. this mapping is said to be one-2 if corresponding to each point of xy-plane.



Exp Determine the region D' of the w-plane into which the triangular region enclosed by the lines $x=0$, $y=0$ $x+y=2$ in the 2-plane is mapped under the transformation $w = 3z$.

Sol. $z = x+iy$ and $w = 3z$

$u + iv = 3(x+iy) = 3x + i3y$

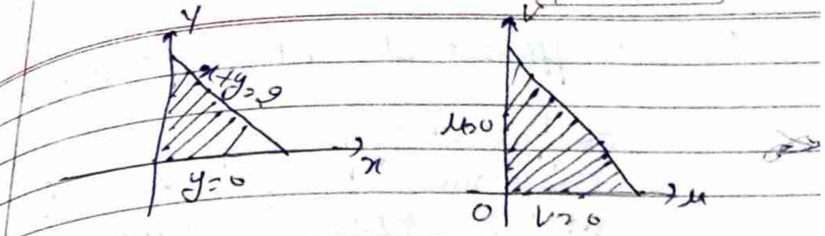
$u = 3x$, $v = 3y$

when $x=0$, $u=0$ and when $y=0$, $v=0$

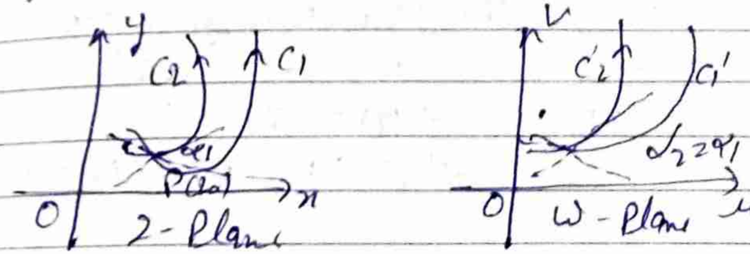
$u + v = 3(x+y) = 6$

$\therefore x+y = 2$

Hence the line $x=0, y=0$ and $x+y=2$ are transformed resp into the lines $u=0$ $v=0$ and $u+v=6$



Conformal mapping:- Suppose two C_1 and C_2 in the 2-plane intersect at the point $P(z_0 = x_0 + iy_0)$ at an angle α_1, then the transformation $f(z)$ which maps C_1 and C_2 resp. on to the curves C_1' and C_2' in w-plane intersect at $P'(w_0 = u_0 + iv_0)$ at an angle α_2 is said to be conformal if it preserves the sense of rotation as well as the magnitude of the angle $\alpha_1 = \alpha_2$



Coefficient of Magnification:-

Coefficient of magnification for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is $|f'(\alpha + i\beta)|$

Angle of Rotation:-

Angle of rotation for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is $\arg[f'(\alpha + i\beta)]$

Exp. Find the coff. of magnification and angle of rotation at $z = 2+i$ for the transformation $w = z^2$

Sol. The given transformation is

$$w = z^2 = f(z)$$

Then $f'(z) = 2z$

$$f'(2+i) = 2(2+i) = 4+2i$$

Now, coff of magnification

$$= |f'(z)|$$

$$= |f'(2+i)| = |4+2i|$$

$$= \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

Angle of rotation $= \arg[f'(z)]$

$$\arg(4+2i)$$

$$\tan^{-1}\frac{2}{4} = \tan^{-1}\frac{1}{2}$$

Mobius transformation or Bilinear transform

A transformation of the form

$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

is called a bilinear transformation, where a, b, c, d are complex constants. Such type of transformation was first studied by Mobius and hence it is sometimes called Mobius transformation.

From (1) we have

$$w(cz+d) = az+b$$

$$cwz + wd - az - b = 0 \quad \text{--- (2)}$$

Evidently it is linear both in w and z. So it is called bilinear.

Let w_1 and w_2 be the corresponding points to z_1 and z_2 in (1) then

$$w_1 = \frac{az_1+b}{cz_1+d}, \quad w_2 = \frac{az_2+b}{cz_2+d}$$

$$w_2 - w_1 = \frac{az_2+b}{cz_2+d} - \frac{az_1+b}{cz_1+d}$$

$$= \frac{(ad-bc)(z_2-z_1)}{(cz_1+d)(cz_2+d)}$$

$$= 0 \text{ if } ad-bc = 0$$

$$\therefore w_1 = w_2 \text{ if } ad-bc = 0$$

This shows that either w is constant or meaningless.

Hence if $ad-bc \neq 0$ then w has

has diff. values for diff values of z

If $ad-bc \neq 0$ it is called the determinant of the transformation.

If $ad-bc = 1$ then it is called normalised transformation.

Nature of Mobius transformation.

1. A Mobius transformation with one fixed point z_0 is called parabolic and is expressible

$$\frac{1}{w-z_0} = \frac{1}{z-z_0} + h \quad \text{if } z_0 \neq \infty$$

$$w = z + h \quad \text{if } z_0 = \infty$$

2. A Mobius transformation with two distinct fixed points z_1 and z_2 is expressible as

$$\frac{w-z_1}{w-z_2} = k\frac{(z-z_1)}{z-z_2} \quad \text{if } z_1, z_2 \neq \infty$$

If $z_2 = \infty$ then it becomes $w - z_1 = k(z - z_1)$

A transformation with two diff points is called hyperbolic if $k>0$ and elliptic if $k = e^{i\alpha}$ and $\alpha \neq 0$

loxodromic if $k = ae^{i\alpha}$, when $a \neq 1$, $\alpha \neq 0$, a and α are both real number and $a>0$

Exp Find the Mobius transformation which maps the points $z_1 = 2$, $z_2 = i$, $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$

Sol. The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

putting $z_1 = 2$, $z_2 = i$, $z_3 = -2$ and $w_1 = 1$, $w_2 = i$, $w_3 = -1$

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\Rightarrow \frac{w-1}{w+1} = \frac{(z-2)(2+i)}{(z+2)(2-i)} \times \left(\frac{1-i}{1+i}\right)$$

$$\frac{w-1}{w+1} = \frac{z-2}{z+2}\left(\frac{4-1+4i}{4+1}\right)\left(\frac{1-1-2i}{1+1}\right)$$

$$\frac{w-1}{w+1} = \left(\frac{4-3i}{5}\right)\left(\frac{z-2}{z+2}\right)$$

$$\frac{w-1+w+1}{(w-1)-(w+1)} = \frac{(4-3i)(z-2)+5(z+2)}{(4-3i)(z-2)-5(z+2)}$$

By Comp. and div.

$$w = \frac{3z(3-i)+2(1+3i)}{12(3-i)-6(3-i)}$$

$$= \frac{(3z+2i)(3-i)}{(12-6)(3-i)}$$

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$$w = \frac{3z+2i}{iz-6} \quad \text{Ans}$$

Exp. Find the image of $|w| = k$ and $\arg(w) = \alpha$ under the transformation $w = \frac{z-a}{z+a}$ where a is a real positive number.

Sol.
$$w = \frac{z-a}{z+a} \quad \text{— (1)}$$

Let $w = \rho e^{i\theta}$ — (2)

$\Rightarrow |w| = \rho \quad \arg(w) = \theta$ — (3)

Now,
$$|w| = k$$

$$\Rightarrow \left|\frac{z-a}{z+a}\right| = k$$

$$\Rightarrow |z-a| = k|z+a|$$

$$\Rightarrow |x+iy-a| = k|x+iy+a|$$

$$\Rightarrow (x-a)^2 + y^2 = k^2[(x+a)^2 + y^2]$$

$$\Rightarrow x^2+a^2-2ax+y^2 = k^2x^2+k^2a^2+2axk^2 + k^2y^2$$

$$x^2(1-k^2)+y^2(1-k^2) - 2ax(1+k^2) + a^2(1-k^2) = 0$$

$$\Rightarrow x^2+y^2 - 2ax\left(\frac{1+k^2}{1-k^2}\right) + a^2 = 0$$

which represents a family of co-axial circles

$$\rho e^{i\theta} = \frac{z-a}{z+a} \quad \text{(From (1) and (2))}$$

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Taking log on both sides, we have

$$\log\rho + i\theta = \log[(x-a)+iy] - \log[(x+a)+iy]$$

$$= \frac{1}{2}\log\{(x-a)^2+y^2\} + i\tan^{-1}\frac{y}{x-a} - \frac{1}{2}\log\{(x+a)^2+y^2\} - i\tan^{-1}\frac{y}{x+a}$$

$$\left[\because \log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}\right]$$

Comparing the imaginary part on both sides, we have

$$\theta = \tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a} = \tan^{-1}\frac{2ay}{x^2+y^2-a^2}$$

$\arg(w) = \alpha$ Given

$\theta = \alpha$ [From 3]

$$\tan^{-1}\frac{2ay}{x^2+y^2-a^2} = \alpha$$

$$x^2+y^2-2ay\cot\alpha - a^2 = 0$$

which are co-axial

Critical Mappings.

Chapter - 7th

Linear Fractional Transformations

(i) Consider the transformation $w = z^n$, $n \in Z$

$\frac{dw}{dz} = nz^{n-1} \neq 0$ for all $z \neq 0$

thus the mapping is conformal at all the points except at $z=0$

set $w = Re^{i\phi}$ $z = re^{i\theta}$

$\Rightarrow Re^{i\phi} = r^n e^{in\theta}$

$R = r^n$ $\phi = n\theta$

(a) If $r = k$ constant then $R = k^n$

Thus the circle with centre at origin and radius $|k|$ in z-plane is transformed into a circle with centre at origin and radius $|k|^n$ in w-plane

(ii) Consider the transformation $w = z^2$

The transformation is conformal except at $z = 0$

Now $w = z^2$

$u + iv = (x+iy)^2$

$= x^2 - y^2 + 2ixy$

$u = x^2 - y^2$ $v = 2xy$

(a) If $x = k$ where k is some constant then

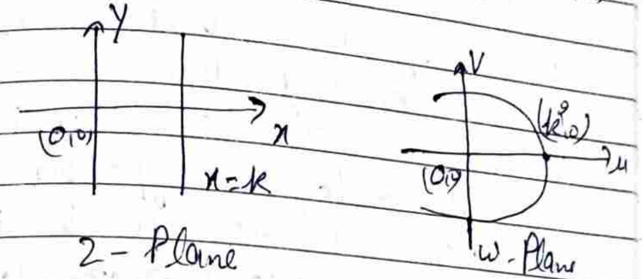
$u = k^2 - y^2$ $v = 2ky$

$v^2 = 4k^2(k^2 - u)$

$v^2 = -4k^2(u - k^2)$

which represents a left handed parabola with vertex $(k^2, 0)$ in w-plane

Hence the line $x = k$ in z-plane maps onto a left handed parabola with vertex $(k^2, 0)$ and focus $(0,0)$ in w-plane.

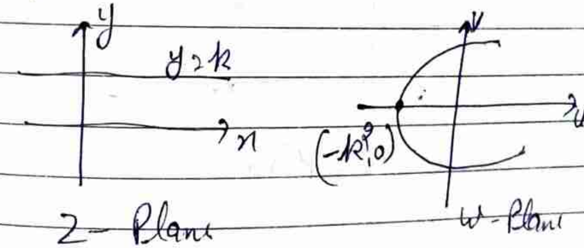


(b) If $y = k$ where k is some constant then

$u = x^2 - k^2$, $v = 2kx$

$v^2 = 4k^2(u + k^2)$

which represents a right handed parabola with vertex $(-k^2, 0)$ and focus $(0,0)$ in w-plane.



(ii) Consider the transformation $z = \sqrt{w}$

$z^2 = w$

$u + iv = (x+iy)^2$

$u + iv = x^2 + i^2y^2 + 2ixy$

$= x^2 - y^2 + 2ixy$

$u = x^2 - y^2, \quad v = 2xy$

(a) If $u = k$ when k is some constant then

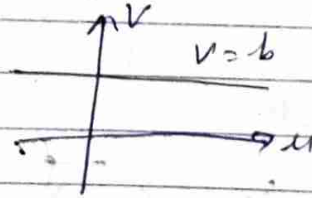
$x^2 - y^2 = k$ which represent a rectangular hyperbola

Hence the line $u = k$ in w-plane maps on to a rectangular hyperbola in z-Plane.

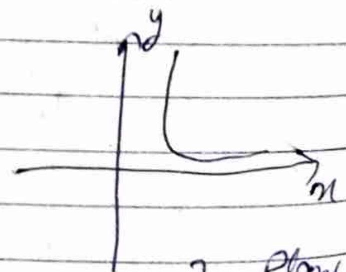
(b) If $v = k$ when k is some constant, then

$2xy = k \Rightarrow xy = \frac{k}{2}$

which represent a curve having asymptote $x = 0, y = 0$



w - Plane



z - Plane

Exp Find the region of the w-plane into which the region $\frac{1}{2} \le x \le 1$ and $\frac{1}{2} \le y \le 1$ is mapped by the transformation $w = z^2$

Sol. the given transformation is $w = z^2$

$u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u = x^2 - y^2 \quad v = 2xy$

when $x = \frac{1}{2}$

$u = \frac{1}{4} - y^2 \quad v = y$

Eliminating y, we have

$v^2 = -(u - \frac{1}{4}) \quad - (1)$

which represents a left handed parabola with vertex $(\frac{1}{4}, 0)$ and latus rectum 1.

when $x = 1$ we have

$u = (1 - y^2) \quad v = 2y$

eliminating y we have

$v^2 = -4(u - 1) \quad - (2)$

which represents a left handed parabola with vertex $(1, 0)$ and latus rectum 4.

when $y = \frac{1}{2}$ we have

$u = x^2 - \frac{1}{4}, \quad v = x$

Eliminating x, we have

$$v^2 = u + \frac{1}{4}$$

which represents a right handed parabola with vertex $(-\frac{1}{4}, 0)$ and latus rectum 1.

when $y = 1$ we have

$$u = x^2 - 1, \quad v = 2x$$

Eliminating x we have

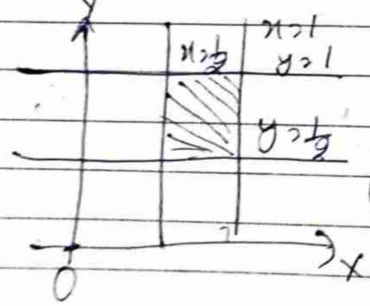
$$v^2 = 4(u+1) \quad \text{--- (4)}$$

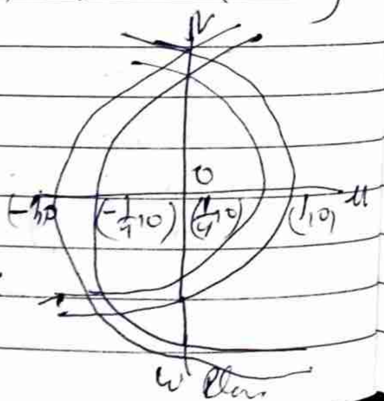
which represents a right handed parabola with vertex $(-1, 0)$ and latus rectum 4.

Hence the rectangular region bounded by the lines $x = \frac{1}{2}$, $x = 1$ and $y = \frac{1}{2}$, $y = 1$ maps in to the domain bounded by the parabolas.

$$v^2 = -(u - \tfrac{1}{4}) \quad v^2 = 4(u-1)$$

$$v^2 = u + \tfrac{1}{4}, \quad v^2 = 4(u+1)$$





Exp. Show that the transformation $w = z^2$ transforms the circle $|z-a| = k$ in the z-plane

(i) to a limacon in the w-plane if $a \neq k$

(ii) to a cardioid in the w-plane if $a = c$

Sol. The given transformation $w = z^2$

The given circle is $|z-a| = k$

$$z = (a + ke^{i\theta})^2$$

Consider,

$$w - a^2 + k^2 = (a + ke^{i\theta})^2 - a^2 + k^2$$

$$= k^2 e^{2i\theta} + 2ake^{i\theta} + k^2$$

$$= ke^{i\theta}\left[ke^{i\theta} + 2a + ke^{-i\theta}\right]$$

$$= ke^{i\theta}\left[k(e^{i\theta} + e^{-i\theta}) + 2a\right]$$

$$= ke^{i\theta}\left[2k\cos\theta + 2a\right]$$

$$= 2ke^{i\theta}(k\cos\theta + a)$$

$$\therefore \; w - (a^2 - k^2) = 2ke^{i\theta}(a + k\cos\theta) \quad \text{--- (1)}$$

Transformation pole in the w-plane to point $a^2 - k^2$, eqn (1) becomes

$$w = 2ke^{i\theta}(a + k\cos\theta)$$

$$Re^{i\phi} = 2ke^{i\theta}(a + k\cos\theta)$$

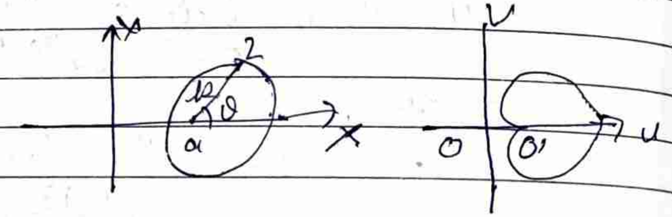
$$R = 2k(a + k\cos\theta) \text{ and } \phi = \theta$$

This image of circle $|z-a| = k$ under $w = z^2$ is

$$R = 2k(a + k\cos\phi)$$

which is the eqn of a limacon in the w-plane.

In particular, if $k = a$ the limacon reduces to $R = 2a^2(1+\cos\phi)$, which is the eqn of a cardiod.



Exp. Find the image of $|z| \leq 1$ in the z-plane under the transformation $w(z+i)^2 = 1$

Sol. The given transformation is

$$w(z+i)^2 = 1$$

$$(z+i)^2 = \frac{1}{w}$$

$$z + i = \frac{1}{\sqrt{w}}$$

$$z = \frac{1}{\sqrt{w}} - i$$

$$= \frac{1}{\sqrt{R}} e^{\frac{-i\phi}{2}} - i \quad [\because w = Re^{i\phi}]$$

$$z = \frac{1}{\sqrt{R}}\cos\frac{\phi}{2} - i\left(\frac{1}{\sqrt{R}}\sin\frac{\phi}{2} + 1\right)$$

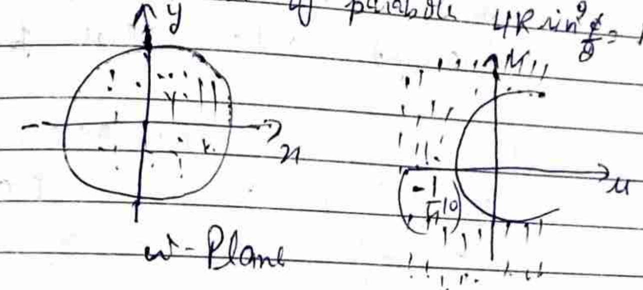
Consider

$$\frac{1}{R} + \frac{2}{\sqrt{R}}\sin\frac{\phi}{2} + 1 \leq 1$$

$$\frac{1}{R} + \frac{2}{\sqrt{R}}\sin\frac{\phi}{2} \leq 0$$

$$4R\sin^2\frac{\phi}{2} \geq 1$$

Thus unit circle $|z| = 1$ corresponds to parabola $4R\sin^2\frac{\phi}{2} = 1$ in the w-plane and the interior of unit circle in z-plane corresponds to the exterior of parabola $4R\sin^2\frac{\phi}{2} = 1$



* To find all the Mobius transformations which map the unit circle $|z| \leq 1$ into the unit circle $|w| \leq 1$

Proof:- Let $w = \frac{az+b}{cz+d}$ such that $ad - bc \neq 0$

be any Mobius transformation which maps the unit circle $|z| \leq 1$ into the circle $|w| \leq 1$

$$w = \frac{a}{c} \cdot \frac{z + \frac{b}{a}}{z + \frac{d}{c}} \quad \text{--- (2)}$$

Clearly $c \neq 0$ otherwise the point at ∞ in the two planes would correspond.

If $w = 0$ then (2) $\Rightarrow z = -\frac{b}{a}$

If $w = \infty$ then (2) $\Rightarrow z = -\frac{d}{c}$

Now the transformation (2) maps a circle of z-plane into a circle of w-plane and inverse points transform into inverse points.

Thus $w = 0, \infty$ the inverse points of $|w| = 1$ are the transforms of $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$ resp.

Hence we may write $-\frac{b}{a} = \alpha$, $-\frac{d}{c} = \frac{1}{\bar{\alpha}}$ such that $|\alpha| < 1$

From (2) we get

$$w = \frac{a}{c}\left[\frac{z-\alpha}{z-\left(\frac{1}{\bar{\alpha}}\right)}\right]$$

$$w = \frac{a\bar{\alpha}}{c}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right) \quad \text{--- (3)}$$

then point $z = 1$ on the boundary of $|z| = 1$ must correspond to a point on the boundary of $|w| = 1$ so that

$$1 = |w| = \left|\frac{a\bar{\alpha}}{c}\right| \left|\frac{1-\alpha}{\bar{\alpha}-1}\right| = \left|\frac{a\bar{\alpha}}{c}\right|$$

we can write $\frac{a\bar{\alpha}}{c} = e^{i\lambda}$

$\therefore$ From (3)

$$w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$$

where $|\alpha| < 1$

which is the required transformation.

verification:-

We have $w\bar{w} - 1$

$$= \left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)\left(\frac{\bar{z}-\bar{\alpha}}{\alpha\bar{z}-1}\right) - 1$$

$$= \frac{(z-\alpha)(\bar{z}-\bar{\alpha}) - (\bar{\alpha}z-1)(\alpha\bar{z}-1)}{(\bar{\alpha}z-1)(\alpha\bar{z}-1)}$$

$$|w|^2 - 1 = \frac{(z\bar{z}-1)(1-\alpha\bar{\alpha})}{|\bar{\alpha}z-1|^2}$$

$$|w|^2 - 1 = \frac{(|z|^2-1)(1-|\alpha|^2)}{|\bar{\alpha}z-1|^2} \quad \text{-} (5)$$

Also $|\alpha|<1$ so that $1-|\alpha|^2>0$

From (5) it is clear that

(i) $|w|=1$ corresponds to $|z|=1$

(ii) $|w|<1$ corresponds to $|z|<1$

Hence $|z|\leq 1$ is transformed to $|w|\leq 1$ by virtue of transformation (4)